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A Novel Adaptive Order/Parameter Identification Method for Variable Order Systems Application in Viscoelastic Soft Tissue Modeling

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Abstract

This paper presents an adaptive system identification approach to identify the order and parameters of a specific type of variable order systems, which, as a motivating example, describes the stress-strain relation of viscoelastic materials. First, the concept of non-integer order modeling will be introduced. Next, the proposed order/parameter identification approach will be presented. Afterwards, a simulation study is performed to validate the identification approach. Finally, the method will be applied on real data gathered from an experimental study for further validation.

Keywords: Adaptive Identification, Order Identification, Parameter Estimation, Variable Order Systems, Soft Tissue Modeling

1. Introduction

The concept of non-integer calculus is an extension of the traditional one by considering a non-integer value as the order of derivation or integration. Based on this, non-integer order differential equations and non-integer order systems can be defined. The non-integer order has been used in modeling the memory in electronic devices [1, 2], viscoelastic damping [3, 4], the human's ability to forget and remember [5, 6, 7], and lung tissues [8], control [9, 10, 11], and chaos modeling and control [12]. The order of a non-integer order system is not required to be constant. It can vary depending on time or the states of the system, and even can have its own dynamics. The definition of variable order derivation and integration operators was first developed in [13].

The concept of variable-order calculus has been studied from different aspects, e.g., in [14, 15, 16, 17]. Furthermore, from an application point of view, evidence has been reported to relate the order of derivation to certain physical quantities [18, 19].

The non-integer derivation operator can be defined in various ways [20, 21], the most common of which are the definitions in the sense of Caputo and Riemann-Liouville [20]. The following shows a left sided non-integer order derivation of order $\alpha(t)$ of signal $x(t)$ in the sense of Caputo [14]:

$$\begin{gathered} {}_0^C D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)} \frac{d}{d\tau} x(\tau) d\tau \\ 0 < \alpha(t) < 1, \forall t \geq 0 \end{gathered} \tag{1}$$

Here $\Gamma(.)$ is the extension of the factorial function to non-integer arguments:

$$\Gamma(w) = \int_0^\infty r^{w-1} e^{-r} dr \tag{2}$$

Choosing the Caputo definition -as used in this paper- leads to direct use of the initial condition, just like the traditional case [20]. The use of non-integer order calculus with constant order for describing viscoelastic materials is very common [22, 23, 24, 25, 26]. The behavior of a viscoelastic material varies between viscous and elastic ones. However, the behavior is strongly dependent on many parameters. In fact, according to the tension amplitude, its frequency, its rate, etc., the material can show a specific amount of elasticity or viscosity [27]. Because of the huge number of affecting parameters and the complexity of the way they influence the material stress-strain relationship, it is difficult to figure out how much the material is elastic or viscous at a certain condition. However, most of the time, the behavior can be practically considered as a function of time. In fact, since in a dynamical environment, stress, strain, force, etc. are all functions of time, even in the case that the state of the material (i.e. the amount of its viscosity or elasticity) is a function of some of these, it can be considered as an implicit function of time. Hence, it makes sense to use time varying order model to describe such materials.

Variable order dynamics are widely used in modeling various physical phenomena. Such operators are used to develop mechanical laws in [28]. In [29], the theory of viscoelasticity and the abilities of variable order calculus build a framework for modeling viscoelastic behavior. In [30], variable order differential equations are used to describe anomalous diffusion modeling. The effect of tension on cable status is modeled in [31] using variable order dynamics.

In the above mentioned papers, when non-integer order modeling is used (especially in cases with varying order), the order or its functionality of the states is supposed to be known [31, 4]. Here, an identification method is also proposed to identify the model order and parameters effectively.

Since the order is a key characteristic of a non-integer order system for having a precise model describing specific dynamics, it is vital to identify it with an acceptable level of precision. Some papers have already proposed estimation schemes for the order of a non-integer order system, assuming a **constant order**. In [32], the order is treated just like other constant system parameters in the estimator, which was not real time. A numerical method is used in [33] to find the constant order of a non-integer system. In [34], a discrete-time technique is used to estimate the constant order of a system using a Kalman filter. A frequency domain system identification approach is proposed in [35] based on continuous distributed-order. As an application, the fractional model describing the lead acid battery is proposed in [36]. In [37], an adaptive estimation process is introduced to estimate the constant order of non-integer order systems. While the unknown order is supposed to be constant in all of the above, the order is allowed to be time-varying in this paper. This is a significant and useful generalization. In fact, even when the order is a function of the states, it can be considered as an implicit function of time. Hence, the method proposed in this paper can be utilized in all cases involving varying order dynamics. This is a main novelty of the proposed method. Allowing the order to vary with time gives the pair of the identification approach and the model the ability of describing and identifying various complicated phenomena such as viscoelastic material behavior. Also, a definite convergence proof is presented in this paper, ensuring that the order identification error can be made arbitrarily small.

The contribution of this paper is two-fold. First, we introduce a time varying order model to describe complex behavior of viscoelastic materials. Actually, since the order is allowed to vary in time, it can effectively model different variations in the material behavior. Such idea gives the model some more levels of precision. However, it leads to a complex problem: The behavior of the material should be modeled and predicted in a real time manner. Hence, an adaptive identification method with an acceptable level of precision should be used. The design of such an identification method is the main contribution of the paper.

The approach proposed in the current paper differs from the former ones in the following aspects:

1. While the order is supposed to be constant in the former papers, the order is allowed to be varying with time in this paper. This is a significant useful generalization. In fact, even when the order is a function of the states, it can be considered as an implicit function of time. Hence, the method proposed in this paper can be utilized in all cases involving varying order dynamics.

2. A definite convergence proof is established in this paper, guaranteeing that the identification error for both order and parameters can be made arbitrarily small.

3. While most of the former papers propose offline methods, the method developed here is an adaptive one. This makes it useful for real time applications.

Accordingly, the paper is organized as follows: After explaining the motivating example in Section II, the preliminaries and problem definition will be introduced in the Section III. As the main outcome of the paper, Section IV is about the order/parameter identification of a variable order model. In Section V, a simulation study validates the theoretical results presented in Section IV. Section VI provides an experimental study followed by conclusion given in Section VII.

2. Motivating Example

A viscoelastic material is **between** the elastic and viscous ones in terms of input/output dynamics. There are several interpretations for the word "between". The stress-strain relation of a pure elastic material (a Hookean spring) is given by

$$\sigma = k\varepsilon \tag{3}$$

where stress and strain are indicated by σ and ε, respectively, and k is the elasticity coefficient. For a pure viscous material (a Newtonian damper), the stress-strain relation is

$$\sigma = b\dot{\varepsilon} \tag{4}$$

where $\dot{\varepsilon} = \frac{d\varepsilon}{dt}$ and b is the viscosity coefficient. The simplest explanation for describing a material between these is to use the Kelvin-Voigt model, i.e., to consider a viscoelastic material as a parallel-series combination of ideal Hookean springs and Newtonian dampers, [38]. This idea leads to a linear relationship between the stress and strain of viscoelastic materials which, in the simplest way, can be written as

$$\sigma = k\varepsilon + b\dot{\varepsilon}. \tag{5}$$

There is a different interpretation for the word "between". In this point of view, (3) and (4) can be rewritten as

$$\sigma = kD_t^0\varepsilon \tag{6}$$

and

$$\sigma = bD_t^1\varepsilon \tag{7}$$

respectively. Here, D_t^n is defined as the n^{th} derivation operator with respect to time and the 0^{th} derivative of a function is considered to be the function itself. Now, if a material behaves between the ones described by (6) and (7), its stress-strain relation can be considered as

$$\sigma = \eta D_t^q\varepsilon, \qquad 0 < q < 1. \tag{8}$$

Equation (8) is the main idea of many recently published papers based on the fractional (or, to be more precise, non-integer) order modeling of viscoelasticity, of which some examples can be found in [39, 40, 41]. In (8), the operator D_t^q is known as the non-integer order derivation operator. Choosing the open interval $q \in (0 \;\; 1)$ is to avoid pure viscous or elastic materials. In fact, it implies that the considered material is definitely a viscoelastic one. In this paper, the great ability of the variable order operator is utilized to model soft tissue behavior. To this aim, first, we propose an analytical adaptive approach to identify the parameters of the above model and then, validate the method by means of applying it on a set of data gathered from a real tissue.

To the best knowledge of the authors, adaptive identification of the order and parameters of a variable order system is not proposed yet, even for a simple model like (8). Accordingly, this paper is the first work introducing a method for identifying the variable order and parameter of a variable non-integer order system. As an application for such systems, the paper also suggests a variable order model for describing a complex material such as soft tissue.

3. Preliminaries

The variable order derivation operator was defined in (1). The inverse of the variable derivation operator is the variable order integration operator of the same order defined as

$$\begin{array}{c} {}_0I_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}x(\tau)d\tau \\ 0 < \alpha(t) < 1, \forall t \geq 0 \end{array} \tag{9}$$

In fact,

$${}_0^C D_t^{\alpha(t)} I_t^{\alpha(t)} x(t) = x(t) \tag{10}$$

Consider the following system:

$${}_0^C D_t^{\alpha^*(t)} x^*(t) = p^{*T} u \tag{11}$$

where $0 < \alpha^*(t) < 1, \forall t$ is the order, p^* is the $n \times 1$ parameter vector, u is the $n \times 1$ input vector, the scalar x^* is the response of the system, T denotes the transpose operator and the superscript $*$ indicates the nominal values. The above equation is equivalent to the following Volterra integral equation [42, 43]:

$$x^*(t) - x^*(0) = \frac{1}{\Gamma(\alpha^*(t))}\int_0^t (t-\tau)^{\alpha^*(t)-1} p^{*T} u(\tau)d\tau \tag{12}$$

According to (12), the relationship between the response of the system and the order is complicated. Hence, order identification is not as easy as the estimation of the other parameters of the system. Fortunately, we can compute the partial derivative of the system response with respect to the order. For any function f, define:

$$z_f(\beta, t) = z_0 + \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} f(\tau)d\tau \tag{13}$$

Then,

$$
\begin{aligned}
\frac{\partial z_f(\beta,t)}{\partial \beta} =& \frac{-\Gamma'(\beta(t)}{\Gamma^2(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} f(\tau)d\tau \\
&+ \frac{1}{\Gamma(\beta(t))}\int_0^t ln(t-\tau)(t-\tau)^{\beta(t)-1} f(\tau)d\tau \\
=& -\frac{\psi(\beta(t))}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} f(\tau)d\tau \\
&+ \frac{1}{\Gamma(\beta(t))}\int_0^t ln(t-\tau)(t-\tau)^{\beta(t)-1} f(\tau)d\tau
\end{aligned}
\tag{14}
$$

where

$$
\psi(\beta) = \frac{d}{d\beta} ln(\Gamma(\beta)) = \frac{\Gamma'(\beta(t)}{\Gamma(\beta(t))}. \tag{15}
$$

Hence, when the history of f in $[0 \;\; t)$ and the value of β at t are known, the above derivative can be computed. It will be shown that the above result helps to design an order/parameter identification method. The next section will provide the theoretical basis for the proposed approach.

4. Main Results

Without loss of generality, consider the model described in (11) with zero initial condition. We will study the effect of the initial conditions later in Remark 4. The following lemma provides the proposed parameter estimation procedure in the case that the time-varying order is known.

Lemma 1. *Consider the following equation which is an approximation for (11) after substituting the unknown value p^* with $p = p(\alpha^*, t)$:*

$$
{}_0^C D_t^{\alpha^*(t)} x(t) = p(\alpha^*, t)^T u \tag{16}
$$

Here, $p(\alpha^*, t)$ is an $n \times 1$ vector computed in real time through the following least squares equation with forgetting factor (LSFF) approximation.

$$
p(\alpha^*, t) = P(\alpha^*, t) \int_0^t e^{-\lambda(t-\tau)} x^*(\tau) \phi(\alpha^*, \tau) d\tau \tag{17}
$$

where $P(\alpha^*, t) = \left(\int_0^t e^{-\lambda(t-\tau)} \phi(\alpha^*, \tau) \phi^T(\alpha^*, \tau) d\tau \right)^{-1}$.

In (17), $0 < \lambda < 1$ *is the forgetting factor and* $\phi(\alpha^*, t) = \phi(\beta, t)|_{\beta=\alpha^*}$ *where* $\phi(\beta, t)$ *is defined as:*

$$\phi(\beta, t) = \frac{1}{\Gamma(\beta(t))} \int_0^t (t-\tau)^{\beta(t)-1} u(\tau) d\tau \tag{18}$$

Then, as long as $\phi(\alpha^*, t)$ *is Persistently Exciting (PE),* $(x - x^*) \to 0$*, or for any arbitrary small positive value of* ϵ *there is* T_ϵ *for which:*

$$t > T_\epsilon \Rightarrow \|x - x^*\| < \epsilon \tag{19}$$

and $p \to p^*$*.*

Proof. Based on (11) and (16) the signals x and x^* can be respectively written as:

$$\begin{aligned} x^*(t) &= \int_0^t \frac{(t-\tau)^{\alpha^*(t)-1}}{\Gamma(\alpha^*(t))} p^{*T} u(\tau) d\tau \\ x(t) &= \int_0^t \frac{(t-\tau)^{\alpha^*(t)-1}}{\Gamma(\alpha^*(t))} p^T u(\tau) d\tau \end{aligned} \tag{20}$$

According to the definition of ϕ, since the values of p and p^* are both independent from τ, the following linear-in-parameter equations are obtained:

$$\begin{aligned} x^* &= p^{*T}\phi \\ x &= p^T(\alpha^*, t)\phi \end{aligned} \tag{21}$$

The rest of the proof is straightforward based on the proof of the regular forgetting factor parameter estimation case [44]. Define the error signal e_1 and parameter error $\tilde{p}$ as $e_1 = x - x*$ and $\tilde{p} = p - p^*$, respectively. Hence, $e_1 = \tilde{p}^T\phi$. By derivating (17) with respect to t it can be concluded that $\dot{\tilde{p}} = -P\phi e_1$ [44]. Define the Lyapunov function $V(\tilde{p}) = \frac{\tilde{p}^T P^{-1}\tilde{p}}{2}$. Then, considering that $\dot{P} = -P\dot{P}^{-1}P$, and the fact that as long as ϕ is PE $P(\alpha^*, t) = \left(\int_0^t e^{-\lambda(t-\tau)}\phi(\alpha^*, \tau)\phi^T(\alpha^*, \tau) d\tau\right)^{-1} > 0$ [44], the time derivative of V is calculated as $\dot{V} = -e_1^2 < 0$ .Hence $\tilde{p} \to 0$, $x \to x^*$ and this completes the proof. □

Remark 1. Lemma 1 shows that, instead of $x^* = p^{*T}\phi$ one may use $x = p(\alpha^*, t)^T\phi$ with small enough error between x and x^*. This substitution will be used in Theorem 1 when the derivation of the output with respect to the parameter is required to approximate the output error using Taylor series.

Remark 2. The set of equations (17) are the well-known LSFF parameter estimation formulation which, as will be seen in Theorem 1, can be used in a real time adaptive manner.

Remark 3. Choosing LSFF instead of other versions of the Least Squares algorithm makes it possible to estimate smooth time varying parameters, as well. In fact, LSFF concentrates on the most recent history of the signals by weighting the recent samples more then the previous ones. This property makes it possible to deal with the time varying parameters.

Remark 4. In Lemma 1 the zero initial condition was assumed for the system. However, in the case that $x^*(0) \neq 0$ it is still available and the model initial condition $x(0)$ is available as well. Hence, we can easily define the auxiliary variables $X^* = x^* - x^*(0)$ and $X = x - x(0)$ satisfying (21). Accordingly, the above lemma and also, the upcoming lemmas and theorems all hold with arbitrary initial conditions. However, for the sake of simplicity, we will consider the system and the model with zeros initial conditions.

In the previous lemma, it was assumed that the order α^* is known. The following lemma provides a more general relationship between the actual and estimated responses when the order is not known and $\hat{\alpha} \neq \alpha^*$ is used as an approximation for it.

Lemma 2. *Suppose that x and $\hat{x}$ are defined in the interval $t \in [0 \quad T]$ as:*

$$\begin{aligned} x^*(t) &= \int_0^t \frac{(t-\tau)^{\alpha^*(t)-1}}{\Gamma(\alpha^*(t))} p^{*T} u(\tau) d\tau \\ \hat{x}(t) &= \int_0^t \frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))} p^T(\hat{\alpha}, t) u(\tau) d\tau \end{aligned} \tag{22}$$

where

$$\begin{aligned} p(\beta, t) &= P(\beta, t) \int_0^t e^{-\lambda(t-\tau)} x^*(\tau) \phi(\beta, \tau) d\tau \\ P(\beta, t) &= \Big(\int_0^t e^{-\lambda(t-\tau)} \phi(\beta, \tau) \phi^T(\beta, \tau) d\tau \Big)^{-1} \end{aligned} \tag{23}$$

Then, for any given ϵ, there is a time T_ϵ such that the value of $x^* - \hat{x}$ can be *evaluated according to the following at any time $t > T_\epsilon$:*

$$x^* - \hat{x} = \left(p^T(\hat{\alpha}, t)\frac{\partial \phi(\beta, t)}{\partial \beta}|_{\beta=\hat{\alpha}} + \frac{\partial p^T(\beta, t)}{\partial \beta}|_{\beta=\hat{\alpha}}\phi(\hat{\alpha}, t)\right)(\alpha^* - \hat{\alpha}) \\ + K(t)(\alpha^* - \hat{\alpha})^2 + e \tag{24}$$

where $|e| \leq \epsilon$ and $K(t) = \frac{\partial^2 y(\beta,t)}{\partial \beta^2}|_{\beta=\alpha_0}$ for some $\alpha_0 \in [min(\alpha, \hat{\alpha}) \quad max(\alpha, \hat{\alpha})]$.

Proof. Define $H(t, \tau, \beta) = \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}$. So, $\phi(\beta, t) = \int_0^t H(t, \tau, \beta)u(\tau)d\tau$. Then, we have:

$$\begin{aligned} x^* &= p^{*T}\phi(\alpha^*, t) \\ \hat{x} &= p^T(\hat{\alpha}, t)\phi(\hat{\alpha}, t) \end{aligned} \tag{25}$$

Define the following auxiliary variables:

$$y(\beta, t) = p^T(\beta, t)\phi(\beta, t)$$

and

$$x = p^T(\alpha^*, t)\phi(\alpha^*, t).$$

Then, $x = y(\alpha^*, t)$ and $\hat{x} = y(\hat{\alpha}, t)$. H is continuous and analytical with respect to β when $0 < \beta < 1$ and $0 < \tau < t$. Therefore, $\frac{\partial H}{\partial \beta}, \frac{\partial^2 H}{\partial \beta^2}$ are both bounded in the interval $0 < \tau < t$. According to the definition of ϕ, $\frac{\partial^k \phi(\beta,t)}{\partial \beta^k} = \int_0^t \frac{\partial^k H(t,\tau,\beta)}{\partial \beta^k}u(\tau)d\tau$. So, while u is bounded and continuous, $\frac{\partial \phi}{\partial \beta}$ and $\frac{\partial^2 \phi}{\partial \beta^2}$ are both bounded and continuous.

Noting that $\frac{\partial P(\beta,t)}{\partial \beta} = -P\frac{\partial P^{-1}(\beta,t)}{\partial \beta}P$, the value of $\frac{\partial p(\beta,t)}{\partial \beta}$ can be calculated through the following equations:

$$\begin{aligned} \frac{\partial p(\beta, t)}{\partial \beta} &= \frac{\partial P(\beta, t)}{\partial \beta}\int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\phi(\beta, \tau)d\tau \\ &\quad + P(\beta, t)\int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\frac{\partial \phi(\beta, \tau)}{\partial \beta}d\tau \\ \frac{\partial P(\beta, t)}{\partial \beta} &= -P(\beta, t)\Big(\int_0^t e^{-\lambda(t-\tau)}\big(\phi(\beta, \tau)\frac{\partial \phi^T(\beta, \tau)}{\partial \beta} \\ &\quad + \frac{\partial \phi(\beta, \tau)}{\partial \beta}\phi^T(\beta, \tau)\big)d\tau\Big)P(\beta, t) \end{aligned} \tag{26}$$

Hence, when $t \in [0 \;\; T]$, as long as $\phi(\beta, t)$ and $\frac{\partial \phi(\beta,t)}{\partial \beta}$ are bounded, so is $\frac{\partial p(\beta,t)}{\partial \beta}$.

Now, Taylor series expansion can be applied on the function y to calculate $x - \hat{x}$ as

$$\begin{aligned} x - \hat{x} &= (\alpha^* - \hat{\alpha}) \frac{\partial y(\beta,t)}{\partial \beta}|_{\beta=\hat{\alpha}} + (\alpha^* - \hat{\alpha})^2 \frac{\partial^2 y(\beta,t)}{\partial \beta^2}|_{\beta=\alpha_0} \\ &= \Big(p^T(\hat{\alpha}, t) \frac{\partial \phi(\beta,t)}{\partial \beta}|_{\beta=\hat{\alpha}} + \frac{\partial p^T(\beta,t)}{\partial \beta}|_{\beta=\hat{\alpha}} \phi(\hat{\alpha}, t) \Big)(\alpha^* - \hat{\alpha}) + K(t)(\alpha^* - \hat{\alpha})^2 \end{aligned} \tag{27}$$

for some $\alpha_0 \in [min(\alpha, \hat{\alpha}) \;\; max(\alpha, \hat{\alpha})]$, where $K(t) = \frac{\partial^2 y(\beta,t)}{\partial \beta^2}|_{\beta=\alpha_0}$.

Since $x^* - \hat{x} = (x - \hat{x}) + (x^* - x)$, where, x is defined as $x = p^T(\alpha^*, t)\phi(\alpha^*, t)$. After defining $e = x^* - x$, we have:

$$\begin{aligned} x^* - \hat{x} = \Big(p^T(\hat{\alpha}, t) \frac{\partial \phi(\beta,t)}{\partial \beta}|_{\beta=\hat{\alpha}} + \frac{\partial p^T(\beta,t)}{\partial \beta}|_{\beta=\hat{\alpha}} \phi(\hat{\alpha}, t) \Big)(\alpha^* - \hat{\alpha}) \\ + K(t)(\alpha^* - \hat{\alpha})^2 + e \end{aligned} \tag{28}$$

where, according to Lemma 1, $|e| < \epsilon$ □

Lemma 3. *Suppose that u is continuous and t belongs to the closed bounded interval $[0 \;\; T]$. Then, in (24), $K = \frac{\partial^2 y(\beta,t)}{\partial \beta^2}|_{\beta=\alpha_0}$ is bounded.*

Proof. Since u is continuous (so is $\|u\|$), according to the Extreme Value Theorem, $|u|$ attains a maximum in the aforementioned interval, here defined as M_u. The functions $\Gamma(\beta), \psi(\beta)$ and $\psi'(\beta) = \frac{d\psi(\beta)}{d\beta}$ are all bounded in $0 < \beta(t) < 1$ and $H \geq 0$. Furthermore, the integrals $\int_0^t |H| d\tau$, $\int_0^t |ln(t-\tau)H| d\tau$ and $\int_0^t |ln^2(t-\tau)H| d\tau$ all converge for $0 < \beta(t) < 1$:

$$\int_0^t |H| d\tau = \int_0^t H d\tau = \frac{t^{\beta(t)}}{\beta(t)\Gamma\big(\beta(t)\big)}$$

$$\int_0^t |ln(t-\tau)H| d\tau = \begin{cases} \frac{t^{\beta(t)}(1-\beta(t)ln(t))}{\beta^2(t)\Gamma\big(\beta(t)\big)}, t \leq 1 \\ \frac{2+t^{\beta(t)}(\beta(t)ln(t)-1)}{\beta^2(t)\Gamma\big(\beta(t)\big)}, t > 1 \end{cases}$$

$$\int_0^t |ln^2(t-\tau)H| d\tau = \frac{t^{\beta(t)}\big(1 + \beta^2(t)ln^2(t) - 2\beta(t)ln(t)\big)}{\beta^3(t)\Gamma(\beta(t))}$$

Now, it should be noted that based on the definition of H,

$$
\begin{aligned}
&\frac{\partial H}{\partial \beta} = (-\psi(\beta) + ln(t-\tau))H \\
&\frac{\partial^2 H}{\partial \beta^2} = -\psi' H + (-\psi(\beta) + ln(t-\tau))\frac{\partial H}{\partial \beta} \\
&\quad = -\psi' H + \psi^2(\beta)H - 2\psi(\beta)ln(t-\tau)H + ln^2(t-\tau)H
\end{aligned}
$$

Hence, for any given T, there are positive scalars $M_H^0(T)$, $M_H^1(T)$, and $M_H^2(T)$ such that $\int_0^t \big|H(t,\tau,\beta)\big|d\tau \le M_H^0(T)$, $\int_0^t \big|\frac{\partial H(t,\tau,\beta)}{\partial \beta}\big|d\tau \le M_H^1(T)$, and $\int_0^t \big|\frac{\partial^2 H(t,\tau,\beta)}{\partial \beta^2}\big|d\tau \le M_H^2(T), \forall t \in [0 \;\; T]$. Therefore,

$$
\begin{aligned}
&\Big\|\phi(\beta,t)\Big\| \le M_H^0 M_u \equiv M_\phi^0 \\
&\Big\|\frac{\partial \phi}{\partial \beta}\Big\| \le M_H^1 M_u \equiv M_\phi^1 \\
&\Big\|\frac{\partial^2 \phi}{\partial \beta^2}\Big\| \le M_H^2 M_u \equiv M_\phi^2
\end{aligned}
$$

Now, after defining $G(\beta,t) = \int_0^t e^{-\lambda(t-\tau)}\big(\phi(\beta,\tau)\frac{\partial \phi^T(\beta,\tau)}{\partial \beta} + \frac{\partial \phi(\beta,\tau)}{\partial \beta}\phi^T(\beta,\tau)\big)d\tau$ and $Q(\beta,t) = \int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\phi(\beta,\tau)d\tau$, based on the definitions of P and p and considering (26) we have:

$$
\begin{aligned}
&\frac{\partial G}{\partial \beta} = \int_0^t e^{-\lambda(t-\tau)}\big(\phi(\beta,\tau)\frac{\partial^2 \phi^T(\beta,\tau)}{\partial \beta^2} \\
&\qquad + 2\frac{\partial \phi(\beta,\tau)}{\partial \beta}\frac{\partial \phi^T(\beta,\tau)}{\partial \beta} + \frac{\partial^2 \phi(\beta,\tau)}{\partial \beta^2}\phi^T(\beta,\tau)\big)d\tau \\
&\frac{\partial Q}{\partial \beta} = \int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\frac{\partial \phi(\beta,\tau)}{\partial \beta}d\tau \\
&\frac{\partial^2 Q}{\partial \beta^2} = \int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\frac{\partial^2 \phi(\beta,\tau)}{\partial \beta^2}d\tau \\
&\frac{\partial^2 P}{\partial \beta^2} = -\frac{\partial P}{\partial \beta}GP - P\frac{\partial G}{\partial \beta}P - PG\frac{\partial P}{\partial \beta} \\
&\frac{\partial^2 p}{\partial \beta^2} = \frac{\partial^2 P}{\partial \beta^2}Q + 2\frac{\partial P}{\partial \beta}\frac{\partial Q}{\partial \beta} + P\frac{\partial^2 Q}{\partial \beta^2}
\end{aligned}
$$

Since ϕ, $\frac{\partial \phi}{\partial \beta}$ and $\frac{\partial^2 \phi}{\partial \beta^2}$ are all bounded and continuous, so are P, G, Q, $\frac{\partial G}{\partial \beta}$, $\frac{\partial Q}{\partial \beta}$, $\frac{\partial^2 Q}{\partial \beta^2}$ and $\frac{\partial^2 Q}{\partial \beta^2}$ and finally p, $\frac{\partial p}{\partial \beta}$ and $\frac{\partial^2 p}{\partial \beta^2}$ are bounded, i.e., there are positive scalars M_p^0, M_p^1 and M_p^2 such that $\|p\| \le M_p^0$, $\Big\|\frac{\partial p}{\partial \beta}\Big\| \le M_p^1$, $\Big\|\frac{\partial^2 p}{\partial \beta^2}\Big\| \le M_p^2, t \in [0 \;\; T]$.

Considering all the above results together, we have:

$$
\begin{aligned}
|K| &= \Big|\frac{\partial^2 y(\beta,t)}{\partial\beta^2}|_{\beta=\alpha_0}\Big| \\
&\le \Big|\Big(\frac{\partial^2 p^T(\beta,t)}{\partial\beta^2}\phi(\beta,t) + 2\frac{\partial p^T(\beta,t)}{\partial\beta}\frac{\partial\phi(\beta,t)}{\partial\beta} + p^T(\beta,t)\frac{\partial^2\phi(\beta,t)}{\partial\beta^2}\Big)|_{\beta=\alpha_0}\Big| \\
&\le M_p^2(T)M_\phi^0(T) + 2M_p^1(T)M_\phi^1(T) + M_p^0(T)M_\phi^2(T) \equiv M_K
\end{aligned}
$$

implying that K is bounded. □

Now, based on the above lemmas, the upcoming theorem provides a simultaneous order/parameter identification method for the system described in (11).

Theorem 1. *Consider the model described by (11) in the interval* $t \in [0 \ \ T]$ *with unknown time varying order* $0 < \alpha^*(t) < 1$, $\|\dot{\alpha}^*(t)\| \le M$, *and unknown parameter vector* p^*. *The following set of adaptation rules identifies the order and the parameter with arbitrary small error:*

$$
\begin{aligned}
&\dot{\hat{p}} = P\phi(x^* - x) \\
&\dot{P} = -\lambda P + P\phi(\hat{\alpha},t)\phi^T(\hat{\alpha},t)P, P(0) = I \\
&\dot{\hat{\alpha}} = \gamma\Lambda(x^* - \hat{x}) \\
&{}_0^C D_t^{\hat{\alpha}(t)}\hat{x}(t) = \hat{p}^T u
\end{aligned}
\tag{29}
$$

where,

$$
\Lambda = \Big(p^T(\hat{\alpha},t)\frac{\partial\phi(\beta,t)}{\partial\beta}|_{\beta=\hat{\alpha}} + \frac{\partial p^T(\beta,t)}{\partial\beta}|_{\beta=\hat{\alpha}}\phi(\hat{\alpha},t)\Big) \tag{30}
$$

and $0 < \lambda < 1$ *is the forgetting factor,* γ *is a positive scalar and* $p(\beta,t) = P(\beta,t)\int_0^t e^{-\lambda(t-\tau)}x^*(\tau)\phi(\beta,\tau)d\tau$, *and* $\phi(\beta,t)$ *is defined in (18). Furthermore, when* u *is in a way that* $\phi(\beta,t)$ *is PE for all functions* $0 < \beta(t) < 1$, *then* $\hat{p} \to p^*$.

Proof. First, it worths mentioning that $\phi(\beta) = z_u(\beta,t), z_0 = 0$ in (13) and $\frac{\partial p}{\partial\beta}$ can be computed using (14) and (15). Also, $\frac{\partial\phi}{\partial\beta}$ can be computed as (26). The integrations are all computed using a numerical method.

By derivating both equations of (23) with respect to t, the first two equations in (29) are obtained which are the real time equivalent of the LSFF algorithm (23) for $\beta = \hat{\alpha}$ [44]. Actually, $\hat{p} = p(\hat{\alpha},t)$. Hence, (24) in Lemma 2 holds.

Consider the Lyapunov function $V = \frac{1}{2}(\alpha^* - \hat{\alpha})^2$. Then,

$$\dot{V} = \dot{\alpha}^*(\alpha^* - \hat{\alpha}) - \gamma\Lambda(\alpha^* - \hat{\alpha})(x^* - \hat{x}) \tag{31}$$

Substituting $x^* - \hat{x}$ from (24), according to Lemma 1, there is T_ϵ such that for $t > T_\epsilon$:

$$\dot{V} \leq (M + \epsilon\gamma\|\Lambda\|)\|\alpha^* - \hat{\alpha}\| - \gamma\Lambda^2(\alpha^* - \hat{\alpha})^2 + \gamma\|\Lambda\|\|K(t)\|\|\alpha^* - \hat{\alpha}\|^3 \tag{32}$$

The term with negative sign can be split to two and all the terms can be grouped as:

$$\begin{aligned} \dot{V} \leq & \left\{(\epsilon\gamma\|\Lambda\| + M)\|\alpha^* - \hat{\alpha}\| - \frac{\gamma}{2}\Lambda^2\|\alpha^* - \hat{\alpha}\|^2\right\} \\ & + \left\{\gamma\|\Lambda\|\|K(t)\|\|\alpha^* - \hat{\alpha}\|^3 - \frac{\gamma}{2}\Lambda^2\|\alpha^* - \hat{\alpha}\|^2\right\} \end{aligned} \tag{33}$$

Suppose that $\Lambda \neq 0$. So, a positive value Λ_0 exists such that $\|\Lambda\| > \Lambda_0$. Then,

1. The first group is negative when $\|\alpha^* - \hat{\alpha}\| > \frac{2M}{\gamma\|\Lambda\|^2} + \frac{2\epsilon}{\|\Lambda\|}$. By setting a small enough value for ϵ, the value of $\epsilon_1 = \frac{2\epsilon}{\Lambda_0}$ can be made arbitrarily small. Furthermore, $\|\alpha^* - \hat{\alpha}\| > \frac{2M}{\gamma\Lambda_0^2} + \epsilon_1$ guarantees that $\|\alpha^* - \hat{\alpha}\| > \frac{2M}{\gamma\Lambda^2} + \frac{2\epsilon}{\|\Lambda\|}$.

2. The second group is negative when $\|\alpha^* - \hat{\alpha}\| < \frac{\|\Lambda\|}{2\|K\|}$. As long as $\|\Lambda\| > 0$, since $\|K\|$ is bounded, there is a positive real number M_K such that $\|\alpha^* - \hat{\alpha}\| < \frac{\Lambda_0}{2M_K}$ yields $\|\alpha^* - \hat{\alpha}\| < \frac{\|\Lambda\|}{2\|K\|}$.

Hence, for $t > T_\epsilon$, $\dot{V}$ is negative inside the interval $\frac{2M}{\gamma\Lambda_0^2} + \epsilon_1 < \|\alpha^* - \hat{\alpha}\| < \frac{\Lambda_0}{2M_K}$. The value of T_ϵ is dependent on κ. It can be adjusted in a way that $T_\epsilon << T$ implying that ϵ_1 rapidly tends to zero and $\dot{V} < 0$ in $\frac{2M}{\gamma\Lambda_0^2} \leq \|\alpha^* - \hat{\alpha}\| \leq \frac{\Lambda_0}{2M_K}$ in the most part of the interval $[0 \;\; T]$.

This implies the locally ultimate boundedness of the error. In fact, since $\dot{V} < 0$ in $\frac{2M}{\gamma\Lambda_0^2} < \|\alpha^* - \hat{\alpha}\|$, as soon as $\alpha - \hat{\alpha}$ tends to leave the area $\|\alpha^* - \hat{\alpha}\| \leq \frac{2M}{\gamma\Lambda_0^2}$, it is pulled back inside. Thus, the error $\alpha^* - \hat{\alpha}$ remains bounded with maximum bound of $\frac{2M}{\gamma\Lambda_0^2}$. Accordingly, if the initial condition $\hat{\alpha}(0)$ is properly chosen, the identification error can be made arbitrarily small. The convergence speed is related to the matrix κ and the scalar γ.

Consider the interval $B_L < \|\alpha^* - \hat{\alpha}\| < B_U$ with the lower bound $B_L = \frac{2M}{\gamma \Lambda_0^2}$ and the upper bound $B_U = \frac{\Lambda_0}{2M_K}$. It should be noted that as long as $B_L < B_U$ the aforementioned interval is never empty, i.e., for $\gamma > \frac{4MM_K}{\Lambda_0^{3^3}}$ there exist a region such that if the initial condition is chosen inside it, the adaptation rules converge to an identification with bounded error. Also, a larger γ leads to a smaller ultimate bound on $\|\alpha^* - \hat{\alpha}\|$. So, a large enough γ ensures the convergence with an acceptable error in (29).

Moreover,

1. For the constant order case where $M = 0$, if the initial value is properly chosen, the identification error asymptotically converges to zero.

2. As $K \to 0$, the upper bound tends to infinity i.e., the identification error will be globally bounded.

3. Since the set $\{\hat{\alpha}, \Lambda(t) = 0\}$ is not an invariant set for the system (i.e. it does not yield $\dot{\hat{\alpha}} = 0, \forall t$), so the trajectory will leave this set. Hence, only the case $\Lambda \neq 0$ is considered in the stability analysis.

Consequently, $\hat{\alpha} \to \alpha^*$ and, due to the continuity of $p(\beta, t)$ with respect to β, $p(\hat{\alpha}, t) \to p(\alpha^*, t)$. When $\phi(\beta, t)$ is PE for all β, then $\phi(\hat{\alpha}, t)$ is PE and according to Lemma 1, $p(\alpha^*, t) \to p^*$. Hence, $p(\hat{\alpha}, t) \to p(\alpha^*, t) \to p^*$ and this completes the proof.

□

The above theorem proposes a scheme to identify the order and parameters of the variable order system in (11). Fig. 1 shows this procedure as a block diagram.

5. Simulation Study

In this section, the identification method introduced in (29) is applied on a multiple-input, single-output system of the form (11). It is noteworthy to mention that in order to demonstrate the effectiveness of the proposed method, we have chosen a noisy system with an additive measurement noise.

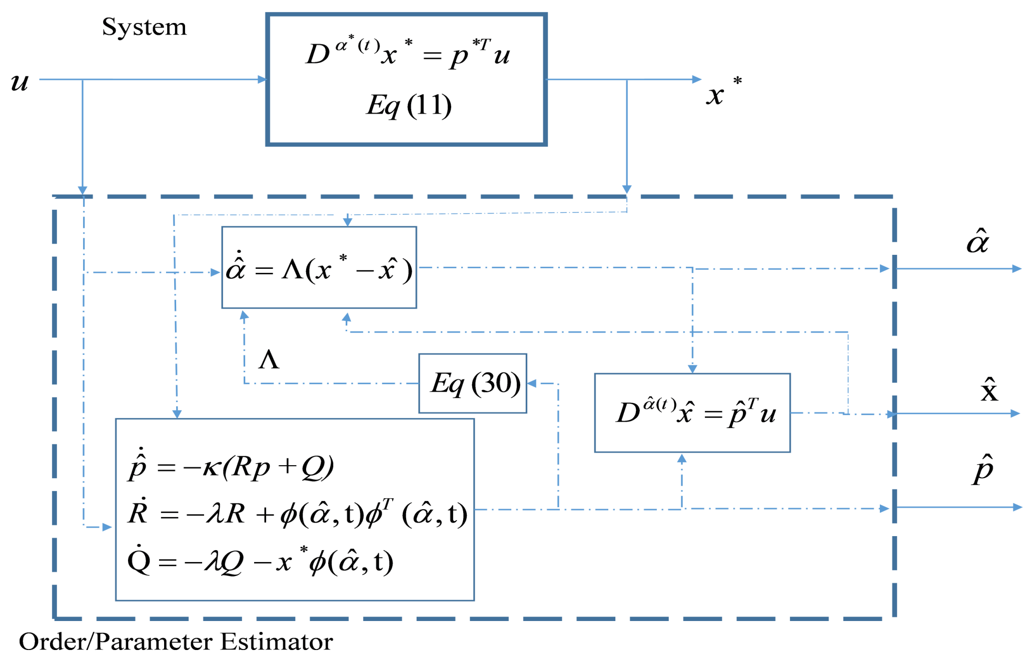

Figure 1: Block diagram of the proposed method for the order/parameter identification

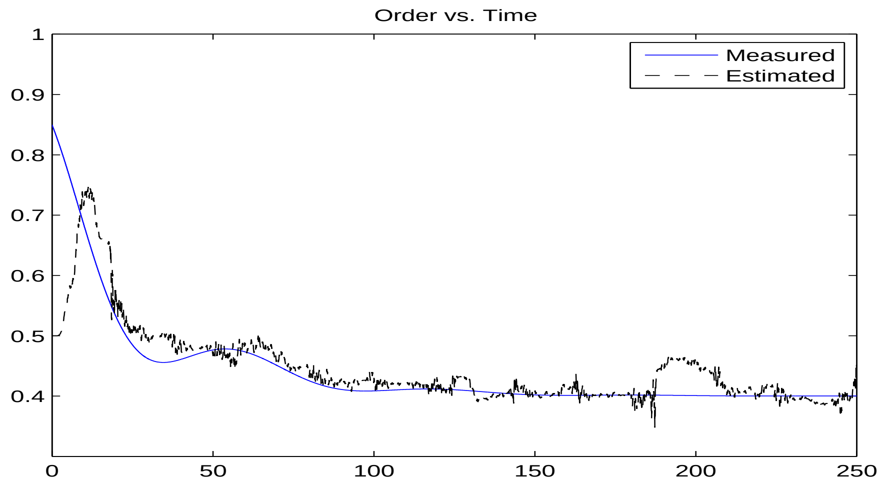

Figure 2: The actual and identified order (α^* and $\hat{\alpha}$)

Consider the following:

$$
\begin{aligned}
&{}_0^C D_t^{\alpha^*(t)} x^*(t) = p^{*T} u,\ \ \nu \sim N(0, 0.25) \\
&u = \begin{bmatrix} sin(0.5t) \\ cos(0.2t) \end{bmatrix} \\
&\alpha^*(t) = 0.15 e^{-0.02t}\big(cos(0.2t) + 2\big) + 0.4 \\
&p^* = \begin{bmatrix} 3 + 4\big(1 - e^{-0.1(t-150)}\big)\Theta(t-150) \\ -5 \end{bmatrix} \\
&\lambda = 0.9
\end{aligned}
\tag{34}
$$

where $\Theta(.)$ denotes the Heaviside step function and ν is a white Gaussian noise. Accordingly, there is a smooth jump in the first entry of the parameter vector at $t = 150$. The actual and identified order, output and parameters are shown in Fig. 2, Fig. 3 and Fig. 4, respectively. Figs 2-4 show that the proposed algorithm efficiently identifies the order and the parameters even in a noisy situation. It identifies the order, the output and parameters with bounded error. This simulation study validates the effectiveness of the theoretical methods. In the next Section, the stress-strain relationship for a real beef tissue will be modeled and identified using the proposed method.

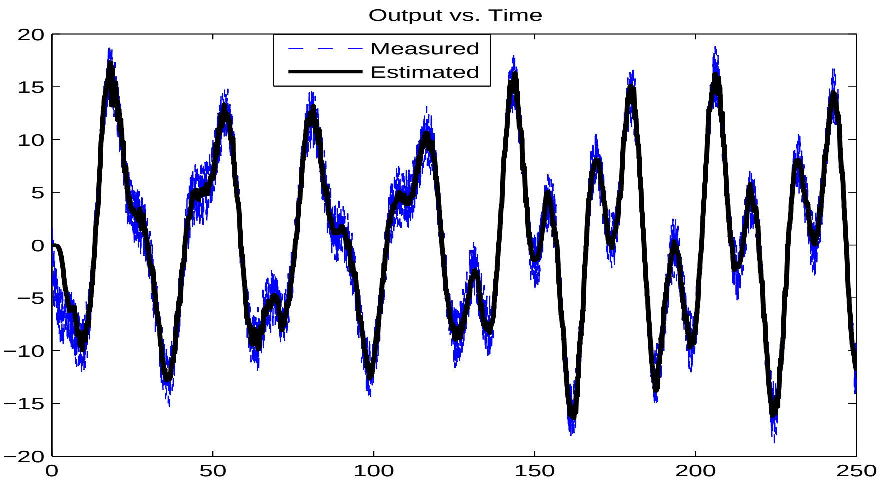

Figure 3: The actual and identified output (x^* and $\hat{x}$)

6. Experimental Study

As mentioned in Section I, describing the dynamic behavior of viscoelastic materials can be done using the variable order model $\sigma = \eta D_t^{q(t)}\varepsilon, 0 < q(t) < 1$ which is of the form (11) with $\alpha^* = q$, $x^* = \varepsilon$, $u = \sigma$, and $p^* = \frac{1}{\eta}$. The material we have chosen is a symmetrical slice of real beef tissue. It is indented in one dimension using a needle insertion robot shown in Fig. 5. The setup consists of a robotic system with two degrees of freedom (DOF) for translational and rotational motions of the needle. Forces and torques are measured at the needle base using a 6-DOF force sensor. After we replace the needle with a blunt indenter, the setup can be used to apply controlled force or displacement to the tissue for indentation tests. The apparatus can record images of the needle inside tissue to track the needle position. However, we have used the encoder data to record the displacement and calculate the strain. Since the tissue is symmetric, we will just consider the indentation direction to get one dimensional stress-strain data. The data gathering process is done through obtaining force and displacement data from the force sensor and the encoder and converting them to stress and strain, respectively, using the one dimensional equations

$$\sigma = \frac{F}{a}, \varepsilon = \frac{d}{l} \tag{35}$$

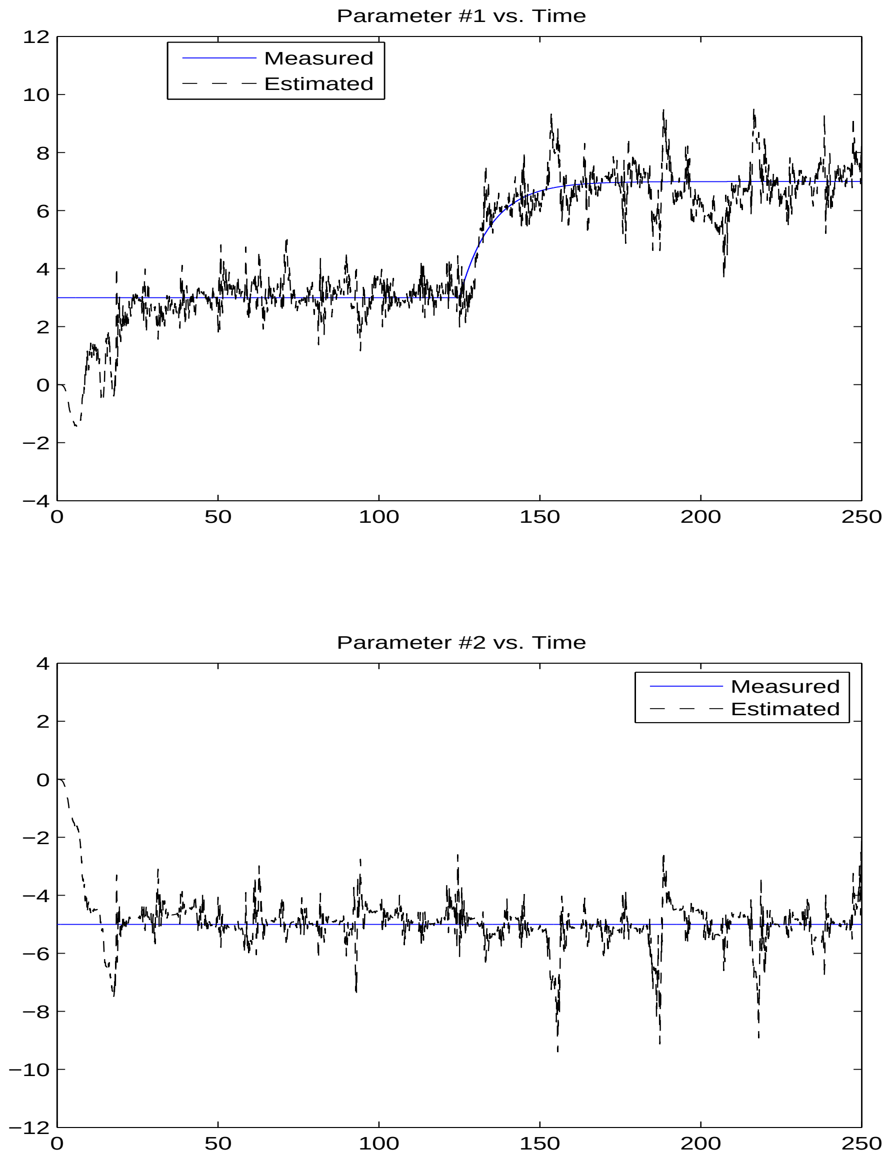

Figure 4: The actual and identified parameters. Top: p_1^* vs. $\hat{p}_1$, Bottom: p_2^* vs. $\hat{p}_2$

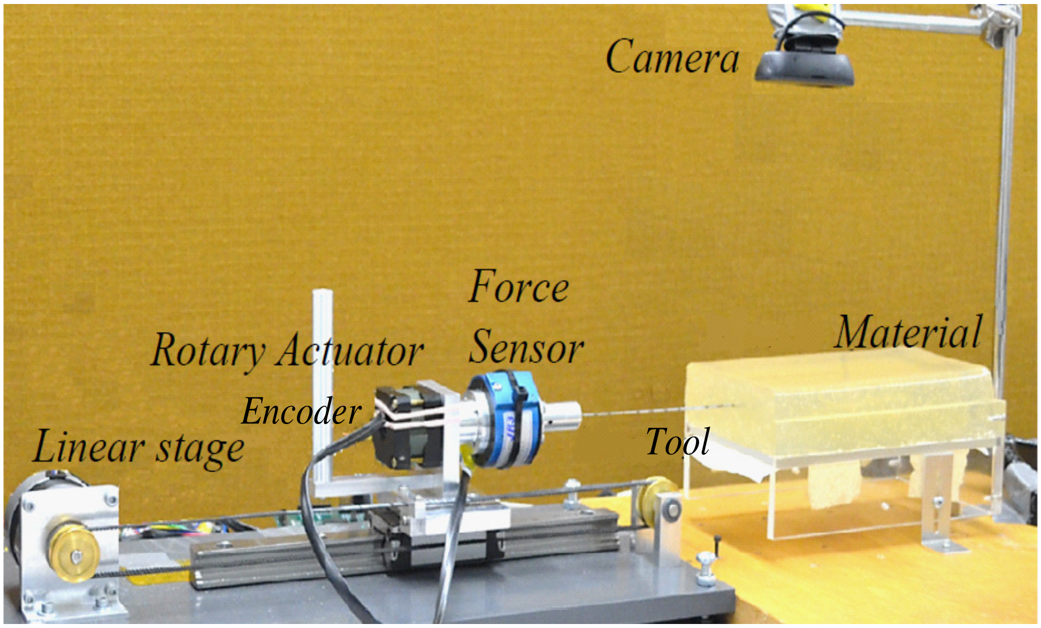

Figure 5: Data gathering apparatus

where F is force, d is displacement, a is area, l is tissue length, σ is stress and ϵ is strain.

We have used controlled tissue displacement with various profiles. Biological tissue is a highly complex system. Accordingly, small changes in displacement profiles may lead to huge difference in the measured force. This strange behavior makes it difficult to identify it through an experiment and used the estimated parameters for another experiment. Fig. 6 shows that two displacement profile in similar ranges have been applied to same tissue, the first one with and the second one without pauses. Although the displacement range and the type of tissue are the same the way the displacement is applied causes a big different in tissue behavior. In the left hand side plots (Experiment 1) the stress remains until the final time. i.e., the tissue is behaving like an elastic material, while in the right hand side plots (Experiment 2) the stress is being gradually disappeared, similar to viscous materials. Consequently, the modeling and identification method should be able to model both behaviors. Hence, it makes more sense to identify the tissue in each experiment in a real time way. This is the reason why the adaptive identification method is needed in practical applications.

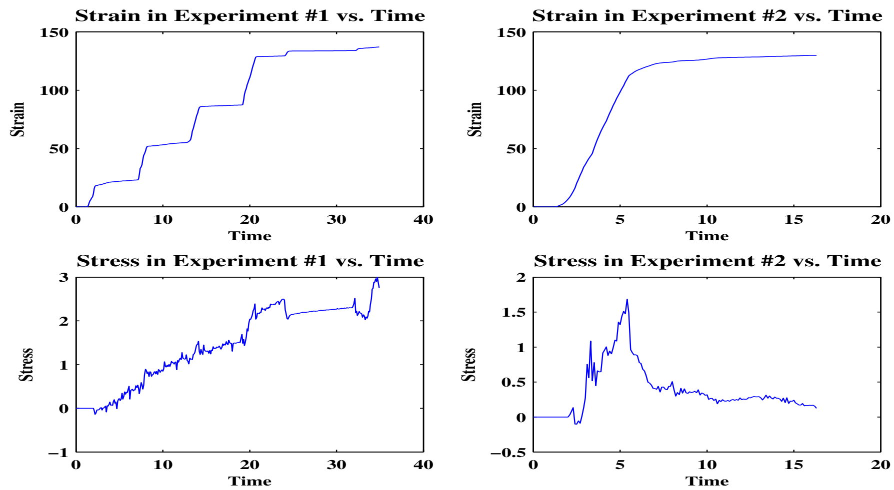

Figure 6: Stress and strain for a tissue in two different experiments. Left hand side: Experiment 1 with paused applied displacement. Right hand side: Experiment 2 with non-paused applied displacement.

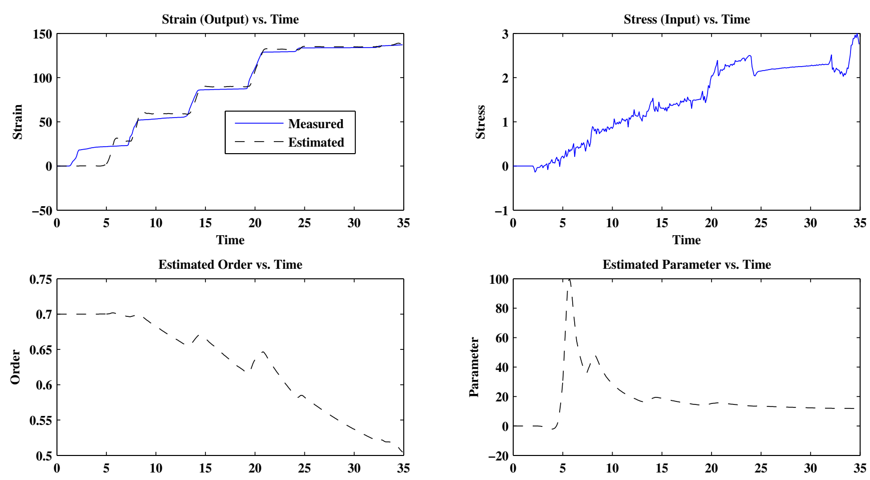

Figure 7: Applying the proposed model and identification method on data gathered from experiment 1. Top-Left: Measured and Identified Strain (Output). Top-Right: Stress (Input). Bottom-Left: Identified Order. Bottom-Right: Estimated Parameter

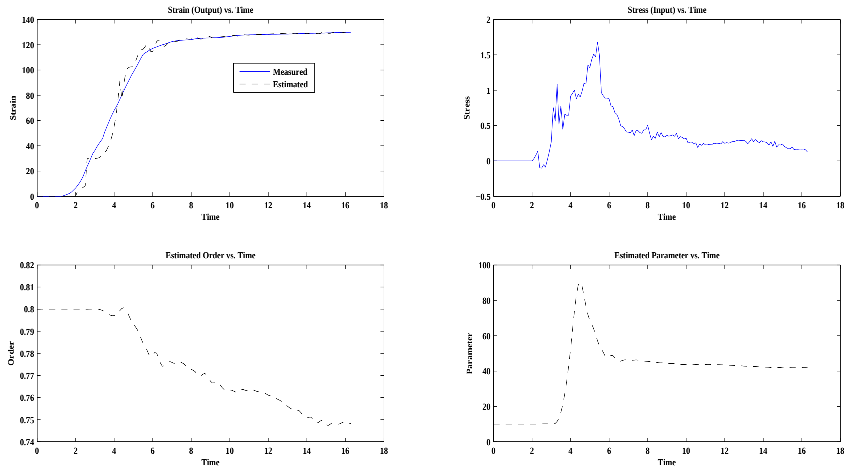

Figure 8: Applying the proposed model and identification method on data gathered from experiment 2. Top-Left: Measured and Identified Strain (Output). Top-Right: Stress (Input). Bottom-Left: Identified Order. Bottom-Right: Estimated Parameter

Now, considering model (8), we find different behaviors for the order q and parameter η in each experiment. The method proposed in Theorem 1 is applied on the data from the above experiments. Fig. 7 and 8 show the results. It can be seen that the output identification error is small. Also, although the tissue behavior is different in each experiment, the proposed model and identification approach effectively identifies the order and parameter in both cases.

7. Conclusion

In this paper, an adaptive order/parameter identification approach is introduced for identifying the order and parameters of a variable order system. After proving some Lemma, Theorem 1 provides the main result. This method is validated using a simulation study. Afterwards, an experimental study is done based on the theoretical results. The experimental study uses an apparatus recording indentation force-displacement data.

Fitting the proposed model on the stress-strain data gathered from a real beef tissue and applying the identification approach validate the modeling concept and show the effectiveness of the identification method. Future works in this topic may be done pertaining to extending the presented results in order to design an order/state/parameter identification method for variable order system. Practical extension may be done regarding to modeling and identification of multi-dimensional viscoelastic materials.

[1] Volos C, Pham VT, Zambrano-Serrano E, Munoz-Pacheco J, Vaidyanathan S, Tlelo-Cuautle E. Analysis of a 4-d hyperchaotic fractional-order memristive system with hidden attractors. *Advances in Memristors, Memristive Devices and Systems*. Springer, 2017; 207–235.

[2] Maundy B, Elwakil A, Gift S. On the realization of multiphase oscillators using fractional-order allpass filters. *Circuits, Systems, and Signal Processing* 2012; **31**(1):3–17.

[3] Wharmby AW, Bagley RL. Generalization of a theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology* 2013; **57**(5):1429–1440.

[4] Diaz G, Coimbra C. Nonlinear dynamics and control of a variable order oscillator with application to the van der pol equation. *Nonlinear Dynamics* 2009; **56**(1-2):145–157.

[5] Tabatabaei SS, Yazdanpanah MJ, Tavazoei M, Karimian A. On dynamic models of human emotion. *20th Iranian Conference on Electrical Engineering (ICEE)*, 2012; 874–878.

[6] Tabatabaei SS, Yazdanpanah MJ, Tavazoei MS. Incommensurate order fractional optimal control: Application to treatment of psychiatric disorders. *21st Iranian Conference on Electrical Engineering (ICEE)*, 2013; 1–5.

[7] Tabatabaei SS, Yazdanpanah MJ, Jafari S, Sprott JC. Extensions in dynamic models of happiness: effect of memory. *International Journal of Happiness and Development* 2014; **1**(4):344–356.

[8] Ionescu CM, De Keyser R. Relations between fractional-order model parameters and lung pathology in chronic obstructive pulmonary disease. *IEEE Transactions on Biomedical Engineering* 2009; **56**(4):978–987.

[9] Yin C, Chen Y, Zhong Sm. Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems. *Automatica* 2014; **50**(12):3173–3181.

[10] Yin C, Cheng Y, Chen Y, Stark B, Zhong S. Adaptive fractional-order switching-type control method design for 3d fractional-order nonlinear systems. *Nonlinear Dynamics* 2015; **82**(1-2):39–52.

[11] Yin C, Stark B, Chen Y, Zhong Sm, Lau E. Fractional-order adaptive minimum energy cognitive lighting control strategy for the hybrid lighting system. *Energy and Buildings* 2015; **87**:176–184.

[12] Yin C, Zhong Sm, Chen Wf. Design of sliding mode controller for a class of fractional-order chaotic systems. *Communications in Nonlinear Science and Numerical Simulation* 2012; **17**(1):356–366.

[13] Samko SG, Ross B. Integration and differentiation to a variable fractional order. *Integral transforms and special functions* 1993; **1**(4):277–300.

[14] Lorenzo CF, Hartley TT. Variable order and distributed order fractional operators. *Nonlinear Dynamics* 2002; **29**(1-4):57–98.

[15] Valério D, Da Costa JS. Variable-order fractional derivatives and their numerical approximations. *Signal Processing* 2011; **91**(3):470–483.

[16] Sierociuk D, Malesza W, Macias M. On a new definition of fractional variable-order derivative. *14th International Carpathian Control Conference (ICCC)*, 2013; 340–345.

[17] Sierociuk D, Malesza W, Macias M. Switching scheme, equivalence, and analog validation of the alternative fractional variable-order derivative definition. *52nd IEEE Annual Conference on Decision and Control (CDC)*, 2013; 3876–3881.

[18] Sheng H, Sun H, Coopmans C, Chen Y, Bohannan G. A physical experimental study of variable-order fractional integrator and differentiator. *The European Physical Journal Special Topics* 2011; **193**(1):93–104.

[19] Sierociuk D, Podlubny I, Petras I. Experimental evidence of variable-order behavior of ladders and nested ladders. *Control Systems Technology, IEEE Transactions on* 2013; **21**(2):459–466.

[20] Podlubny I. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, vol. 198. Academic press, 1998.

[21] Kilbas A, Marzan S. Nonlinear differential equations with the caputo fractional derivative in the space of continuously differentiable functions. *Differential Equations* 2005; **41**(1):84–89.

[22] Meral F, Royston T, Magin R. Fractional calculus in viscoelasticity: an experimental study. *Commun. Nonlinear. Sci. Numer. Simul.* 2010; **15**(4):939–945.

[23] Koeller R. Applications of fractional calculus to the theory of viscoelasticity. *J. Appl. Mech.* 1984; **51**(2):299–307.

[24] Bagley RL, Torvik PJ. On the fractional calculus model of viscoelastic behavior. *J. Rheol.* 1986; **30**(1):133–155.

[25] Di Paola M, Pirrotta A, Valenza A. Visco-elastic behavior through fractional calculus: an easier method for best fitting experimental results. *Mech. Matter.* 2011; **43**(12):799–806.

[26] Bazhlekova E, Bazhlekov I. Viscoelastic flows with fractional derivative models: Computational approach by convolutional calculus of dimovski. *Fract. Calc. Appl. Anal.* 2014; **17**(4):954–976.

[27] Thien NP. Understanding viscoelasticity: Basics of rheology 2002.

[28] Coimbra CF. Mechanics with variable-order differential operators. *Annalen der Physik* 2003; **12**(11-12):692–703.

[29] Ramirez LE, Coimbra CF. A variable order constitutive relation for viscoelasticity. *Annalen der Physik* 2007; **16**(7-8):543–552.

[30] Sun H, Chen W, Chen Y. Variable-order fractional differential operators in anomalous diffusion modeling. *Physica A: Statistical Mechanics and its Applications* 2009; **388**(21):4586–4592.

[31] Bhrawy A, Zaky M. Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. *Nonlinear Dynamics* 2015; **80**(1-2):101–116.

[32] Hussain S, Elbergali A. Fractional order estimation and testing, application to swedish temperature data. *Environmetrics* 1999; **10**(3):339–349.

[33] Victor S, Malti R, Garnier H, Oustaloup A. Parameter and differentiation order estimation in fractional models. *Automatica* 2013; **49**(4):926–935.

[34] Sierociuk D, Dzielinski A. Fractional kalman filter algorithm for the states, parameters and order of fractional system estimation. *International Journal of Applied Mathematics and Computer Science* 2006; **16**(1):129–140.

[35] Hartley TT, Lorenzo CF. Fractional-order system identification based on continuous order-distributions. *Signal processing* 2003; **83**(11):2287–2300.

[36] Sabatier J, Aoun M, Oustaloup A, Grégoire G, Ragot F, Roy P. Fractional system identification for lead acid battery state of charge estimation. *Signal processing* 2006; **86**(10):2645–2657.

[37] Rapaic MR, Pisano A. Variable-order fractional operators for adaptive order and parameter estimation. *Automatic Control, IEEE Transactions on* 2014; **59**(3):798–803.

[38] Jamison C, Marangoni R, Glaser A. Viscoelastic properties of soft tissue by discrete model characterization. *J. Manuf. Sci. Eng.* 1968; **90**(2):239–247.

[39] Mainardi F. An historical perspective on fractional calculus in linear viscoelasticity. *Fract. Calc. Appl. Anal.* 2012; **15**(4):712–717.

[40] Shen JJ, Li CG, Wu HT, Kalantari M. Fractional order viscoelasticity in characterization for atrial tissue. *Korea-Aust. Rheol. J.* 2013; **25**(2):87–93.

[41] Mainardi F. *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models.* World Scientific, 2010.

[42] Esmaeili S, Shamsi M, Dehghan M. Numerical solution of fractional differential equations via a volterra integral equation approach. *Open Physics* 2013; **11**(10):1470–1481.

[43] Samko SG. Fractional integration and differentiation of variable order. *Analysis Mathematica* 1995; **21**(3):213–236.

[44] Ioannou PA, Sun J. *Robust adaptive control*. Courier Corporation, 2012.