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A Finite-Time Adaptive Order Estimation Approach for Non-integer Order Nonlinear Systems

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Abstract

This paper proposes some methods for estimating the order of nonlinear systems having non-integer order. First, the stability of time-varying order systems is studied. Afterward, the multivariable systems with time-varying incommensurate order are studied and an estimation scheme is proposed to approximate the order. In the next step, considering the pseudo-states of a system to be unknown, an order/pseudo-state estimator is designed for a category of nonlinear systems. It is shown that the method is extendible to form order/pseudo-state estimators for other classes of nonlinear systems using other traditional nonlinear observers. One of the advantages of the proposed methods is that a compact time interval is enough to guarantee bounded estimation error.

Keywords: Adaptive Order Estimation, Nonlinear Systems, Variable Order Systems, State Observers

1. Introduction

Considering the order of derivation and integration beyond the limit of integers, calculus is generalized to the case with non-integer order [1]. Non-integer order dynamical systems have wide applications in physics and engineering [1, 1]. The main notability of these systems is the order. The order is a unique attribute in making the non-integer order systems more powerful, more flexible, and more general compared to the traditional ones. The order is not necessarily a constant number. It may vary with respect to time or the systems pseudo-states. Such generalization leads to the variable-order calculus, introduced in [2]. The concept has been investigated from different aspects [1]. The operators are redefined in [3] to be able to handle discontinuous order and form the switching order derivative [4]. Several papers studied the response of the variable order differential equations to introduce the conditions for which the response exists and is unique [5, 6]. Optimality conditions are studied from different sights of view [7, 8] and optimal control methods are developed [9]. Several variable order models are introduced to interpret the order of derivation as some physical quantities e.g. the memory in electronic devices [10], the memory index in human's emotion [11, 12, 13], the effect of strain in viscoelastic materials [14] and soft tissue [15, 16, 17], diffusion and sub-diffusion [18, 19], etc.

In studying traditional integer order systems, there is no such thing as order. However, as soon as we deal with non-integer order, a method should also be provided for estimating it. The response of a system is highly correlated with its order. Minor changes in order may fundamentally change the behavior of the system. A stable system may become unstable with a slight increase in order [1]. Accordingly, detecting the order is the first step is in control and estimation [9, 20]. As mentioned, the order sometimes indicates a physical quantity (e.g. the impact of memory, the amount of viscoelasticity, etc.). In such cases, accurate modeling is tightly tied to the identification of the order, otherwise, one of the quantities involved in the problem is not yet modeled. After suggesting a gray box

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model, it is required to estimate the parameters, and the most important parameter is the order. It is after estimating the order that we can report whether a system is definitely of non-integer order or not. Either it is constant or variable. Therefore, providing a method for estimating the order is a step that must be taken immediately after non-integer order modeling. It complements the definitions, properties, and capabilities provided by non-integer order systems, unless, non-integer order modeling has no practical value and is just some calculations on the paper.

Several methods are proposed focusing on estimation and identification of non-integer order systems, among which, some have considered the order a constant parameter and used an offline or numerical approach to estimate it [21, 22]. Kalman filter is used in [23] to estimate the constant order using a discrete method. In [24] a heuristic method is used to obtain a non-integer order transfer function describing voltammetry electronic tongue. The method is then verified using the experimental data. A signal similarity-based approach is proposed in [25] using a regression kernel method with occupation to approximate non-integer order dynamics as a linear combination of occupation kernels. A parameter estimation method is used in [26] to estimate the parameters of available order describing a class of batteries.

The main advantage of the approach suggested in this paper is the online estimation. In fact, instead of requiring a batch set of data, the method provides the estimation in a real-time manner. Such an approach is firstly used in [27] for estimating the order, where, a derivation operator is suggested which is similar to the variable order derivation operator. It provides an auxiliary system, performing along with the main system, which is a linear SISO system with commensurate order. The order is constant and the system is interpreted by a transfer function. Using the auxiliary system, an error signal is calculated established on which the order estimator is developed. Using the transfer function indicates that the direct input-output relationship is given. Therefore, the pseudo-state estimation problem is not stated. A similar system described by the input-output relationship is studied in [17]. However, the order is considered time-varying and the parameter vector is unknown, as well. A two-stage approach is developed to estimate the order and the parameters. The parameter estimation method is inspired from the recursive least squares famous method. However, an in-between auxiliary system is used to connect twofold of the estimation process. For the first time, real experimental data are used to verify the non-integer order modeling, along with the proposed identification process. The paper includes an experimental study, where, a non-integer order model is fitted on a set of data describing soft tissue deformation. The dynamical behavior of a battery is modeled in[28] when the order is considered non-integer. Afterward, a method is used to estimate the order of the system and its parameters, using real data. The issue of simultaneous estimating of the order and pseudo-states is firstly proposed in [29]. The system is time-varying order described in state-space, with unknown pseudo-states and order. Several basic theories about the stability of variable order systems are proven to develop the estimator. The proposed estimator guarantees bounded error for both the order and the pseudo-states as long as the system is linear with commensurate order. A framework for estimating the parameters of a wide variety of linear time-invariant systems (not necessarily representable in pseudo-state form) is suggested in [30]. One of the studied systems is the non-integer order system with unknown constant order. A method is proposed which basically rearranges the estimation to an optimization problem. It then uses the gradient method to solve the problem with an adaptive scheme. The issue of parameter identification for a class of systems is studied in [31]. The order is considered commensurate. Two filtering algorithms are proposed to estimate the parameters. Afterward, the order adaptation law is given. Adaptive parameter identification is also studied in [32]. In the recent paper, the main idea is to use an extended version of the iterative least squares method to estimate the parameters of non-integer order systems. Indeed, to handle the measurement noise, a bias compensated RLS algorithm is developed. Although some of the above papers have studied the estimation of the non-integer order systems, however, they mainly deal with constant order, linear, and commensurate order systems, where, the main purpose of the current paper is suggesting some effective methods for nonlinear variable order systems. In-line with [27, 29, 17] an adaptive estimation approach is considered here. In [27] the non-integer order system is SISO. A transfer function is used to model it, and the order is constant. The system studied in [29] is linear commensurate order and [17] studies a scalar time-varying integrator. However, the present paper deals with

1. The order estimation of the general multivariable nonlinear incommensurate time-varying order system with known pseudo-states, and,

2. Estimating the order of the multivariable nonlinear time-varying order (with commensurate order) systems with unknown pseudo-states, while the pseudo-states are estimated simultaneously.

Consequently, the main difference of the methods proposed in this paper in comparison to the similar ones is basically the fact that the theorems suggested here are capable of handling nonlinear systems with possibly unknown pseudo-states. Also, the incommensurate order case is considered here, where the former methods mainly deal with commensurate linear systems. The following table provides a comparison between this work and the formers. It shows that whether the methods suggested for estimating the order can be applied on systems with nonlinear dynamics/Incommensurate order/Variable order/Unknown pseudo-states.

Table 1: Comparing methods presented for adaptive order estimation

Paper	Nonlinear	Incommensurate	Multivariable	Variable order	Unknown pseudo-states
[27]	No	No	No	No	No
[17]	No	No	No	Yes	No
[29]	No	No	Yes	Yes	Yes
[30]	No	No	Yes	No	No
Present work	Yes	Yes	Yes	Yes	Yes

The methods proposed here lead to a bounded order estimation error in a compact temporal interval. Consequently, they are useful for control goals, when the order is required to be known to design the control input, by adjusting the estimation time scale smaller than the control time scale.

Accordingly, the paper is organized as follows:

In the second section some fundamental definitions as well as some lemmas are mentioned. Then, in Section 3, first, the nonlinear incommensurate case is studied, then, considering the pseudo-states to be unknown, a proof is given to show the effectiveness of the traditional nonlinear state observers to be used alongside an order estimator and eventually, the discussion is addressed in Section 4.

2. Preliminaries

Definition 1. *In this paper the Caputo derivation is used [33]:*

$$ {}_0^C D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma\left(1-\alpha(t)\right)} \int_0^t (t-\tau)^{-\alpha(t)} \frac{dx(\tau)}{d\tau} d\tau + \Psi_c^x(t), \; 0 < \alpha(t) < 1, \forall t \geq 0 \tag{1} $$

where, $\Gamma(\beta) = \int_0^\infty \theta^{\beta-1} e^{-\theta} d\theta$.

$\Psi_c^x(t) = \frac{1}{\Gamma\left(1-\alpha(t)\right)} \int_{-c}^0 \left(t-\tau\right)^{-\alpha(t)} \frac{dx(\tau)}{d\tau} d\tau$ *is the initializing function. To avoid discontinuity, mainly in the time-delay systems, it reflects the values of* x *in* $[-c \;\; 0]$ *before the starting time of the derivation operator,* $t = 0$ *[34, 35]. Since the systems studied here are all considered at rest in* $0 < t$*, the initializing function* $\Psi_c^x(t)$ *is equal to zero and ignored in the rest of the paper. Therefore, (1)* *is reduced to (2).*

$$
{}_0^C D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma\big(1-\alpha(t)\big)} \int_0^t (t-\tau)^{-\alpha(t)} \frac{dx(\tau)}{d\tau} d\tau \tag{2}
$$

Definition 2. *The definition of the integration of the function x with the variable order $\alpha(t)$ is [33]*

$$
\begin{gathered}
{}_0 I_t^{\alpha(t)} x(t) = \tfrac{1}{\Gamma(\alpha(t))} \int_0^t (t-\tau)^{\alpha(t)-1} x(\tau) d\tau \\
0 < \alpha(t) < 1, \forall t \geq 0
\end{gathered} \tag{3}
$$

Considering the operators defined in (1) and (2);

$$
{}_0^C D_t^{\alpha(t)} I_t^{\alpha(t)} f = f \tag{4}
$$

Definition 3. *A dynamical system with non-integer order may be defined as a series of coupled differential equations possibly with different orders of derivation[36, 1]. Here, the following definition is used:*

$$
\begin{gathered}
\begin{cases}
{}_0^C D_t^{\alpha_1(t)} x_1(t) = f_1\big(x_1(t), \ldots, x_n(t), U(t)\big) \\
\vdots \\
{}_0^C D_t^{\alpha_n(t)} x_n(t) = f_n\big(x_1(t), \ldots, x_n(t), U(t)\big)
\end{cases} \\
y(t) = g\big(x_1(t), \ldots, x_n(t), U(t)\big)
\end{gathered} \tag{5}
$$

where, $0 < \alpha_i < 1, i = 1, 2, \ldots, n$ are the orders, $X = [x_1 \ \ldots \ x_n]^T$ is the state vector, or to be more precise, the pseudo-state vector, $U = [u_1 \ \ldots \ u_p]^T$ is the input, and $y = [y_1(t) \ \ldots \ y_q(t)]^T$ is the output. Each equation of (5) is equal to the following Volterra integral equation: [37]:

$$
x_i(t) = x_i(0) + \frac{1}{\Gamma(\alpha_i(t))} \int_0^t (t-\tau)^{\alpha_i(t)-1} f_i(x(\tau), u(\tau)) d\tau, \ i = 1, 2, \ldots, n \tag{6}
$$

A shorter notation for system (5) is

$$
{}_0^C D^{\bar{\alpha}(t)} X = F(X, U), \ y = g(X, U) \tag{7}
$$

where $F(X, U) = [f_1(X, U) \ldots f_n(X, U)]^T$, and $\bar{\alpha}(t) = [\alpha_1(t) \ldots \alpha_n(t)]^T$ is the order vector. A commensurate order system is a system in which all the entries of the order vector are equal. Furthermore, when $F(.,.)$ and $g(.,.)$ are two linear functions of x, u the system (7) is linear. Using constant matrices $A, \ B, \ C$ and D with appropriate dimensions, the linear system is interpreted as:

$$
\begin{gathered}
{}_0^C D^{\bar{\alpha}(t)} x = Ax + Bu \\
y = Cx + Du
\end{gathered} \tag{8}
$$

It is noteworthy to mention that although the argument (t) may be omitted for the sake of convenience, however, the order is time-varying all along this paper.

Lemma 1. *The inequality ${}_0^C D^{\gamma(t)}(x^T P x) \leq 2x^T P {}_0^C D^{\gamma(t)} x$ holds as long as the non-integer order derivative ${}_0^C D^{\gamma(t)} x$ exists and the matrix P is Hermitian and positive definite.*

Proof. The lemma is shortly proven in [29], however, a more precise proof is provided here.

Decompose P into $P = P^{\frac{1}{2}}P^{\frac{1}{2}}$ and define $w = P^{\frac{1}{2}}x$. Obviously, $P^{\frac{1}{2}}$ is positive definite matrix. It is also Hermitian.

Therefore;

$$2w^T(t){}_0^C D^{\gamma(t)} w(t) - {}_0^C D^{\gamma(t)}\big(w^T(t)w(t)\big) \geq 0 \tag{9}$$

Using (2) and then using the integration by parts approach, the left-hand side of (9) is simplified to:

$$\begin{aligned}
&2w^T(t){}_0^C D^{\gamma(t)} w(t) - {}_0^C D^{\gamma(t)}\big(w^T(t)w(t)\big)\\
&= \frac{2w^T(t)}{\Gamma(1-\gamma(t))}\int_0^t (t-\tau)^{-\gamma(t)}\frac{d}{d\tau}w(\tau)d\tau - \frac{1}{\Gamma(1-\gamma(t))}\int_0^t (t-\tau)^{-\gamma(t)}\frac{d}{d\tau}\big(w^T(\tau)w(\tau)\big)d\tau\\
&= \frac{1}{\Gamma(1-\gamma(t))}\int_0^t (t-\tau)^{-\gamma(t)}\frac{d}{d\tau}\Big(2w^T(t)w(\tau) - w^T(\tau)w(\tau)\Big)d\tau\\
&= \frac{1}{\Gamma(1-\gamma(t))}\int_0^t (t-\tau)^{-\gamma(t)} 2\Big(w^T(t) - w^T(\tau)\Big)\frac{d}{d\tau}w(\tau)d\tau\\
&= \frac{-1}{\Gamma(1-\gamma(t))}\left(\frac{\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{(t-\tau)^{\gamma(t)}}\right)\Big|_{\tau=0}^{\tau=t} + \frac{\gamma(t)}{\Gamma(1-\gamma(t))}\int_0^t \frac{\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{(t-\tau)^{\gamma(t)+1}}d\tau
\end{aligned} \tag{10}$$

1. Since $\Gamma(1-\gamma(t))$ and $(t-\tau)^{\gamma(t)+1}$ are non-negative for $0 < \gamma(t) < 1$ and $\tau \leq t$, and $\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big) \geq 0$ therefore $\frac{\gamma(t)}{\Gamma(1-\gamma(t))}\int_0^t \frac{\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{(t-\tau)^{\gamma(t)+1}}d\tau \geq 0$.

2. Define $Y(t,\tau) = \frac{-1}{\Gamma(1-\gamma(t))}\left(\frac{\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{(t-\tau)^{\gamma(t)}}\right)$. Then,

$$Y(t,0) = \frac{-1}{\Gamma(1-\gamma(t))}\left(\frac{\big(w(t)-w(0)\big)^T\big(w(t)-w(0)\big)}{t^{\gamma(t)}}\right) \leq 0 \tag{11}$$

To Calculate $Y(t,\tau)$ at $\tau = t$ the H'opital formula is used:

$$\begin{aligned}
Y(t,t) &= \lim_{\tau\to t} Y(t,\tau) = \lim_{\tau\to t}\left(\frac{\frac{d}{d\tau}\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{\frac{d}{d\tau}(t-\tau)^{\gamma(t)}}\right)\\
&= \left(\frac{-2\big(w(t)-w(\tau)\big)^T\frac{dw(\tau)}{d\tau}}{-\gamma(t)(t-\tau)^{\gamma(t)-1}}\right)\Bigg|_{\tau=t}\\
&= \left(\frac{2}{\gamma(t)}\big(w(t)-w(\tau)\big)^T\frac{dw(\tau)}{d\tau}(t-\tau)^{1-\gamma(t)}\right)\Bigg|_{\tau=t} = 0
\end{aligned} \tag{12}$$

Accordingly,

$$\begin{aligned}
\frac{-1}{\Gamma(1-\gamma(t))}\left(\frac{\big(w(t)-w(\tau)\big)^T\big(w(t)-w(\tau)\big)}{(t-\tau)^{\gamma(t)}}\right)\Bigg|_{\tau=0}^{\tau=t} &= Y(t,t) - Y(t,0)\\
&= 0 - Y(t,0) \geq 0
\end{aligned} \tag{13}$$

Considering (11) to (13), (10) is non-negative. Hence, $2w^T{}_0^C D^{\gamma(t)}w - {}_0^C D^{\gamma(t)}(w^T w) \geq 0$ and the proof is completed. □

Theorem 1. *For the variable order system ${}_0^C D^{\beta(t)}x = f(x), x(0) = x_0, 0 < \beta(t) < 1, \forall t$. Consider $V \geq 0$ such that $V(x) = 0 \iff x = 0$ and $V(x) > 0, x \neq 0$. In such conditions Lyapunov stability of the equilibrium point $x = 0$ holds if ${}_0^C D^{\beta(t)}(V) \leq 0$ in neighborhood $\Delta \subset R^n$ around $x = 0$. Furthermore, asymptotically stability holds when ${}_0^C D^{\beta(t)}(V) < 0, x \neq 0$.*

Proof. The theorem provides an extension for the second Lyapunov theorem of stability to the variable order systems. Proof is given in [29]. □

Corollary 1. *For the system ${}_0^C D^{\alpha(t)}x = f(x),\ f(0) = 0$ the origin is Lyapunov stable providing that it is stable in the integer order equivalent of the main system, i.e., the system $\dot{x} = f(x)$ with a Lyapunov function of the form $V = \frac{1}{2}x^T Px$, where $P > 0$.*

Proof. With the Lyapunov function $V = \frac{1}{2}x^T Px$, stability of the integer order system implies that $\dot{V} = \frac{\partial V}{\partial x}\dot{x} = x^T Pf(x) \leq 0$. According to Lemma 1, ${}_0^C D^{\alpha(t)}V \leq x^T P\, {}_0^C D^{\alpha(t)}x = x^T Pf(x) \leq 0$. □

Corollary 2. *As long as A is Hurwitz, the linear system ${}_0^C D^{\beta(t)}x = Ax$ is stable. Also, asymptotically stability necessarily hold in such conditions, i.e. $x \to 0$ as $t \to \infty$.*

Proof. Proof is given in [29]. □

Considering Equations (5) and (6), define $z_\phi(t, \beta)$ as the following integral equation:

$$z_\phi(t, \beta) = \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}\phi(\tau)d\tau \tag{14}$$

Differentiating (14), with respect to β yields

$$\frac{\partial z_\phi(t, \beta)}{\partial \beta} = -\frac{\psi(\beta(t))}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}\phi(\tau)d\tau + \frac{1}{\Gamma(\beta(t))}\int_0^t ln(t-\tau)(t-\tau)^{\beta(t)-1}\phi(\tau)d\tau \tag{15}$$

where $\psi(\beta) = \frac{d}{d\beta}ln(\Gamma(\beta)) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$. Based on (14) and (15), as long as the history of $\phi(\tau), \tau \in [0\ \ t]$ and $\beta(t)$ are available, $z_\phi(t, \beta)$, $\frac{\partial z_\phi(t,\beta)}{\partial \beta}$ is computable.

Lemma 2. *Define*

$$\begin{aligned} x(t) &= x(0) + \int_0^t \frac{(t-\tau)^{\alpha(t)-1}\phi(\tau)}{\Gamma(\alpha(t))}d\tau \\ \hat{x}(t) &= \hat{x}(0) + \int_0^t \frac{(t-\tau)^{\hat{\alpha}(t)-1}\phi(\tau)}{\Gamma(\hat{\alpha}(t))}d\tau \end{aligned} \tag{16}$$

Define $e(t) = x(t) - \hat{x}(t)$. Based on (16) and considering the definition of z_ϕ in (14), it is calculated as:

$$e(t) = e(0) + \frac{\partial z_\phi(t, \beta)}{\partial \beta}\Big|_{\beta=\hat{\alpha}}(\alpha(t) - \hat{\alpha}(t)) + K(t)\big(\alpha(t) - \hat{\alpha}(t)\big)^2 \tag{17}$$

where $K(t) = \int_0^t \frac{\partial^2 H(\tau,t,\beta)}{\partial \beta^2}\big|_{\beta=\alpha_0}\phi(\tau)d\tau,\ \alpha_0(t) \in \big[min\big(\hat{\alpha}(t), \alpha(t)\big)\ \ max\big(\hat{\alpha}(t), \alpha(t)\big)\big]$.

Proof. Consider the continuous function $H\big(t, \tau, \beta\big) = \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}$. As long as $0 < \tau < t$ and $0 < \beta < 1$, $H(.,.,.)$ is analytical with respect to its third argument, β. Accordingly, using the Taylor Series Expansion Theorem, $\exists \alpha_0(t) \in \big[min\big(\hat{\alpha}(t), \alpha(t)\big)\ \ max\big(\hat{\alpha}(t), \alpha(t)\big)\big]$

$$\begin{aligned} e(t) =& e(0) + \int_0^t \Big(\frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))} - \frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))}\Big)\phi(\tau)d\tau \\ =& e(0) + \int_0^t \Big(\frac{\partial H(\tau, t, \beta)}{\partial \beta}\Big|_{\beta=\hat{\alpha}}(\alpha(t) - \hat{\alpha}(t)) + \frac{\partial^2 H(\tau, t, \beta)}{\partial \beta^2}\Big|_{\beta=\alpha_0}(\alpha(t) - \hat{\alpha}(t))^2\Big)\phi(\tau)d\tau \end{aligned} \tag{18}$$

α and $\hat{\alpha}$ are independent on τ, therefore, (18) can be rewritten as:

$$e(t) = e(0) + (\alpha(t) - \hat{\alpha}(t)) \int_0^t \frac{\partial H(\tau,t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}} \phi(\tau) d\tau + (\alpha(t) - \hat{\alpha}(t))^2 \int_0^t \frac{\partial^2 H(\tau,t,\beta)}{\partial \beta^2}|_{\beta=\alpha_0} \phi(\tau) d\tau \tag{19}$$

Based on (14), $\int_0^t \frac{\partial H(\tau,t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}} \phi(\tau) d\tau = \frac{\partial z_\phi(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}$. Consequently,(19) is reduced to the shorter form (20) and the proof is completed.

$$e(t) = \frac{\partial z_\phi(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}} (\alpha(t) - \hat{\alpha}(t)) + K(t)(\alpha(t) - \hat{\alpha}(t))^2 \tag{20}$$

□

Lemma 3. *For continuous function ϕ, K in (17) is bounded with the bound $|K| < M_K T^3$ for some $M_K > 0$ as long as $t \in [0\ T]$, $T > 1$.*

Proof. See Appendix. □

Lemma 4. *The inequality $a\theta^3 + b\theta^2 + c\theta < -d\theta, \theta > 0, a > 0$ holds when $\frac{-b-\sqrt{\Delta}}{2a} < \theta < \frac{-b+\sqrt{\Delta}}{2a}$ providing that $\Delta = b^2 - 4a(c + d) > 0$.*

Proof. Defining $\Pi(\theta) = \theta^3 + b\theta^2 + (c+d)\theta$, the proof is straightforward after factorizing $\Pi(\theta)$ as $\Pi(\theta) = \theta(\theta - \frac{-b-\sqrt{\Delta}}{2a})(\theta - \frac{-b+\sqrt{\Delta}}{2a})$. □

Lemma 5. *Consider $\beta_L = \frac{\gamma\eta-\sqrt{\Delta}}{2\gamma|K|}, \beta_U = \frac{\gamma\eta+\sqrt{\Delta}}{2\gamma|K|}$ with $\Delta = \gamma^2\eta^2 - 4\gamma|K|(m+M)$ and non-negative values of $\gamma, \eta, |K|, M, m$, when $\exists K_0, |K| < K_0, \exists \eta_0, \eta > \eta_0$, and $\gamma > \frac{4(m+M)K_0}{\eta_0}$. Then,*

$$\frac{\partial \beta_L}{\partial \eta} \leq 0,\ \frac{\partial \beta_L}{\partial |K|} \geq 0,\ \frac{\partial \beta_U}{\partial \eta} \geq 0,\ \frac{\partial \beta_U}{\partial |K|} \leq 0$$

Proof. See Appendix. □

3. Main Results

In this Section, several theorems are proposed suggesting some methods for estimating the order of nonlinear systems in the most general case, i.e., multivariable nonlinear incommensurate time-varying order systems, with known pseudo-states and nonlinear commensurate time-varying order systems, with unknown pseudo-states.

Theorem 2. *Consider x as the measurable response of the system ${}_0^C D^{\alpha(t)} x = f\big(x, u\big)$ with the initial condition $x(0) = x_0$ and the estimated signal $\hat{x}(t) = x_0 + \int_0^t \frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))} f(x(\tau), u(\tau)) d\tau$. The order is differentiable, i.e., $\exists M > 0, \|\dot{\alpha}\| \leq M$. The adaptation rule $\dot{\hat{\alpha}} = \lambda(x - \hat{x})$ makes the error $\alpha - \hat{\alpha}$ get bounded within a finite time, where, $\lambda = \gamma sgn\Big(\frac{\partial z_\phi(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)^T$, $\phi(\tau) = f(x(\tau), u(\tau))$, γ is a positive real design parameter, and $sgn(.)$ is the sign function and $sgn([a_1 \dots a_n] = [sgn(a_1) \dots sgn(a_n)])$.*

Proof. Define $\eta(t) = \left|\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\right|$. Obviously $\eta(t)$ is non-negative for all t. Consider the Lyapunov function $V = \frac{1}{2}(\alpha - \hat{\alpha})^2$. According to Eq. 17 (with $e(0) = 0$, since $x(0) = \hat{x}(0)$);

$$\begin{aligned}\dot{V} &= \dot{\alpha}(\alpha - \hat{\alpha}) - \lambda(x - \hat{x})(\alpha - \hat{\alpha}) \\ &\leq |K||\lambda|\|\alpha - \hat{\alpha}\|^3 - \lambda \frac{\partial z_\phi(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}} \|\alpha - \hat{\alpha}\|^2 + M\|\alpha - \hat{\alpha}\| \\ &= \gamma|K|\|\alpha - \hat{\alpha}\|^3 - \gamma\eta\|\alpha - \hat{\alpha}\|^2 + M\|\alpha - \hat{\alpha}\|\end{aligned} \tag{21}$$

After adding $-d\|\alpha-\hat{\alpha}\|$ to both sides of (21), according to Lemma 4, $\dot{V}<-d\|\alpha-\hat{\alpha}\|$ when $\beta_L=\frac{\gamma\eta-\sqrt{\Delta}}{2\gamma|K|}<\|\alpha-\hat{\alpha}\|<\frac{\gamma\eta+\sqrt{\Delta}}{2\gamma|K|}=\beta_U$, $\Delta=\gamma^2\eta^2-4\gamma|K|(M+d)$. Supposing $\eta>0$, there exists $\eta_0>0$ such that $\eta>\eta_0$. Also, $|K|<M_KT^3(=K_0$ in Lemma 5). Therefore, defining $\Delta_0=\gamma^2\eta_0^2-4\gamma M_KT^3(M+d)$, based on Lemma 5, setting $m=d$, β_L is ascending with respect to $|K|$ and descending with respect to η, hence $\beta_L\leq\frac{\gamma\eta_0-\sqrt{\Delta_0}}{2\gamma M_KT^3}=B_L$. Furthermore, β_U is ascending with respect to η and descending with respect to $|K|$. Therefore, $B_U=\frac{\gamma\eta_0+\sqrt{\Delta_0}}{2\gamma M_KT^3}\leq\beta_U$. Consequently, $B_L<\|\alpha-\hat{\alpha}\|<B_U\rightarrow\beta_L<\|\alpha-\hat{\alpha}\|<\beta_U\Rightarrow\dot{V}<-d\|\alpha-\hat{\alpha}\|=-d\sqrt{2}V^{\frac{1}{2}}$.

Obviously, B_L,B_U are constant and $B_U-B_L=\frac{\sqrt{\Delta_0}}{\gamma M_KT^3}$. Accordingly, large γ (i.e. $\gamma>\frac{4(M+d)M_KT^3}{\eta_0^2}$) ensures that $B_L<B_U$ certifying that the aforementioned interval exists.

It means that the set $S_{\hat{\alpha}}=\{\hat{\alpha},B_L<\|\alpha-\hat{\alpha}\|<B_U\}$ (which is never empty) is an invariant set for $t\in[0\ \ T]$. Hence, there is a non-empty neighborhood around the actual order, α, where the trajectory would not leave this neighborhood in $t\in[0\ \ T]$, also, since $\dot{V}<-d\sqrt{2}V^{\frac{1}{2}}$ the error $\|\alpha-\hat{\alpha}\|$ gets smaller and tends to its lower bound, B_L. Also, since $\dot{V}<-d\sqrt{2}V^{\frac{1}{2}}<0$ in the above interval, as soon as the trajectory escapes the interval $\|\alpha-\hat{\alpha}\|\leq B_L$, it is pulled back inside. Accordingly, $B_L<\|\alpha_0-\hat{\alpha}(0)\|$, where, $\alpha_0-\hat{\alpha}(0)$ is the initial estimation error. Furthermost, the convergence time is finite, because;

$\dot{V}<-d\sqrt{2}V^{\frac{1}{2}}\rightarrow V^{-\frac{1}{2}}dV<-d\sqrt{2}dt\rightarrow\int_{\frac{1}{2}\|\alpha_0-\hat{\alpha}(0)\|^2}^{\frac{1}{2}{B_L}^2}V^{-\frac{1}{2}}dV<-d\sqrt{2}\int_0^{T_1}dt\rightarrow T_1<\frac{\|\alpha_0-\hat{\alpha}(0)\|^2-B_L^2}{d}$

T_1 is finite and its upper bound is positive. This means that the finite-time convergence of error to the bound $\|\alpha-\hat{\alpha}\|\leq B_L$ is guaranteed when T (and therefore γ) is set large enough in a way that $T>T_1$. □

Remark 1. $\gamma>\frac{4(M+d)M_KT^3}{\eta_0^2}\rightarrow\gamma>\frac{4(M+d)|K|}{\eta^2}$*, hence,* Δ *is positive,* β_L,β_U *are real and* $\beta_L<\beta_U$*.*

Remark 2. *For the case* $\eta=0$ *(which based on (15) may occur for some values of t satisfying* $\psi(\beta(t))=\frac{\int_0^t ln(t-\tau)(t-\tau)^{\beta(t)-1}\phi(\tau)d\tau}{\int_0^t(t-\tau)^{\beta(t)-1}\phi(\tau)d\tau}$*) it should be noted that the trajectory leaves the set* $\{\hat{\alpha}:\eta(t)=0\}$ *because is not an invariant set (i.e. it does not satisfy* $\dot{\hat{\alpha}}=0,\forall t$*), therefore it does not violate the convergence.*

Remark 3. γ *is a very important design parameter. In fact, the boundedness of* K *requires temporal compactness, i.e. the results are valid for* $T<\infty$*. Since the invariant set* $S_{\hat{\alpha}}$ *would be never empty, however, it requires* γ *to be set sufficiently large. Increasing* T *(and, increasing* γ *after it) gives the error* $\alpha-\hat{\alpha}$ *enough time to get its lower bound* B_L*.*

Also, $\lim_{\gamma\rightarrow\infty}B_L=\lim_{\gamma\rightarrow\infty}\frac{\gamma\eta_0-\sqrt{\Delta_0}}{2\gamma M_KT^3\eta_0}=\lim_{\gamma\rightarrow\infty}\frac{\gamma\eta_0-(\gamma\eta_0-2M_K(M+d)T^3)}{2\gamma M_KT^3\eta_0}=\lim_{\gamma\rightarrow\infty}\frac{M+d}{\gamma\eta_0}=0$*. Hence, larger* γ *leads to smaller estimation error.*

Theorem 2 proposes an adaptation rule for order estimation of nonlinear non-integer order systems whose pseudo-states are available. In Theorem 3, the results are extended to the incommensurate multivariable case.

Theorem 3. *(Order Estimation in Multivariable Incommensurate Order Case) For the system described in Eq. (5) with* $\|\dot{\hat{\alpha}}(t)\|\leq M$*, the adaptation rules* $\dot{\hat{\alpha}}_i=\lambda_i(x_i-\hat{x}_i),i=1,...,n$*, with the following definitions lead to a finite-time bounded error for all the estimated orders.*

$$\begin{aligned}&\phi_i(\tau)=f_i(x_1(\tau),...,x_n(\tau),u(\tau))\\&\lambda_i=\gamma_i sgn\Big(\frac{\partial z_{\phi_i}(t,\beta)}{\partial\beta}|_{\beta=\hat{\alpha}_i}\Big),\gamma_i>0\\&\hat{x}_i=x_i(0)+\frac{1}{\Gamma(\hat{\alpha}_i(t))}\int_0^t(t-\tau)^{\hat{\alpha}_i(t)-1}\phi_i(\tau)d\tau\end{aligned}\tag{22}$$

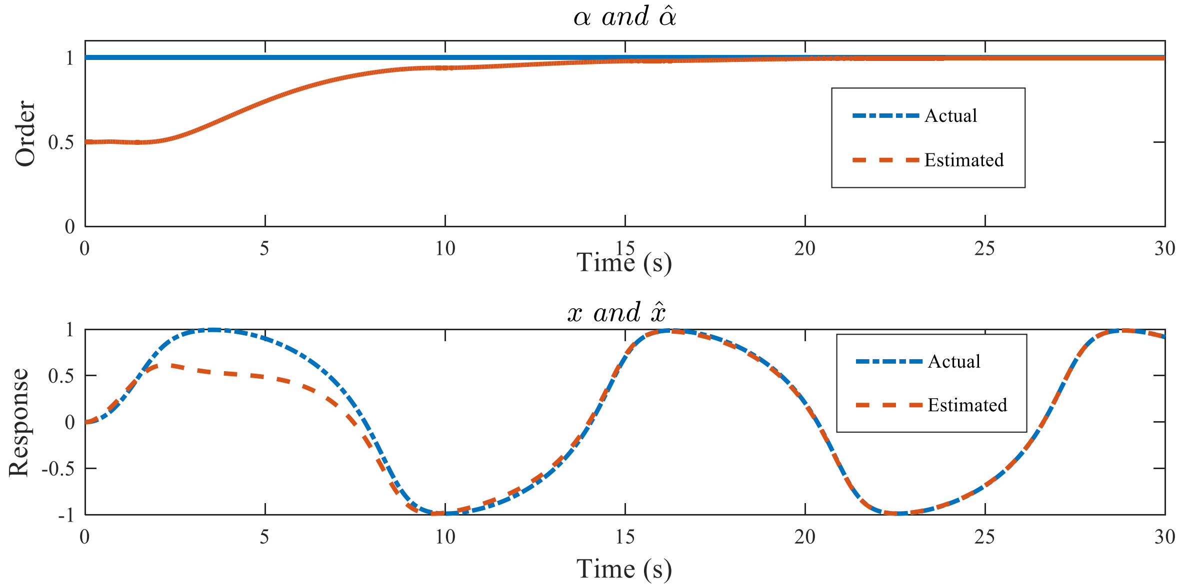

Figure 1: The actual and estimated order and state for the integer order system (23)

Proof. Changing the Lyapunov function to $V = \frac{1}{2}\sum_{i=1}^{n} V_i, V_i = (\alpha_i - \hat{\alpha}_i)^2$, the proof would be similar to the proof of the former theorem. $\square$

Therefore, an adaptive order estimator is designed. As discussed before, the proposed method is somehow able to verify if a system is of non-integer order. Figure 1 indicates the convergence of the estimated order to 1, albeit with a bounded error, when the case study is the integer order system $\dot{x} = -x^3 + u$ with zero initial condition, $x(0) = \hat{x}(0) = 0$ and, $u = sin(0.5t)$, $\hat{\alpha}(0) = 0.5$. As mentioned, the order is a function of time all over the paper. In fact, in this example it is considered as $\mathbf{1}(t)$. In general, the above theorem provides an approach for estimating the order of nonlinear multivariable incommensurate order systems. Figures 2 and 3 show the simulation results considering the following case study:

$$\begin{aligned}
&{}_0^C D^{\bar{\alpha}(t)} x = F(x,u), \bar{\alpha}(t) = \left(\begin{smallmatrix} 0.7-0.2e^{-0.02t} sin(0.3t) \\ 0.45e^{-0.1t}+0.2 \end{smallmatrix} \right) \\
&x(0) = \hat{x}(0) = \left(\begin{smallmatrix} 0.1 \\ -0.2 \end{smallmatrix} \right), \hat{\alpha}(0) = \left(\begin{smallmatrix} 0.2 \\ 0.2 \end{smallmatrix} \right) \\
&u(t) = 1 - e^{-0.1t} + sin(t) \\
&F(x,u) = \left(\begin{smallmatrix} -x_1+u \\ -x_1 cos(x_2)-0.2u \end{smallmatrix} \right)
\end{aligned} \tag{23}$$

Figure 2 shows that the order estimator (22) efficiently works and leads to bounded error and Figure 3 shows the convergence of $\hat{x} \to x$.

As long as the pseudo-states are available, the method presented in Theorem 3 works for all systems. However, when the pseudo-states are not available, in addition to the order adaptation rule, a pseudo-state observer is needed, as well. In this case, the stability proof should be given for the coupled order/pseudo-state observer. Based on Corollary 1, stability of an integer order system with Lyapunov function of the form $\frac{1}{2}x^T Px, P > 0$ leads to stability of its non-integer order equivalent. In this regard, as long as the order

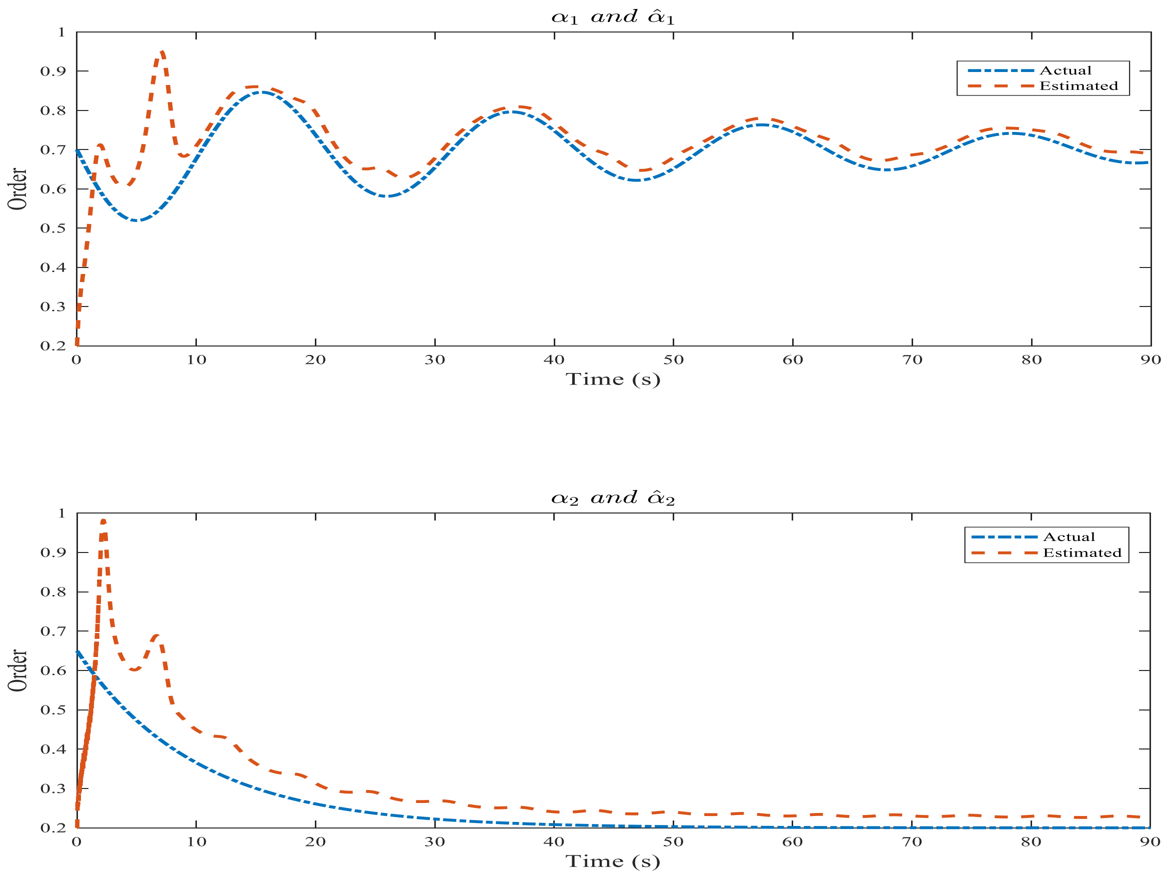

Figure 2: The original and estimated orders for system (23). top: α_1, bottom: α_2

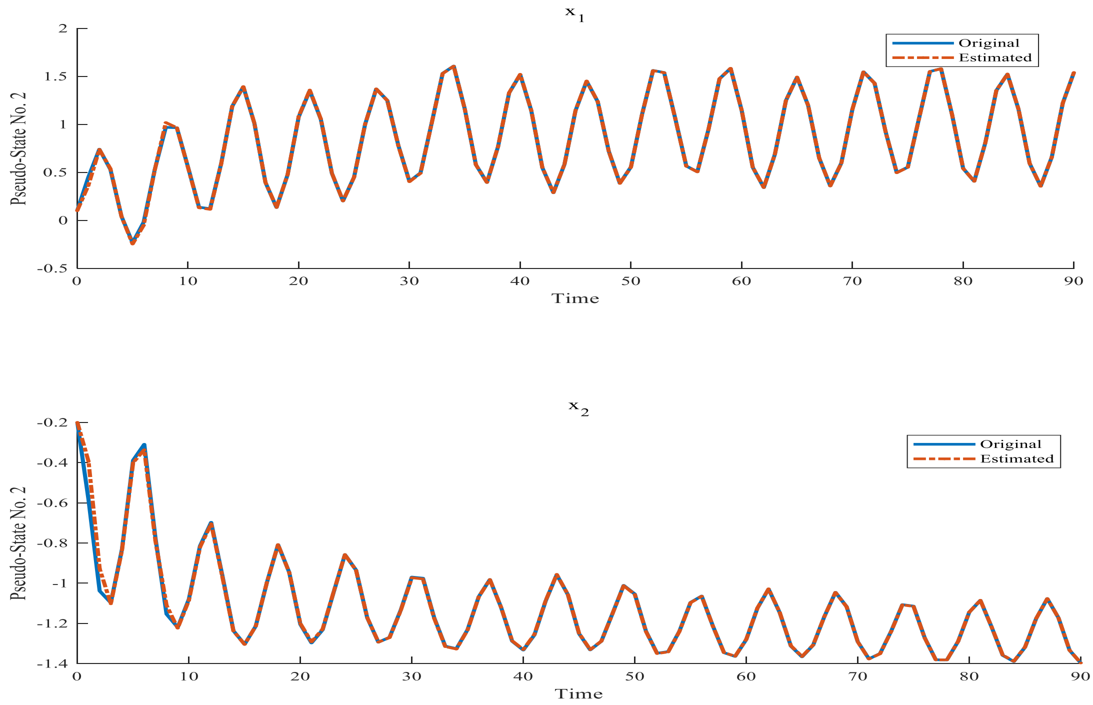

Figure 3: The actual and estimated pseudo-states for system (23). top: x_1 vs. $\hat{x}_1$, bottom: x_2 vs. $\hat{x}_2$

is known, most of the traditional state observers work in the non-integer order world. However, what it would be with unknown order? The stability must be proven when the order estimator is also included. The following lemmas and theorems prove that the traditional observers can be extended to order/pseudo-state estimators for the non-integer order case with an appropriate order adaptation rule. After proving some lemmas, the high gain observer is studied, as an important nonlinear observer.

Lemma 6. *Consider the systems* ${}_0^C D^{\beta(t)} e_1 = Ae_1 + \delta(e_1) + v$ *and* ${}_0^C D^{\beta(t)} e_2 = Ae_2 + v$*. A is Hurwitz and* $\|\delta(e_1)\| < c\|e_1\|$ *for small enough* $c > 0$*. If v is bounded; 1. e_2 is bounded. 2.* $\|e_1 - e_2\|$ *is bounded.*

Proof. Since A is Hurwitz, the positive definite Q exists such that $A + A^T = -Q$. Consider the Lyapunov function $W = \frac{1}{2} e_2^T e_2$ for the second system. Then,

$$ {}_0^C D^{\beta(t)} W \leq e_2^T {}_0^C D^{\beta(t)} e_2 \leq -e_2^T Q e_2 + \|e_2\|\|v\| = -\|e_2\|^2 \|Q\| + \|e_2\|\|v\| \tag{24} $$

Therefore, as $\|e_2\|$ escapes the bound $\frac{\|v\|}{\|Q\|}$ right hand side of (24) would be negative and the trajectory would be pulled back. Hence, e_2 is bounded in the bound $\|e_2\| < \frac{\|v\|}{\|Q\|}$. Since v is bounded, $\exists M; \|v\| < M$, so, $\|e_2\| < \frac{M}{\|Q\|}$ and this proves part 1.
Now, define $\zeta = e_1 - e_2$. Hence, ${}_0^C D^{\beta(t)} \zeta = A\zeta + \delta(e_1)$. Consider the Lyapunov function $V = \frac{1}{2}\zeta^T \zeta$. Consequently;

$$ \begin{aligned} {}_0^C D^{\beta(t)} V &\leq \zeta^T {}_0^C D^{\beta(t)} \zeta \\ &= -\zeta^T Q \zeta + \zeta^T \delta(\zeta + e_2) \leq -\zeta^T Q \zeta + \zeta^T c\|\zeta + e_2\| \\ &\leq -\|\zeta\|^2 \|Q\| + c\|\zeta\|^2 + c\|\zeta\|\|e_2\| \\ &\leq -\|\zeta\|^2 (\|Q\| - c) + c\frac{M}{\|Q\|}\|\zeta\| \end{aligned} \tag{25} $$

Consequently, for $c < \|Q\|$, outside the bound $\|\zeta\| > \frac{cM}{\|Q\|(\|Q\|-c)}$ the last term in (25) is negative. Therefore, ζ would remain in the bound $\zeta < \frac{cM}{\|Q\|(\|Q\|-c)}$. It can be concluded that the error bound can be small enough by increasing $\|Q\|$, which is in turn, in direct relation with the absolute value of the eigenvalues of A. □

Lemma 7. *When A is Hurwitz,*

1. The system ${}_0^C D^{\beta(t)} e = Ae + v, e(0) = e_0$ as $t \to \infty$ is equivalent to ${}_0^C D^{\beta(t)} e = v, e(0) = 0$.

2. The signal e defined as $e = e_0 + {}_0I_t^{\beta(t)}(Ae) + w, w(0) = 0$. tends to w as t tends to ∞.

Proof. 1. In the first system, e is the summation of the responses of ${}_0^C D^{\beta(t)} e = Ae, e(0) = e_0$ and ${}_0^C D^{\beta(t)} e = v, e(0) = 0$. According to Corollary 2, the first part tends to 0 as $t \to \infty$, therefore, the main response tends to the second part.

2. Based on the Volterra equivalent integral (equation (6)) and considering (4), after applying the operator ${}_0^C D^{\beta(t)}$ on both sides of the integral equation mentioned in part 2, we have ${}_0^C D^{\beta(t)} e = Ae + v, e(0) = e_0$, where $v = {}_0^C D^{\beta(t)} w$. Now, part 1 implies that as $t \to \infty$ the system is equivalent to ${}_0^C D^{\beta(t)} e = v, e(0) = 0$. We have also $w(0) = 0$ leading to ${}_0^C D^{\beta(t)}(e - w) = 0, e(0) - w(0) = 0$, which results to $e - w = 0, t \to \infty$. □

Corollary 3. *If $e = {}_0I_t^{\beta(t)}(Ae + \delta) + w, w(0) = 0$ with $\|\delta\| < c\|e\|$, as $t \to \infty$, $\exists \mu, \|e - w\| < \mu$ and μ is related to the eigenvalues of A.*

Proof. The above relationship is equivalent to the system ${}_0^C D^{\beta(t)} e = Ae + \delta + {}_0^C D^{\beta(t)} w$. Comparing with the system ${}_0^C D^{\beta(t)} e' = Ae' + {}_0^C D^{\beta(t)} w$ as $t \to \infty$; $e' \to w, also, \exists \mu : \|e - e'\| < \mu$, therefore, $\|e - w\| < \mu$. Additionally, based on Lemma 6, while the eigenvalues of A are far enough from the imaginary axis μ can be adjusted small enough. □

Theorem 4. *(High Gain Order/Pseudo-State Estimator) Consider the following system for $t \in [0 \;\; T]$, suppose that $\exists M, \|\dot{\alpha}\| < M$.*

$$\begin{cases} {}_0^C D_t^{\alpha(t)} x_1 = x_2 \\ {}_0^C D_t^{\alpha(t)} x_2(t) = f(x, u) \end{cases} \qquad (26)$$
$$y(t) = x_1$$

$f(.,.)$ is Lipschitz with respect to its first argument, i.e. $\exists c, \|f(x, u) - f(\hat{x}, u)\| < c\|x - \hat{x}\|$. The following equations with high enough gains h_1, h_2 lead to a bounded estimation error for both pseudo-states and order within a finite time.

$$\begin{cases} {}_0^C D_t^{\hat{\alpha}(t)} \hat{x}_1 = \hat{x}_2 + h_1(x_1 - \hat{x}_1) \\ {}_0^C D_t^{\hat{\alpha}(t)} \hat{x}_2(t) = f(\hat{x}, u) + h_2(x_1 - \hat{x}_1) \end{cases}$$
$$\dot{\hat{\alpha}} = \gamma \, sgn\Big(\frac{\partial z_{\hat{\phi}_1}(t, \beta)}{\partial \beta}|_{\beta = \hat{\alpha}}\Big)(x_1 - \hat{x_1}), \gamma > 0 \qquad (27)$$
$$\hat{\phi}_1(\tau) = \hat{x}_2 + h_1(x_1 - \hat{x}_1)$$

Proof. Using 26;

$$
\begin{aligned}
x_1 &= x_1(0) + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1} x_2(\tau)d\tau \\
x_2 &= x_2(0) + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1} f\big(x(\tau),u(\tau)\big)d\tau \\
\hat{x}_1 &= \hat{x}_1(0) + \frac{1}{\Gamma(\hat{\alpha}(t))}\int_0^t (t-\tau)^{\hat{\alpha}(t)-1} \Big((\hat{x}_2(\tau) + h_1(x_1-\hat{x}_1)\Big)d\tau \\
\hat{x}_2 &= \hat{x}_2(0) + \frac{1}{\Gamma(\hat{\alpha}(t))}\int_0^t (t-\tau)^{\hat{\alpha}(t)-1} \Big(f\big(\hat{x}(\tau),u(\tau)\big) + h_2(x_1-\hat{x}_1)\Big)d\tau
\end{aligned}
\tag{28}
$$

Now, define $e = [\varepsilon_1 \quad \varepsilon_2]^T, \varepsilon_1 = x_1 - \hat{x}_1, \varepsilon_2 = x_2 - \hat{x}_2$. Hence, (28) is rewritten as

$$
\begin{aligned}
\varepsilon_1 &= \varepsilon_1(0) + {}_0I_t^{\alpha(t)} x_2 - {}_0I_t^{\hat{\alpha}(t)}\big(\hat{x}_2 - h_1(x_1-\hat{x}_1)\big) \\
\varepsilon_2 &= \varepsilon_2(0) + {}_0I_t^{\alpha(t)} f(x,u) - {}_0I_t^{\hat{\alpha}(t)}\big(f(\hat{x},u) - h_2(x_1-\hat{x}_1)\big)
\end{aligned}
\tag{29}
$$

Adding and subtracting ${}_0I_t^{\alpha(t)}\big(\hat{x}_2 - h_1(x_1-\hat{x}_1)\big)$ and ${}_0I_t^{\alpha(t)}\big(f(\hat{x},u) - h_2(x_1-\hat{x}_1)\big)$ to the above equalities, respectively, defining $\hat{\phi}_2 = f(\hat{x}, u + h_2(x_1-\hat{x}_1))$, and considering Lemma 2, (29) yields

$$
\begin{aligned}
\varepsilon_1 &= \varepsilon_1(0) + {}_0I_t^{\alpha(t)}\big(-h_1\varepsilon_1\big) + {}_0I_t^{\alpha(t)}\varepsilon_2 + (\alpha-\hat{\alpha})\frac{\partial z_{\hat{\phi}_1(t,\beta)}}{\partial\beta}\Big|_{\beta=\hat{\alpha}} + K_1(t)(\alpha-\hat{\alpha})^2 \\
\varepsilon_2 &= \varepsilon_2(0) + {}_0I_t^{\alpha(t)}\big(f(x,u) - h_2\varepsilon_2 - f(\hat{x},u)\big) + (\alpha-\hat{\alpha})\frac{\partial z_{\hat{\phi}_2}(t,\beta)}{\partial\beta}\Big|_{\beta=\hat{\alpha}} + K_2(t)(\alpha-\hat{\alpha})^2
\end{aligned}
\tag{30}
$$

By defining

$$
\begin{aligned}
&A = \begin{bmatrix} -h_1 & 1 \\ 0 & -h_2 \end{bmatrix}, \delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\big(f(x,u) - f(\hat{x},u)\big), w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&w_1 = (\alpha-\hat{\alpha})\frac{\partial z_{\hat{\phi}_1(t,\beta)}}{\partial\beta}\Big|_{\beta=\hat{\alpha}} + (\alpha-\hat{\alpha})^2 K_1(t) \\
&w_2 = (\alpha-\hat{\alpha})\frac{\partial z_{\hat{\phi}_2}(t,\beta)}{\partial\beta}\Big|_{\beta=\hat{\alpha}} + (\alpha-\hat{\alpha})^2 K_1(t)
\end{aligned}
\tag{31}
$$

equation (30) can be rearranged to

$$
e = e(0) + {}_0I_t^{\alpha(t)}(Ae + \delta) + w \tag{32}
$$

where, in (31) and (32), $\|\delta\| < c\|e\|$ and the eigenvalues of A can be arbitrarily set by adjusting the design parameters h_1, h_2. According to Corollary 3, $e \to w$ (and $\varepsilon_1 \to w_1$) with a bounded error. Both the error bound and the convergence rate can be set small by setting the eigenvalues of A far enough from the imaginary axis, i.e., for arbitrarily chosen $\epsilon_0 > 0$ there exists T_0 such that $\|\varepsilon_1 - w_1\| < \epsilon_0$ for $t > T_0$. If the design parameters h_1, h_2 are adjusted high enough ensuring $T_0 << T$, then $\|\varepsilon_1 - w_1\| \approx 0$ almost everywhere on $t \in (0 \ \ T]$.

Now, define $V(\hat{\alpha}) = \frac{1}{2}(\alpha - \hat{\alpha})^2$ and $\eta(t) = \left|\frac{\partial z_{\hat{\phi}_1}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\right|$. Therefore, considering (27)

$$\begin{aligned}\dot{V}(\hat{\alpha}) &= \dot{\alpha}(\alpha - \hat{\alpha}) - \gamma sgn\Big(\frac{\partial z_{\hat{\phi}_1}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)\varepsilon_1 \\ &\leq \Bigg(\gamma\epsilon_0 + M\Bigg)\|\alpha - \hat{\alpha}\| - \gamma\eta(\alpha - \hat{\alpha})^2 + \gamma\|K_1(t)\|\|\alpha - \hat{\alpha}\|^3\end{aligned} \tag{33}$$

According to Lemma 3, $\exists M_1, \|K_1\| < M_1T^3$. Also, suppose that $\eta > \eta_0$. Considering $\Delta_0 = \gamma^2\eta_0^2 - 4\gamma M_1T^3(M + d + \gamma\epsilon_0)$, $\Delta = \gamma^2\eta^2 - 4\gamma|K_1|(M + d + \gamma\epsilon_0)$, consequently, $\dot{V} < -dV^{\frac{1}{2}}$ in the interval

$$\beta_L = \frac{\gamma\eta - \sqrt{\Delta}}{2\gamma|K_1|} < \|\alpha - \hat{\alpha}\| < \frac{\gamma\eta + \sqrt{\Delta}}{2\gamma|K_1|} = \beta_U \tag{34}$$

Based on Lemma 5 with $m = \gamma\epsilon_0 + d$, $\beta_L \leq \frac{\gamma\eta_0 - \sqrt{\Delta_0}}{2\gamma M_1T^3} = B_L$ and $B_U = \frac{\gamma\eta_0 + \sqrt{\Delta_0}}{2\gamma M_1T^3} \leq \beta_U$. Therefore, $\{\hat{\alpha}, B_L < \|\alpha - \hat{\alpha}\| < B_U\}$ is an invariant set when $\gamma > \frac{4M_1T^3(M+\gamma\epsilon_0+d)}{\eta_0^2} \Rightarrow \gamma > \frac{4M_1T^3(M+d)}{\eta_0^2 - 4M_1T^3\epsilon_0}$. Hence, within a finite time $T_1 < \frac{\|\alpha_0 - \hat{\alpha}(0)\|^2 - B_L^2}{d}$, $\|\alpha - \hat{\alpha}\|$ tends to $\frac{2(M+d)}{\gamma\eta_0} + \frac{2\epsilon_0}{\eta_0}$ for $\gamma > max\{0, \frac{4M_1T^3(M+d)}{\eta_0^2 - 4M_1T^3\epsilon_0}\}$. □

Remark 4. *Consider (27). Obviously, in developing the dynamic of $\hat{x}$ the variables $\hat{x}_1, \hat{x}_2$ are involved, with the order $\hat{\alpha}$, which are all available estimated values. In the third equation, which estimates the order, the variable x_1 is the output of the main system ($y = x_1$) and measurable. Based on (14), in calculating $\frac{\partial z_{\hat{\phi}_1}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}$, as the term is evaluated at $\beta = \hat{\alpha}$, only ϕ_1 and $\hat{\alpha}$ are required, where, $\hat{\alpha}$ is available and, according to the fourth equation. Finally, ϕ_1 can be calculated using $\hat{x}$ and $y = x_1$.*

Remark 5. *γ, ϵ_0 are design parameters. Setting γ large and ϵ_0 small guarantees the invariant set in (34) to be non-empty, also, results in small order estimation error.*

Remark 6. *This method can be directly extended to the case in which x_1, x_2 are both vectors.*

Remark 7. *Here, the high gain observer is extended to high gain order/pseudo-state estimator as an example for nonlinear observers. The key to design the order/pseudo-state estimator is equation (32) leading to $e_1 \approx (\alpha - \hat{\alpha})\frac{\partial z_{\hat{\phi}_1(t,\beta)}}{\partial \beta}\Big|_{\beta=\hat{\alpha}} + (\alpha - \hat{\alpha})^2K_1(t)$. In this regard, traditional observers whose error can be interpreted in similar way have the potential to be extended to an order/pseudo-state estimator for the non-integer order equivalent system with an appropriate definition for ϕ_1 in (27).*

Figure 4 depicts the proposed simultaneous order and pseudo-states estimator in a block diagram. When the pseudo-states are available, the estimation system is reduced to the block diagram shown in Figure 5.

Comparing two methods, as long as the pseudo-states are available, the method efficiently works for any system, including linear or nonlinear, scalar or multivariable, constant or variable order. However, when the pseudo-states are unknown, the proposed method can estimate order and pseudo-states simultaneously, however, the system must be of commensurate order. Also, there must be an observer with Lyapunov function of the form $\frac{1}{2}x^TPx, P > 0$ which is able to estimate the states of the traditional integer-order equivalent of the system.

The proposed method is utilized to estimate the order and the pseudo-states of system (35). There is also an additive measurement

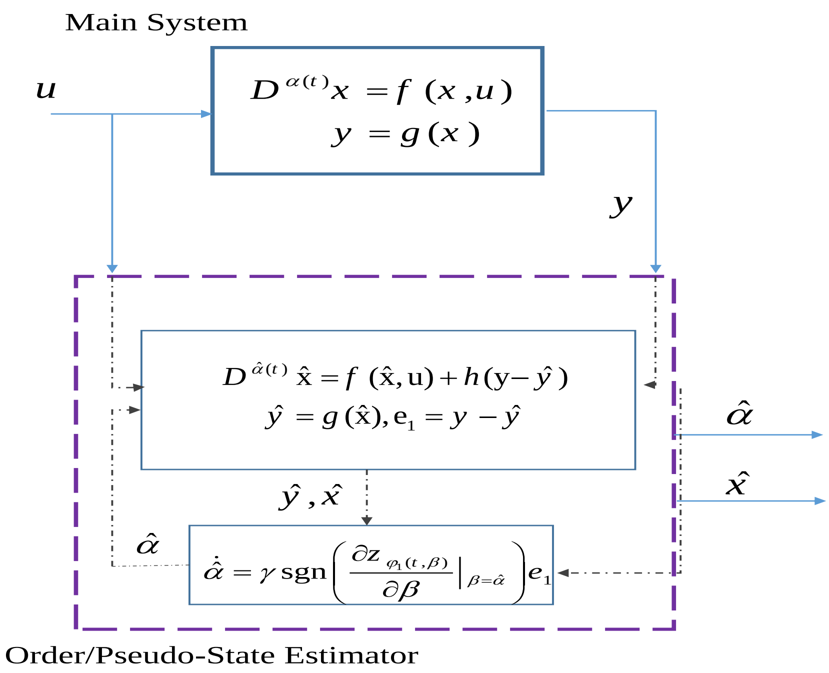

Figure 4: Signal flow and block diagram of the pseudo-state/order estimator

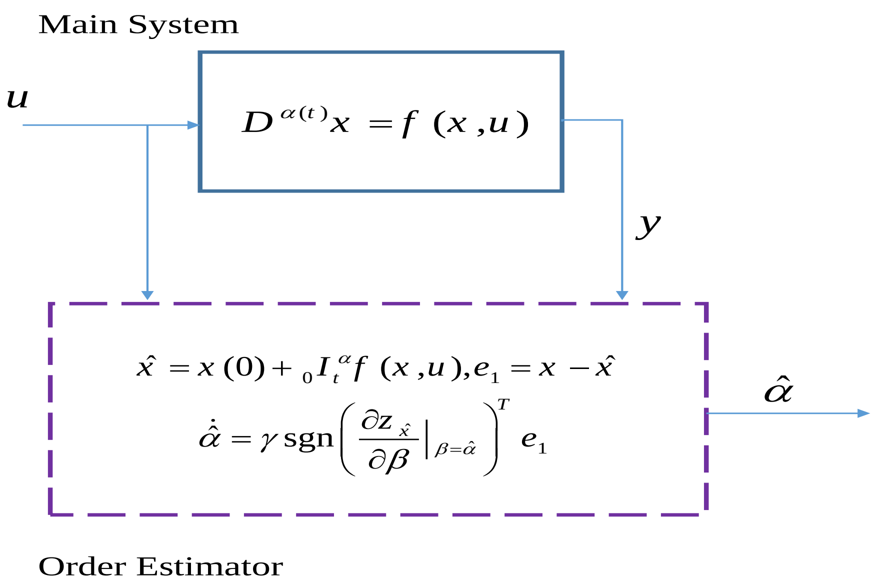

Figure 5: signal flow and block diagram of the order estimator when the pseudo-states are known

white Gaussian noise on the output.

$$
\begin{aligned}
&\begin{cases} {}_0^C D_t^{\alpha(t)} x_1 = x_2 \\ {}_0^C D_t^{\alpha(t)} x_2(t) = -x_1 sin(x_2) + 2u \end{cases} \\
&\alpha(t) = 0.35e^{-0.1t}(cos(0.3t) + 0.7) \\
&y(t) = x_1 + n, \\
&n \sim N(0, 10^{-2}) \\
&u = 1 - e^{-0.1t} sin(0.6t) \\
&x(0) = \left(\begin{smallmatrix} 0.5 \\ 0.5 \end{smallmatrix}\right), \hat{x}(0) = \left(\begin{smallmatrix} 0.1 \\ 0.1 \end{smallmatrix}\right), \hat{\alpha}(0) = 0.85
\end{aligned}
\tag{35}
$$

The design parameters are considered as $h_1 = 20$ and $h_2 = 200$ and $\gamma = 0.5$.

Figure 6 shows that the method effectively estimates the order and pseudo-states with an acceptable error, even with noisy measurement. Although theory warns to set γ very large, however, simulations work even with $\gamma = 0.5$, implying that theory may be too conservative.

4. Conclusion

The paper proposes some methods for estimating the order, mainly focusing on the most general case, i.e. the system with time-varying order. After proving some lemmas and theorems, the main results are presented step by step. First, the scalar case with the available pseudo-states was solved. Then, the method was extended to the multivariable case and some simulation studies verified the effectiveness of the order estimator. Afterward, the pseudo-states were considered unknown. It was claimed that the traditional observers can be extended in appropriate ways to estimate the pseudo-states of a non-integer order system with unknown order and pseudo-states to build a pseudo-state/order estimator. The idea was proven for the high gain observer as a prevalent case study. It was also claimed that all other integer order state observers whose convergence is proven using a Lyapunov function of the form $\frac{1}{2}x^T Px, P > 0$ can be used instead of the high gain observer by making some minor changes in the order adaptation rule.

As a possible future work, the proposed order estimation methods can be improved by considering the incommensurate order case with unknown pseudo-states. Also, to decrease the effect of the initial estimation value and reaching to global stability, the methods may be enhanced by enlarging the domain of attraction. The method may be generalized to be capable of estimating the order, the pseudo-states, and the parameters of the incommensurate variable order systems, as the most general case of dynamical systems. Also, the methods are useful for a vast variety of systems e.g. viscoelastic behavior, soft tissue deformation, etc. to determine if they are of integer or non-integer order and to estimate their order and pseudo-states.

5. Appendix

Proof of Lemma 3:

Due to continuity of ϕ and $|\phi|$, it certainly has a bound (M_ϕ) in $[0\ \ T]$. The functions $\Gamma(.),\ \psi(.)$, and $\psi'(.) = \frac{d\psi(\beta)}{d\beta}$ are bounded for $0 < \beta < 1$ and $H \geq 0$. Also, $\exists\kappa_0 > 0 \Rightarrow |tlnt| \leq \kappa_0 t^2$, because $-\frac{1}{e} \leq tlnt \leq t^2$, also, $\exists\kappa_1 > 0, |tln^2 t| < t^3$. Furthermore, the integrals $\int_0^t |H| d\tau$, $\int_0^t |ln(t-\tau)H| d\tau$ and $\int_0^t |ln^2(t-\tau)H| d\tau$ all exist and

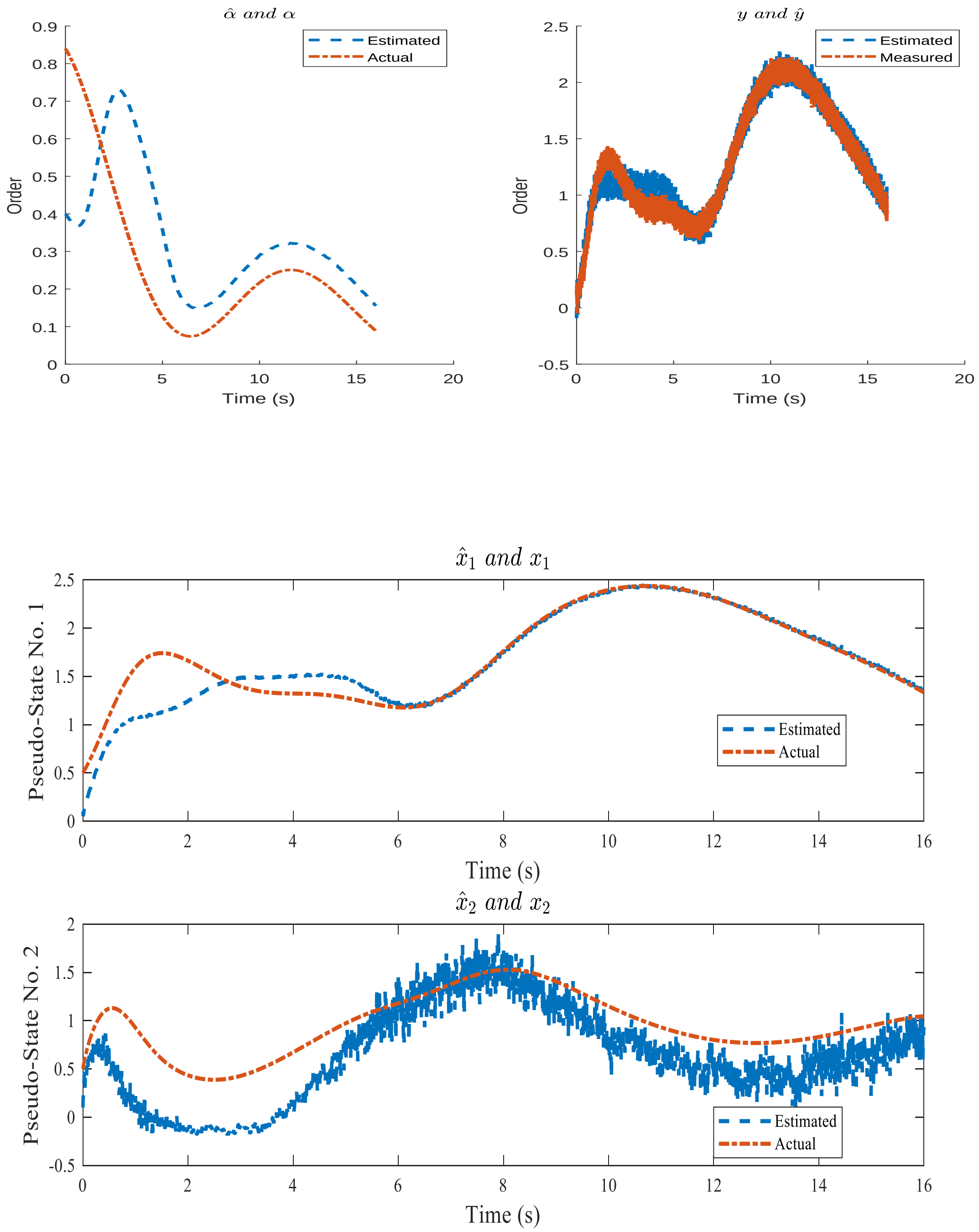

Figure 6: The actual and estimated order, output and pseudo-states using order/pseudo-state estimator for the noisy system (35). top left: α vs. $\hat{\alpha}$, top right: y vs. $\hat{y}$, bottom: x vs. $\hat{x}$

$$\int_0^t |H| d\tau = \int_0^t H d\tau = \frac{t^{\beta(t)}}{\beta(t)\Gamma\big(\beta(t)\big)} \Rightarrow \exists A_0 > 0, \int_0^t |H| d\tau \leq A_0 t$$

$$\int_0^t |ln(t-\tau)H| d\tau = \begin{cases} \frac{t^{\beta(t)}(1-\beta(t)ln(t))}{\beta^2(t)\Gamma\big(\beta(t)\big)}, t \leq 1 \\ \frac{t^{\beta(t)}(\beta(t)ln(t)-1)+2}{\beta^2(t)\Gamma\big(\beta(t)\big)}, t > 1 \end{cases}$$

$$\Rightarrow \exists B_0, C_0, D_0 > 0, \int_0^t |ln(t-\tau)H| d\tau \leq B_0 + C_0 t + D_0 t^2$$

$$\int_0^t |ln^2(t-\tau)H| d\tau = \frac{t^{\beta(t)}\big(1+\beta^2(t)ln^2(t) - 2\beta(t)ln(t)\big)}{\beta^3(t)\Gamma(\beta(t))}$$

$$\Rightarrow \exists E_0, F_0, G_0 > 0, \int_0^t |ln^2(t-\tau)H| d\tau \leq E_0 t + F_0 t^2 + G_0 t^3$$

Note that $K(t) = \int_0^t \frac{\partial^2 H(\tau,t,\beta)}{\partial \beta^2}|_{\beta=\alpha_0} \phi(\tau) d\tau$, and,

$$\frac{\partial^2 H}{\partial \beta^2} = -\psi'(\beta)H + \psi^2(\beta)H - 2\psi(\beta)ln(t-\tau)H + ln^2(t-\tau)H$$

Therefore,

$|K(t)| \leq \int_0^t |\frac{\partial^2 H(\tau,t,\beta)}{\partial \beta^2}|_{\beta=\alpha_0} |\phi(\tau)| d\tau \leq M_\phi \Big((|\psi| + \psi^2) A_0 t + 2|\psi|(B_0 + C_0 t^2) + D_0 + E_0 t^2 + t^3 \Big)$

Accordingly, for $t \leq T$, $\exists M_H > 0$ such that $|K| \leq M_\phi M_H t^3 < M_\phi M_H T^3 \equiv M_K T^3$ proving the lemma.

Proof of Lemma 5:

First, it should be noted that $\gamma > \frac{4K_0(m+M)}{\eta_0^2} \to \gamma > \frac{4|K|(m+M)}{\eta^2} \to \Delta > 0$. Also, $\Delta_\eta = \frac{\partial \Delta}{\partial \eta} = 2\gamma^2\eta$, $\Delta_{|K|} = \frac{\partial \Delta}{\partial |K|} = -4\gamma(m+M)$.

The first partial derivation is $\frac{\partial \beta_L}{\partial \eta} = \frac{(\gamma - \frac{\Delta_\eta}{2\sqrt{\Delta}})}{2\gamma|K|}$, and,

$$\Delta \leq \gamma^2\eta^2 \to \sqrt{\Delta} \leq \gamma\eta \to \frac{\gamma\eta}{\sqrt{\Delta}} \geq 1 \to 1 - \frac{\gamma\eta}{\sqrt{\Delta}} \leq 0 \to \gamma - \frac{2\gamma^2\eta}{2\sqrt{\Delta}} \to \gamma - \frac{\Delta_\eta}{2\sqrt{\Delta}} \to \frac{\partial \beta_L}{\partial \eta} \leq 0.$$

The second term is positive because:

$$\frac{\partial\beta_L}{\partial|K|} = \frac{\frac{-\Delta_{|K|}}{2\sqrt{\Delta}}(2\gamma|K|) - 2\gamma(\gamma\eta - \sqrt{\Delta})}{4\gamma^2|K|^2\eta^2} = \frac{\frac{-\Delta_{|K|}}{\sqrt{\Delta}}|K| - 2\gamma\eta + 2\sqrt{\Delta}}{4\gamma|K|^2\eta^2}$$

$$\frac{4(m+M)^2|K|^2}{\eta^2} \geq 0$$
$$\to \gamma^2\eta^2 - 4\gamma|K|(m+M) \leq \gamma^2\eta^2 - 4\gamma|K|(m+M) + 4(m+M)^2|K|^2$$
$$\to \Delta \leq (\gamma\eta - \frac{2(m+M)|K|}{\eta})^2 \quad (I)$$
$$\gamma\eta > \frac{4(m+M)|K|}{\eta} \geq \frac{2(m+M)|K|}{\eta} \to (\gamma\eta - \frac{2(m+M)|K|}{\eta}) > 0 \quad (II)$$
$$(I),(II) \to \sqrt{\Delta} \leq \gamma\eta - 2|K|(m+M) \to -\sqrt{\Delta} + \gamma\eta - \frac{2|K|(m+M)}{\eta} \geq 0$$
$$\to -2\gamma\eta\sqrt{\Delta} + 2\gamma^2\eta^2 - 4|K|(m+M)\gamma \geq 0$$
$$\to 4\gamma(m+M)|K| - 2\gamma\eta\sqrt{\Delta} + 2(\gamma^2\eta^2 - 4|K|(m+M)\gamma) \geq 0$$
$$\to 4\gamma(m+M)|K| - 2\gamma\eta\sqrt{\Delta} + 2\Delta \geq 0$$
$$\to \frac{4\gamma(m+M)|K|}{\sqrt{\Delta}} - 2\gamma\eta + 2\sqrt{\Delta} \geq 0 \to \frac{-\Delta_{|K|}}{\sqrt{\Delta}} - 2\gamma\eta + 2\sqrt{\Delta} \geq 0$$
$$\to \frac{\partial\beta_L}{\partial|K|} \geq 0$$

$\frac{\partial\beta_U}{\partial\eta} = \frac{\gamma + \frac{\gamma^2\eta}{\sqrt{\Delta}}}{2\gamma|K|}$ which is obviously positive. Also, $\frac{\partial\beta_U}{\partial|K|} = \frac{\frac{\Delta_{|K|}}{\sqrt{\Delta}}|K| - 2\gamma\eta - 2\sqrt{\Delta}}{4\gamma|K|^2}$, and

$$-4\gamma(m+M)|K| - 2\gamma\eta\sqrt{\Delta} - 2\Delta \leq 0$$
$$\to -\frac{4\gamma(m+M)|K|}{\sqrt{\Delta}} - 2\gamma\eta - 2\sqrt{\Delta} \leq 0$$
$$\to \frac{\Delta_{|K|}|K|}{\sqrt{\Delta}} - 2\gamma\eta - 2\sqrt{\Delta} \leq 0 \to \frac{\partial\beta_L}{\partial|K|} \leq 0$$

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