Nonlinear Trilateral Teleoperation Stability Analysis subjected to Time-varying Delays

Farzad Hashemzadeh, Mojtaba Sharifi and Mahdi Tavakoli, *Member, IEEE*

Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada

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Abstract

A trilateral teleoperation system facilitates the collaboration of two users to share control of a single robot in a remote environment. While various applications of shared-control trilateral haptic teleoperation systems have recently emerged, they have mostly been studied in the context of single-DOF, LTI robotic systems. On the other hand, robotic manipulators with multiple degrees of freedom (DOF) and therefore nonlinear dynamics have recently found many applications such as in robotic-assisted surgery and therapy, space exploration and navigation systems. In this paper, considering the full nonlinear dynamical models of multi-DOF robots, stability analysis of a dual-user haptic teleoperation system is considered over a communication network subjected to asymmetrical time varying delays and through a dominance factor suitable for trainer-trainee applications. Stability in free motion and contact motion and asymptotic position tracking of the trilateral haptic teleoperation system in free motion are proven via Lyapunov stability analysis and Barbalats lemma where operators and the environment are assumed to be passive. Simulation and experimental results concerning robot position tracking and user-perceived forces for three 2-DOF robots and experimental analysis of user-perceived stiffnesses for three 3-DOF robots validate the theoretical findings pertaining to the system stability and demonstrate the efficiency of the proposed controller.

Keywords: haptics-assisted training; dual-user teleoperation; trilateral system; dominance factor; Lyapunov stability analysis

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1. Introduction

One of the emerging applications of haptics and teleoperation involves trilateral systems in which two users collaborate in performing a task using a robot while receiving haptic feedback. Examples of these applications are in haptics-assisted surgical training (Chebbi et al. (2007), Greer et al. (2008) and Nudehi et al. (2005)) and robot-assisted rehabilitation (Culmer et al. (2010), Carignan & Olsson (2004), Gupta & OMalley (2006) and Mussa-Ivaldi & Patton (2000)).The challenge of controller design for such systems is in guaranteeing the system stability at the same time as enhancing the effectiveness of collaboration by enabling the trainer to transfer/retract partial or full task authority to/from the trainee in a natural and intuitive way.

Robotic manipulators with multiple DOFs are ubiquitous in various applications and inevitably involve nonlinear dynamics. In past research, it is assumed that multi-DOF nonlinear robotic systems can be decoupled to 1-DOF systems for stability analysis of bilateral systems (Speich & Goldfarb (2005)). However, given model uncertainties and the fact that human operator(s) and environment(s) coupled to the haptic devices are also often multi-DOF nonlinear systems, full decoupling is next to impossible.

A shared control architecture for haptics-assisted training in minimally invasive surgery is proposed in Nudehi et al. (2005) in which 1-DOF LTI models for robots are assumed. A six-channel, dual-user teleoperation system for interaction between the users through a dominance factor is proposed in Khademian & Hashtrudi-Zaad (2012) in which the robots dynamical models are again 1-DOF LTI and a new framework for the coupled stability analysis of linear dual-user teleoperation is considered in Razi & Hashtrudi-Zaad (2014). Haptic-enabled training approaches are discussed in Shahbazi et al. (2013) and in Shahbazi et al. (2012) for a 1-DOF LTI multi-master/single-slave system.

Considering linear dynamics for robots, the stability analysis of a trilateral haptic collaboration is studied in Li et al. (2013c) using extending the Llewellyn's criterion. The stability analysis of a trilateral teleoperation by splitting desired task between two master robots is considered in Li et al. (2013b). Conservatism analysis between absolute stability and passivity criteria in linear trilateral teleoperation is studied in Li et al. (2013d) and extension of the Zeheb-Walach absolute stability criteria for n-port networks is considered in Razi & Hashtrudi-Zaad (2012) that can be applied to linear trilateral teleoperation. Controller synthesis in a bilateral teleoperation of a composed system with single local manipulator and multiple cooperative remote manipulators are considered in Aldana et al. (2012) and trilateral teleoperation control of kinematically redundant robotic manipulators is studied in Malysz & Sirouspour (2011).

As far as LTI trilateral haptic systems are concerned, two different ways to study the stability are passivity (Raisbeck (1954), Shahbazi et al. (2010), Panzirsch et al. (2012) and Mendez & Tavakoli (2010)) and absolute stability. Our research group has introduced an extension of Llewellyn's criterion for absolute stability analysis of single-DOF, LTI bilateral haptic systems to single-DOF, LTI trilateral haptic systems (Li et al. (2013a)). We have also performed other extensions of the stability analysis to 3-DOF, LTI bilateral systems and to the more general case of multi-DOF multi-lateral LTI systems (Li et al. (2014)). All of the above have been done in the context of LTI systems.

Design of stable and high-performance trilateral teleoperation systems involving robots with multi-DOF nonlinear dynamics has not received much attention yet. Time-varying time delays add to the complexity of the problem. In the literature of bilateral teleoperation under nonlinear robot dynamics and time-varying delays, P+D and PD like controllers are used widely to guarantee asymptotic stability of the closed-loop systems and zero convergence of the tracking errors (Lee & Spong (2006), Kim et al. (2005), Polushin et al. (2008) and Hua & Liu (2010)). A stability analysis of a bilateral teleoperation system with actuator saturation and nonlinear dynamical models for robots and time varying delays in communication channel was studied in Hashemzadeh et al. (2013). Extension of the above to trilateral nonlinear teleoperation systems subject to time-varying delays remains to be done.

In this paper, assuming the operators and environment are passive, a PD like controller to guarantee the stability of a trilateral dual-user system in the presence of multi-DOF nonlinear dynamics for all three robots and time-varying communication delays in all communication channels is proposed. The system has two master robots for the two users and one slave robot to perform the desired task on an environment. In the trilateral teleoperation system, the goal is that two users collaboratively control a robot in order to perform a task. Based on a Lyapunov stability analysis and using Barbalats lemma, theorems are given to analyze the stability and asymptotic position tracking of the proposed trilateral system. Simulation and experimental results show the validity of the theoretical findings and the efficiency of the proposed controller.

The rest of the paper is organized as follows. In Section 2, mathematical preliminaries are stated. In Section 3, the proposed controller for delay-free nonlinear trilateral teleoperation is proposed. Section 4 studies the generalization of the proposed controller in the presence of time-varying delays. In Section 5, simulation and experimental results are provided followed by the conclusions in Section 6.

Notation. We denote the set of real numbers by $R = (-\infty, \infty)$, the set of positive real numbers by $R_{>0} = (0, \infty)$, and the set of nonnegative real numbers by $R_{\geq 0} = [0, -\infty)$. Also, $||X||_{\infty}$ and $||X||_2$ stand for the Euclidian $\infty - norm$ and $2 - norm$ of a vector $X \in R^{(n\times 1)}$, and $|X|$ denotes element-wise absolute value of the vector$X$. The L_∞ and L_2 norms of a time function $f : R_{\geq 0} \rightarrow R^{n\times 1}$ are shown as $||f||_{L_\infty} = sup_{t\in[0,\infty)} ||f(t)||_\infty$ and $||f||_{L_2} = \sqrt{\int_a^b f(x)\ dx}$, respectively. The L_∞ and L_2 spaces are defined as the sets $\{f : R_{\geq 0} \rightarrow R^{n\times 1}, ||f||_{L_\infty} < +\infty\}$ and $\{f : R_{\geq 0} \rightarrow R^{n\times 1}, ||f||_{L_2} < +\infty\}$, respectively. For simplicity, we refer to $||f||_{L_\infty}$ as $||f||_\infty$ and to $||f||_{L_2}$ as $||f||_2$. We also simplify the notation $\lim_{t\rightarrow\infty} f(t) = 0$ to $f(t) \rightarrow 0$.

2. Preliminaries

Consider the n-DOF master 1, master 2 and slave robots to have the following nonlinear dynamics, respectively:

$$M_1 (q_1(t))\, \ddot{q}_1 + C_1 (q_1(t), \dot{q}_1)\, \dot{q}_1 + G_1 (q_1(t)) = \tau_1(t) - \tau_{h_1}(t) \tag{1}$$

$$M_2 (q_2(t))\, \ddot{q}_2 + C_2 (q_2(t), \dot{q}_2)\, \dot{q}_2 + G_2 (q_2(t)) = \tau_2(t) - \tau_{h_2}(t) \tag{2}$$

$$M_s (q_s(t))\, \ddot{q}_s(t) + C_s (q_s(t), \dot{q}_s(t))\, \dot{q}_s(t) + G_s (q_s(t)) = \tau_s(t) - \tau_e(t) \tag{3}$$

Here, q_i, $\dot{q}_i$ and $\ddot{q}_i \in R^{(n\times 1)}$ for $i \in \{1, 2, s\}$ are the joint positions, velocities and accelerations of master 1, master 2 and slave, respectively. Also, $M_i(q_i(t)) \in R^{(n\times n)}$,$C_i(q_i(t), \dot{q}_i(t)) \in R^{(n\times n)}$, and $G_i(q_i(t)) \in R^{(n\times 1)}$ are the inertia matrices, the Coriolis/centrifugal matrices, and the gravitational vectors for the three robots. Moreover, τ_{h_1}, τ_{h_2} and $\tau_e \in R^{(n\times 1)}$ are the torques applied by the first and second human operators and the environment on their respective robots, respectively. Lastly, τ_1, τ_2 and $\tau_s \in R^{(n\times 1)}$ are the control signals (torques) for the master 1, master 2 and the slave robots, respectively. Properties of the nonlinear dynamic models (1)-(3), which will be used in this paper, are (Kelly et al. (2006) and Spong et al. (2006)) :

P-1. For a manipulator with revolute joints, the inertia matrix $M(q)$ is symmetric positive-definite and has the following upper and lower bounds:

$$0 < \lambda_{min}\left(M\left(q(t)\right)\right) I \leq M\left(q(t)\right) \leq \lambda_{Max}\left(M\left(q(t)\right)\right) I \leq \infty$$

where $I \in R^{(n\times n)}$ is the identity matrix and λ denotes the eigenvalue of a matrix.

P-2. For a manipulator, the relation between the Coriolis/centrifugal and the inertia matrices is as follows:

$$\dot{M}\left(q(t)\right) = C\left(q(t), \dot{q}(t)\right) + C^T\left(q(t), \dot{q}(t)\right)$$

This is equivalent to $\dot{M}\left(q(t)\right) - 2C\left(q(t), \dot{q}(t)\right)$ being skew-symmetric.

P-3. For a manipulator with revolute joints, there exists a positive bounding the Coriolis/centrifugal term as follows:

$$||C\left(q(t), x(t)\right) y(t)||_2 \leq ||x(t)||_2||y(t)||_2$$

P-4. The time derivative of $C\left(q(t), \dot{q}(t)\right)$ is bounded if $\dot{q}(t)$ and $\ddot{q}(t)$ are bounded.

3. Proposed trilateral teleoperation laws

As described before, a dual-user teleoperation system comprises two master robots as haptic interfaces for the two users and one slave robot to perform a desired task on an environment. This finds application in many real-world scenarios such as when the aim is to train a novice trainee (user 1) to do a task under haptic guidance from a mentor (user 2). The key in haptic guidance is the capability of authority sharing between the two users. As elaborated by Khademian & Hashtrudi-Zaad (2010) and Nudehi et al. (2005), in the so-called complementary linear combination (CLC) authority sharing, the reference position and force for each robot are sums of positions and forces of the other two robots weighted by a parameter $0 \leq \alpha \leq 1$ that specifies their relative control authorities. Therefore, α affects how the trainee and the mentor collaborate and contribute to the reference position for the slave and what share of force feedback each of them receives. For instance, if $\alpha = 0$, the slave robot will be completely controlled by the mentor and the trainee will receive large force feedback urging him/her to follow the mentors motions. On the other hand, if $\alpha = 1$, the slave robot is completely controlled by the trainee, allowing the mentor to assess the skill level of the trainee by feeling the reflected forces. If $0 < \alpha < 1$, the trainee and the mentor collaborate and each contribute to the slave robot position while receiving some force feedback. So, the process of training can start from a small α, which gives less authority to the trainee, and can gradually increase α to give more authority to the trainee as he/she learns how to do the task.

As another application of dual-user teleoperation systems comprising two master robots and one slave robot, supervised robot-assisted mirror rehabilitation therapy can be considered (Shahbazi et al. (2013)). For hemiparetic disabled patients (who have one functional arm and one impaired arm), master-slave teleoperation can be used to move the impaired arm (holding the slave robot) in accordance with the mirror-image movements of the functional arm (holding the master robot). In this way, the patient takes charge for rehabilitating his/her own impaired arm with great therapeutic benefits. To allow for the intervention of a therapist in terms of both correcting the commanded movements and assessing the patient's motor recovery, a second master device can be added that receives the therapist's commanded movements. In this way, the therapist's arm and the patient's functional arm (i.e., the two master robots) collaboratively control and receive haptic feedback from the patient's impaired limb (i.e., the slave robot). The collaboration of the therapist and the patients functional arm to provide movement therapy to the patients impaired arm via the dual-user teleoperation framework improves the patient's recovery process under supervision of the therapist.

In this paper, we adopt a similar framework. Defining $\alpha \in (0, 1)$ as the dominance factor that determines the contribution of the master 1 and master 2 operators on the slave robots motion, let us define the following desired positions and position tracking errors:

$$\begin{aligned} q_{1_d}(t) &\triangleq (\alpha q_s(t) + (1-\alpha) q_2(t)) \\ q_{2_d}(t) &\triangleq (\alpha q_1(t) + (1-\alpha) q_s(t)) \\ q_{s_d}(t) &\triangleq (\alpha q_1(t) + (1-\alpha) q_2(t)) \end{aligned} \tag{4}$$

$$\begin{aligned} e_1(t) &\triangleq q_1(t) - (\alpha q_s(t) + (1-\alpha) q_2(t)) \\ e_2(t) &\triangleq q_2(t) - (\alpha q_1(t) + (1-\alpha) q_s(t)) \\ e_s(t) &\triangleq q_s(t) - (\alpha q_1(t) + (1-\alpha) q_2(t)) \end{aligned} \tag{5}$$

where $q_{1_d}(t)$, $q_{2_d}(t)$ and $q_{s_d}(t)$ are desired positions for the master 1, master 2 and slave robots and $e_1(t)$, $e_2(t)$ and $e_s(t)$ are position tracking errors for the master 1, master 2 and slave robots, respectively. The goal of this paper is to define a controller such that $e_1(t)$, $e_2(t)$ and $e_s(t)$ converge to zero asymptotically.

In Figure 1, the trilateral teleoperation systems authority sharing laws between the three robots are illustrated, where m_1, m_2 and s denote master 1, master 2 and slave, respectively. The three robots send $q_1(t)$, $q_2(t)$, $q_s(t)$, $e_1(t)$ and $e_2(t)$ in the network through shown coefficients α and $1-\alpha$. Note that as $e_s(t)$ is linearly dependent on $e_1(t)$ and $e_2(t)$ as shown later in the proof of Theorem I, we do not need to use $e_s(t)$ in the control laws or in Figure 1.

Note that for the Lyapunov functional we will use in the stability analysis and to guarantee asymptotic zero convergence of the tracking errors, in the proposed control laws for the master 1, master 2 and slave robots in (6-8), we need the terms $\alpha q_s(t)$, $(1-\alpha)q_2(t)$ and $\alpha e_2(t)$ for the master 1, $(1-\alpha)q_s(t)$, $\alpha q_1(t)$ and $(1-\alpha)e_1(t)$ for the master 2 and $\alpha e_1(t)$ and $(1-\alpha)e_2(t)$ for the slave.

Now, we focus on the analysis of stability and asymptotic zero convergence of velocities and position tracking errors in a trilateral teleoperation system. In Hashemzadeh et al. (2013), Proportional plus Damping (P+D) controllers have been introduced for bilateral teleoperation systems subjected to nonlinear dynamics for robots. In this paper, we extend these laws to trilateral teleoperation and propose the following P+D controller involving gravity compensation for the master 1, master 2 and the slave robots:

$$\tau_1(t) = G_1\big(q_1(t)\big) - B_1\dot{q}_1(t) - k_1\big(e_1(t) - \alpha e_2(t)\big) \tag{6}$$

$$\tau_2(t) = G_2\big(q_2(t)\big) - B_2\dot{q}_2(t) - k_2\big(e_2(t) - (1-\alpha)e_1(t)\big) \tag{7}$$

$$\tau_s(t) = G_s\big(q_s(t)\big) - B_s\dot{q}_s(t) - k_s\big(-\alpha e_1(t) - (1-\alpha)e_2(t)\big) \tag{8}$$

In the above, B_1, B_2 and B_s are positive-definite matrices and k_1, k_2 and k_s are positive scalars that act as control gains. Note that the tracking error in the first robot, $e_1(t)$, affects through the gain $(1-\alpha)$ the second robots control signal while the tracking error in the second robot, $e_2(t)$, affects through the gain α the first robots control signal. Both $e_1(t)$ and $e_2(t)$ contribute to the slave robots control signals via gains α and $(1-\alpha)$, respectively.

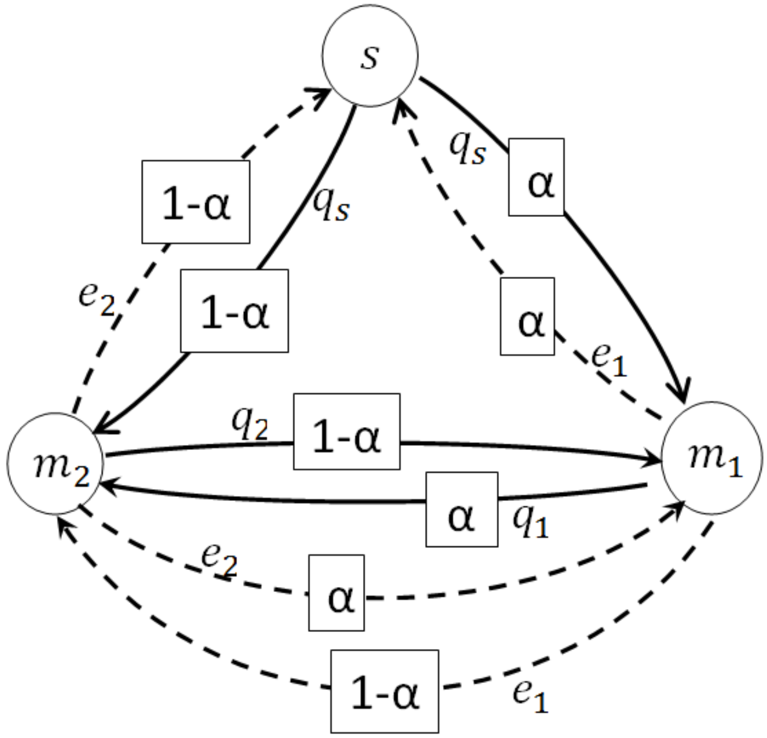

Figure 1: The trilateral teleoperation systems authority sharing connections.

In contact-motion teleoperation, where position tracking errors will inevitably exist, at the steady state the torques exerted by the human operators and the environment can be found as

$$
\begin{aligned}
\tau_{h_1} &= -k_1(e_1 - \alpha e_2) \\
\tau_{h_2} &= -k_2(e_2 - (1-\alpha)e_1) \\
\tau_e &= -k_s(-\alpha e_1 - (1-\alpha)e_2)
\end{aligned}
$$

It is easy to show that, for any $\alpha \in (0,1)$, we have

$$\frac{\tau_{h_1}}{k_1} + \frac{\tau_{h_2}}{k_2} + \frac{\tau_e}{k_s} = 0$$

The above relationship shows that in contact-motion teleoperation, both human operators can sense each others' scaled torque combined with a scaled environments torque. To have force tracking during contact motion, it is possible to find appropriate coefficients k_1, k_2 and k_s. For example, if operator 1 wants to sense the environment torque τ_e but not the operator 2 torque τ_{h_2}, then take $k_1 = k_s$ and a large value for k_2. In such a case, operator 1 will sense the environment torque (i.e., $\tau_{h_1} = \tau_e$) and master 2 robot will be very stiff so that the exerted torque τ_{h_2} from operator 2 cannot have any effect on the e_i and so on the torque that operator 1 senses. Another example is when operators 1 and 2 want to sense each others' torques but not the environment torque, in which case $k_1 = k_2$ and a large value for k_s would be appropriate.

To study the stability and asymptotic free-motion tracking performance of the trilateral system, four theorems are provided below. In Theorem I, stability of the delay-free system is analyzed. In Theorem II, the asymptotic free-motion tracking performance of the delay-free system is considered. In sections III and IV, considering asymmetric time-varying delays in the communication channels, the stability and asymptotic tracking of the trilateral system are studied, respectively.

Theorem I. Assuming that the first and second human operators and the environment are passive, in the trilateral teleoperation system (1)-(3) with controllers (6)-(8) where damping coefficients B_1, B_2 and B_s are positive-definite matrices and k_1, k_2 and k_s are positive scalars, the joint velocities $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_s$ and the joint position errors $(q_1(t) - (\alpha q_s(t) + (1-\alpha)q_2(t)))$, $(q_2(t) - (\alpha q_1(t) + (1-\alpha)q_s(t)))$ and $q_s(t) - (\alpha q_1(t) + (1-\alpha)q_2(t))$ are bounded for any $\alpha \in (0, 1)$.

Proof of Theorem I:

Applying controllers (6)-(8) to the system (1)-(3), we have the following closed-loop dynamics for the three robots:

$$M_1\big(q_1(t)\big)\ddot{q}_1(t) + C_1\big(q_1(t), \dot{q}_1(t)\big)\dot{q}_1 = -\tau_{h_1}(t) - B_1\dot{q}_1(t) - k_1\big(e_1(t) - \alpha e_2(t)\big) \tag{9}$$

$$\begin{aligned} M_2\big(q_2(t)\big)\ddot{q}_2(t) + C_2\big(q_2(t), q_2(t)\big)\dot{q}_2 &= -\tau_{h_2}(t) - B_2\dot{q}_2(t) - k_2\big(e_2(t) \\ &\quad - (1-\alpha)e_1(t)\big) \end{aligned} \tag{10}$$

$$\begin{aligned} M_s\big(q_s(t)\big)\ddot{q}_s(t) + C_s\big(q_s(t), q_s(t)\big)\dot{q}_s &= -\tau_e(t) - B_s\dot{q}_s(t) - k_s\big(-\alpha e_1(t) \\ &\quad - (1-\alpha)e_2(t)\big) \end{aligned} \tag{11}$$

To show the stability of the system (9)-(11), let us define the Lyapunov function candidate

$$V(x(t)) = V_1\big(x(t)\big) + V_2\big(x(t)\big) + V_3\big(x(t)\big) \tag{12}$$

where $x(t) \triangleq [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t), q_s(t), \dot{q}_s(t)]$

$$\begin{aligned}
V_1(x(t)) &= V_{11}\big(x(t)\big) + V_{12}\big(x(t)\big) + V_{1s}\big(x(t)\big) \\
V_{11}(x(t)) &= \frac{1}{2}\big(\frac{k_s}{k_1}\big)\dot{q}_1^T(t) M_1\big(q_1(t)\big)\dot{q}_1(t) \\
V_{12}(x(t)) &= \frac{1}{2}\big(\frac{k_s}{k_2}\big)\dot{q}_2^T(t) M_2\big(q_2(t)\big)\dot{q}_2(t) \\
V_{1s}(x(t)) &= \frac{1}{2}\dot{q}_s^T(t) M_s\big(q_s(t)\big)\dot{q}_s(t)
\end{aligned} \tag{13}$$

$$\begin{aligned}
V_2(x(t)) &= V_{21}\big(x(t)\big) + V_{22}\big(x(t)\big) \\
V_{21}(x(t)) &= \frac{1}{2}k_s\big(e_1(t)\big)^T\big(e_1(t)\big) \\
V_{22}(x(t)) &= \frac{1}{2}k_s\big(e_2(t)\big)^T\big(e_2(t)\big) \\
V_3(x(t)) &= \int_0^t \left(\frac{k_s}{k_1}\dot{q}_1^T(\zeta)\tau_{h_1}(\zeta) + \frac{k_s}{k_2}\dot{q}_2^T(\zeta)\tau_{h_2}(\zeta) + \dot{q}_s^T(\zeta)\tau_e(\zeta)\right) d\zeta \\
&\quad + \kappa_1 + \kappa_2 + \kappa_s
\end{aligned} \tag{14}$$

Note that, based on the assumption of passivity of the two human operators and the environment, there exist positive constants κ_1, κ_2 and κ_s such that $V_3(x(t)) \geq 0$.

Considering (12)-(14) and property P-2 of the robots dynamics, the time derivative of V_{11}, V_{12}, and V_{1s} can be found as

$$\begin{aligned}
\dot{V}_{11} &= \big(\frac{k_s}{k_1}\big)\dot{q}_1^T(t)\Big(- \tau_{h_1}(t) - B_1\dot{q}_1(t) - k_1\big(e_1(t) - \alpha e_2(t)\big)\Big) \\
\dot{V}_{12} &= \big(\frac{k_s}{k_1}\big)\dot{q}_2^T(t)\Big(- \tau_{h_2}(t) - B_2\dot{q}_2(t) - k_2\big(e_2(t) - (1-\alpha)e_1(t)\big)\Big) \\
\dot{V}_{1s} &= \dot{q}_s^T(t)\Big(- \tau_e(t) - B_s\dot{q}_s(t) - k_s\big(- \alpha e_1(t) - (1-\alpha)e_2(t)\big)\Big)
\end{aligned} \tag{15}$$

Also, the time derivatives of V_{21} and V_{22} are

$$\dot{V}_{21} = k_s\dot{q}_1^T(t)\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big) + k_s\dot{q}_2^T(t)(-(1-\alpha))\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big) + k_s\dot{q}_s^T(t)(-\alpha)\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big)$$

$$\dot{V}_{22} = k_s\dot{q}_2^T(t)\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big) + k_s\dot{q}_1^T(t)(-\alpha)\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big) + k_s\dot{q}_s^T(t)(-(1-\alpha))\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big) \tag{16}$$

Using the definitions of $e_1(t)$, $e_2(t)$ and $e_3(t)$ in (5) and considering (15) and (16),$\dot{V}_1 + \dot{V}_2$ can be simplified as

$$\dot{V}_1 + \dot{V}_2 = -(\frac{k_s}{k_1})\dot{q}_1^T(t)\tau_{h_1}(t) - (\frac{k_s}{k_2})\dot{q}_2^T(t)\tau_{h_2}(t) - \dot{q}_s^T(t)\tau_e(t) - (\frac{k_s}{k_1})\dot{q}_1^T(t)B_1\dot{q}_1(t) - (\frac{k_s}{k_2})\dot{q}_2^T(t)B_2\dot{q}_2(t) - \dot{q}_s^T(t)B_s\dot{q}_s(t) \tag{17}$$

The time derivative of $V_3(x(t))$ is

$$\dot{V}_3 = (\frac{k_s}{k_1})\dot{q}_1(t)\tau_{h_1}(t) + (\frac{k_s}{k_2})\dot{q}_2(t)\tau_{h_2}(t) + \dot{q}_s(t)\tau_e(t) \tag{18}$$

Using (17) and (18), it can be seen that

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 = -(\frac{k_s}{k_1})\dot{q}_1^T(t)B_1\dot{q}_1(t) - (\frac{k_s}{k_2})\dot{q}_2^T(t)B_2\dot{q}_2(t) - \dot{q}_s^T(t)B_s\dot{q}_s(t) \tag{19}$$

Then,$\dot{V} \leq 0$. Since $V(x(t)) \geq 0$ and $\dot{V}(x(t)) \leq 0$, all variables in $V(x(t))$ are bounded. Therefore, the velocities $\dot{q}_1$,$\dot{q}_2$,$\dot{q}_s$ and the position errors $e_1(t)$ and $e_2(t)$ are bounded. Note that $e_s(t)$ has a linear dependence on $e_1(t)$ and $e_2(t)$, as $e_s(t) = \gamma_1 e_1(t) + \gamma_2 e_2(t)$ with $\gamma_1 = \frac{\alpha(1-\alpha)+\alpha}{\alpha(1-\alpha)-1}$ and $\gamma_2 = \frac{1-\alpha^2}{\alpha(1-\alpha)-1}$

Since for any $\alpha \in [0,1]$, γ_1 and γ_2 have nonzero bounded values, the boundedness of $e_1(t)$ and $e_2(t)$ will result in the boundedness of $e_s(t)$ and the proof is completed.

■

Theorem II. In the trilateral teleoperation system (1)-(3) with controllers (6)-(8), in free motion tele-manipulation ($\tau_{h_1}(t) = \tau_{h_2}(t) = \tau_e(t) = 0$), the absolute values of the velocities $|\dot{q}_1(t)|$, $|\dot{q}_2(t)|$ and $|\dot{q}_s(t)|$ and the position errors $q_1(t) - (\alpha q_s(t) + (1-\alpha)q_2(t))$, $q_2(t) - (\alpha q_1(t) + (1-\alpha)q_s(t))$ and $q_s(t) - (\alpha q_1(t) + (1-\alpha)q_2(t))$ converge to zero asymptotically.

Proof of Theorem II: Given that $V(x(t)) \geq 0$ and integrating $\dot{V}(x(t))$ in (26), we have:

$$
\begin{aligned}
&-\big(\frac{k_s}{k_1}\big)\int_0^t \dot{q}_1^T(\zeta)B_1\dot{q}_1(t)\ d\zeta - \big(\frac{k_s}{k_2}\big)\int_0^t \dot{q}_2^T(\zeta)B_2\dot{q}_2(t)\ d\zeta - \int_0^t \dot{q}_s^T(\zeta)B_s\dot{q}_s(t)\ d\zeta \\
&= V(t) - V(0) \geq -V(0) \qquad (20)
\end{aligned}
$$

Therefore, $\dot{q}_1(t)$, $\dot{q}_2(t)$, $\dot{q}_s(t) \in L_2$. Given $V(x(t))$ is a lower-bounded decreasing function. Therefore, $\dot{q}_1(t)$, $\dot{q}_2(t)$, $\dot{q}_s(t)$, $\dot{q}_1(t)-(\alpha q_s(t)+(1-\alpha)q_2(t))$, $q_2(t)-(\alpha q_1(t)+(1-\alpha)q_s(t)) \in L_\infty$. Based on Property P-1 and boundedness of the gravity terms $G_1(q_1(t))$, $G_2(q_2(t))$ and $G_s(q_s(t))$, and control laws $\tau_1(t)$, $\tau_2(t)$ and $\tau_s(t)$, it can be seen that $\ddot{q}_1(t)$, $\ddot{q}_2(t)$, $\ddot{q}_s(t) \in L_\infty$. Given that $\dot{q}_1(t) \in L_2$ and $\ddot{q}_1(t) \in L_\infty$, using Barbalats lemma we have that $\dot{q}_1(t) \to 0$. Similarly, it can be reasoned that $\dot{q}_2(t)$ and $\dot{q}_s(t) \to 0$. Now, if $\ddot{q}_1$, $\ddot{q}_2$ and $\ddot{q}_s$ are continuous in time, or equivalently $\dddot{q}_1(t)$, $\dddot{q}_2(t)$,$\dddot{q}_s(t) \in L_\infty$, then $\dot{q}_1(t)$, $\dot{q}_2(t)$,$\dot{q}_s(t) \to 0$ ensuring that $\ddot{q}_1(t)$, $\ddot{q}_2(t)$,$\ddot{q}_s(t) \to 0$ (Barbalats lemma). Let us investigate the boundedness of $\dddot{q}_1(t)$. Using (9),

$$
\begin{aligned}
\ddot{q}_1(t) = \Big(M_1(q_1(t))\Big)^{-1}\Big\{&C_1\big(q_1(t),\dot{q}_1(t)\big)\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Big(\Big(q_1(t) - \big(\alpha q_s(t) \\
&+ (1-\alpha)q_2(t)\big)\Big) - \alpha\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big)\Big)\Big\} \qquad (21)
\end{aligned}
$$

Differentiating both sides with respect to time produces $\dddot{q}_1(t)$ as shown below:

$$\dddot{q}_1(t) = \frac{d}{dt}\Big(M_1(q_1(t))\Big)^{-1}\Big\{C_1\big(q_1(t), \dot{q}_1(t)\big)\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Big(\Big(q_1(t) - \big(\alpha q_s(t)$$
$$+ (1-\alpha)q_2(t)\big)\Big) - \alpha\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big)\Big)\Big\} + \Big(M_1(q_1(t))\Big)^{-1}$$
$$\frac{d}{dt}\Big\{C_1\big(q_1(t), \dot{q}_1(t)\big)\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Big(\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big) -$$
$$\alpha\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big)\Big)\Big\} \tag{22}$$

Note that,

$$\frac{d}{dt}\Big(M_1(q_1(t))\Big)^{-1} = \\ -\Big(M_1(q_1(t))\Big)^{-1}\Big(C_1(q_1(t), \dot{q}_1(t)) + C_1^T(q_1(t), \dot{q}_1(t))\Big)M_1(q_1(t))$$

and based on Properties P-I and P-III and given the boundedness of $\dot{q}_1$, it is easy to see that $d(M_1(q_1(t)))^{-1}/dt$ is bounded. Using Properties P-I, P-III and P-IV and the boundedness of $(q_1(t) - (\alpha q_s(t) + (1-\alpha)q_2(t)))$, $(q_2(t) - (\alpha q_1(t) + (1-\alpha)q_s(t)))$, $\dot{q}_1$, $\ddot{q}_1$, $\dot{q}_2$ and $\dot{q}_s$, it can be seen from (22) that $\dddot{q}_1$ is bounded. Given that $\dot{q}_1(t) \to 0$ and $\dddot{q}_1(t) \in L_\infty$, using Barbalats lemma we have that $\ddot{q}_1(t) \to 0$. Based on the above results and considering (9), (10) and (11), when $t \to \infty$ we have that,

$$\begin{aligned} e_1(t) - \alpha e_2(t) &= 0 \\ e_2(t) - (1-\alpha)e_1(t) &= 0 \\ -\alpha e_1(t) - (1-\alpha)e_2(t) &= 0 \end{aligned} \tag{23}$$

which can be solved to $e_1(t) = e_2(t) = 0$, for every $\alpha \in (0, 1)$. Given that $e_s(t) = \gamma_1 e_1(t) + \gamma_2 e_2(t)$, we can see that $e_s(t) = 0$ when $t \to \infty$ and the proof is completed.

■

Note that the free-motion case considered in Theorem II is in line with other papers in the area of nonlinear teleoperation control when it comes to analyzing position tracking performance (Hashemzadeh et al. (2013)).

4. Trilateral tele-manipulation with time varying delays in the channel

In this section, we study stability and asymptotic zero convergence of velocities and position tracking errors in the trilateral teleoperation system in Figure 2, which is essentially the same as Figure 1 but includes unsymmetrical time-varying delays in all communication channels. In the following, a Proportional plus Damping (P+D) controller that incorporates gravity compensation is proposed for the master 1, master 2 and slave robots:

$$\tau_1(t) = G_1\big(q_1(t)\big) - B_1\dot{q}_1(t) - k_1\Bigg(\Big(q_1(t) - \big(\alpha q_s(t - T_{s1}(t)) + (1-\alpha) \\ q_2(t - T_{21}(t))\big)\Big) - \alpha\Big(q_2(t - T_{21}(t)) - \big(\alpha q_1(t) + (1-\alpha) q_s(t - T_{s1}(t))\big)\Big)\Bigg) \tag{24}$$

$$\tau_2(t) = G_2\big(q_2(t)\big) - B_2\dot{q}_2(t) - k_2\Bigg(\Big(q_2(t) - \big(\alpha q_1(t - T_{12}(t)) + (1-\alpha) \\ q_s(t - T_{s2}(t))\big)\Big) - (1-\alpha)\Big(q_1(t - T_{12}(t)) - \big(\alpha q_s(t - T_{s2}(t)) + (1-\alpha) q_2(t)\big)\Big)\Bigg) \tag{25}$$

$$\tau_s(t) = G_s\big(q_s(t)\big) - B_s\dot{q}_s(t) - k_s\Bigg(- \alpha\Big(q_1(t - T_{1s}(t)) - \big(\alpha q_s(t) + (1-\alpha) \\ (q_2(t - T_{2s}(t))\big)\big)\Big) - (1-\alpha)\Big(q_2(t - T_{2s}(t)) - \big(\alpha q_1(t - T_{1s}(t)) + (1-\alpha) q_s(t)\big)\Big)\Bigg) \tag{26}$$

In the above, $T_{s1}(t)$ and $T_{s2}(t)$ are communication delays from the slave to master 1 and master 2, respectively. Also, $T_{1s}(t)$ and $T_{2s}(t)$ are communication delays from master 1 and master 2 to the slave, respectively. Lastly, $T_{21}(t)$ and $T_{12}(t)$ are communication delays from master 2 to master 1 and from master 1 to master 2, respectively.

Theorem III. Assuming the first and second human operators and the environment are passive, in the trilateral teleoperation system (1)-(3) with controllers (24)-(26), if k_1, k_2 and k_s are positive and satisfy

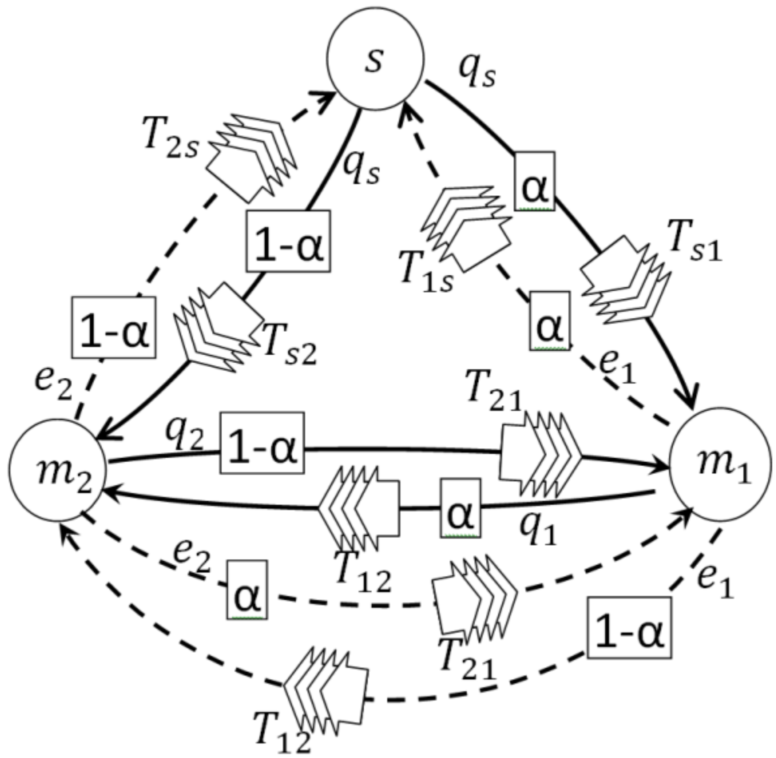

Figure 2: The trilateral teleoperation systems authority sharing connections in with varying delays in network.

$$B_1 - \left(k_1\left(\alpha^2\left(T_{s1_{max}} + T_{1s_{max}}\right) + T_{21_{max}} + T_{12_{max}}\right)\right)I \geq 0$$
$$B_2 - \left(k_2\left((1-\alpha)^2\left(T_{s2_{max}} + T_{2s_{max}}\right) + T_{21_{max}} + T_{12_{max}}\right)\right)I \geq 0$$
$$B_s - \left(k_s\left(\alpha^2\left(T_{1s_{max}} + T_{s1_{max}}\right) + (1-\alpha)^2\left(T_{2s_{max}} + T_{s2_{max}}\right)\right)\right)I \geq 0$$

then the velocities $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_s$ and the position errors $(q_1(t) - (\alpha q_s(t) + (1-\alpha)q_2(t)))$, $(q_2(t)-(\alpha q_1(t)+(1-\alpha)q_s(t)))$ and $(q_s(t)-(\alpha q_1(t)+(1-\alpha)q_2(t)))$ are bounded for any $\alpha \in (0, 1)$ and any bounded time-varying delays $T_{s1}(t)$, $T_{s2}(t)$, $T_{1s}(t)$, $T_{2s}(t)$, $T_{21}(t)$ and $T_{12}(t)$ having upper bounds $T_{s1_{max}}$, $T_{s2_{max}}$, $T_{1s_{max}}$, $T_{2s_{max}}$, $T_{21_{max}}$ and $T_{12_{max}}$, respectively.

Proof of Theorem III:

Applying controller (24)-(26) to the system (1)-(3), we have the following closed-loop dynamics:

$$M_1\big(q_1(t)\big)\ddot{q}_1(t) + C_1\big(q_1(t), \dot{q}_1(t)\big)\dot{q}_1 = -\tau_{h_1}(t) - B_1\dot{q}_1(t) - k_1\Bigg(\bigg(q_1(t) -$$
$$\Big(\alpha q_s(t - T_{s1}(t)) + (1-\alpha)q_2(t - T_{21}(t))\Big)\bigg) - \alpha\bigg(q_2(t - T_{21}(t)) - \Big(\alpha q_1(t)$$
$$+ (1-\alpha)q_s(t - T_{s1}(t))\Big)\bigg)\Bigg) \tag{27}$$

$$M_2\big(q_2(t)\big)\ddot{q}_2(t) + C_2\big(q_2(t), \dot{q}_2(t)\big)\dot{q}_2 = -\tau_{h_2}(t) - B_2\dot{q}_2(t) - k_2\Bigg(\bigg(q_2(t) -$$
$$\Big(\alpha q_1(t - T_{12}(t)) + (1-\alpha)q_s(t - T_{s2}(t))\Big)\bigg) - (1-\alpha)\bigg(q_1(t - T_{12}(t)) -$$
$$\Big(\alpha q_s(t - T_{s2}(t)) + (1-\alpha)q_2(t)\Big)\bigg)\Bigg) \tag{28}$$

$$M_s\big(q_s(t)\big)\ddot{q}_s(t) + C_s\big(q_s(t), \dot{q}_s(t)\big)\dot{q}_s = -\tau_e(t) - B_s\dot{q}_s(t) - k_s\Bigg(- \alpha\bigg(q_1(t - T_{1s}(t))$$
$$- \Big(\alpha q_s(t) + (1-\alpha)(q_2(t - T_{2s}(t)))\Big)\bigg) - (1-\alpha)\bigg(q_2(t - T_{2s}(t)) -$$
$$\Big(\alpha q_1(t - T_{1s}(t)) + (1-\alpha)q_s(t)\Big)\bigg)\Bigg) \tag{29}$$

To show the stability of the system (27)-(29), lets define $x_t = x(t+\psi)$, as the state of the system where $x(t) \triangleq [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t), q_s(t), \dot{q}_s(t)]$, $T_{max} = max(T_{s1_{max}}, T_{s2_{max}}, T_{1s_{max}}, T_{2s_{max}}, T_{12_{max}}, T_{21_{max}})$ and $-T_{max} \leq \psi \leq 0$ (Hale (1993)). The Lyapunov-Krasovskii functional candidate $V(x_t)$ can be defined as

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t) \tag{30}$$

$$
\begin{aligned}
V_1(x_t) &= V_{11}(x_t) + V_{12}(x_t) + V_{1s}(x_t)\\
V_{11}(x_t) &= \frac{1}{2}\Big(\frac{k_s}{k_1}\Big)\dot{q}_1^T(t)M_1\big(q_1(t)\big)\dot{q}_1(t)\\
V_{12}(x_t) &= \frac{1}{2}\Big(\frac{k_s}{k_2}\Big)\dot{q}_2^T(t)M_2\big(q_2(t)\big)\dot{q}_2(t)\\
V_{1s}(x_t) &= \frac{1}{2}\dot{q}_s^T(t)M_s\big(q_s(t)\big)\dot{q}_s(t)
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
V_2(x_t) &= V_{21}(x_t) + V_{22}(x_t)\\
V_{21}(x_t) &= \frac{1}{2}k_s\big(e_1(t)\big)^T\big(e_1(t)\big)\\
V_{22}(x_t) &= \frac{1}{2}k_s\big(e_2(t)\big)^T\big(e_2(t)\big)
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
V_3(x_t) = &\int_0^t \left(\Big(\frac{k_s}{k_1}\Big)\dot{q}_1^T(\zeta)\tau_{h_1}(\zeta) + \Big(\frac{k_s}{k_2}\Big)\dot{q}_2^T(\zeta)\tau_{h_2}(\zeta) + \dot{q}_s^T(\zeta)\tau_e(\zeta)\right)\, d\zeta\\
&+ \kappa_1 + \kappa_2 + \kappa_s
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
V_4(x_t) &= V_{41}(x_t) + V_{42}(x_t) + V_{43}(x_t)\\
V_{41}(x_t) &= (\alpha^2)\Big(\int_{-T_{s1max}}^0\int_{t+\gamma}^t \dot{q}_s^T(\eta)\dot{q}_s(\eta)d\eta d\gamma + \int_{-T_{1smax}}^0\int_{t+\gamma}^t \dot{q}_1^T(\eta)\dot{q}_1(\eta)d\eta d\gamma\Big)\\
V_{42}(x_t) &= ((1-\alpha)^2)\Big(\int_{-T_{s2max}}^0\int_{t+\gamma}^t \dot{q}_s^T(\eta)\dot{q}_s(\eta)d\eta d\gamma + \int_{-T_{2smax}}^0\int_{t+\gamma}^t \dot{q}_2^T(\eta)\dot{q}_2(\eta)d\eta d\gamma\Big)\\
V_{43}(x_t) &= \Big(\int_{-T_{12max}}^0\int_{t+\gamma}^t \dot{q}_1^T(\eta)\dot{q}_1(\eta)d\eta d\gamma + \int_{-T_{21max}}^0\int_{t+\gamma}^t \dot{q}_2^T(\eta)\dot{q}_2(\eta)d\eta d\gamma\Big)
\end{aligned}
\tag{34}
$$

Note that, based on the assumption of passivity of the two human operators and the environment, there exist positive constants κ_1, κ_2 and κ_s such that $V_3(x_t) \geq 0$. Similar to the definition of $e_1(t)$, $e_2(t)$ and $e_s(t)$ in (5), let us define the following errors which can be seen as the measured values of e_1, e_2 and e_s at master 1, master 2 and the slave:

$$\begin{aligned} e_{11} &\triangleq q_1(t) - \big(\alpha q_s(t - T_{s1}(t)) + (1-\alpha) q_2(t - T_{21}(t))\big) \\ e_{12} &\triangleq q_1(t - T_{12}(t)) - \big(\alpha q_s(t - T_{s2}(t)) + (1-\alpha) q_2(t)\big) \\ e_{1s} &\triangleq q_1(t - T_{1s}(t)) - \big(\alpha q_s(t) + (1-\alpha) q_2(t - T_{2s}(t))\big) \end{aligned} \tag{35}$$

also

$$\begin{aligned} e_{21} &\triangleq q_2(t - T_{21}(t)) - \big(\alpha q_1(t) + (1-\alpha) q_s(t - T_{s1}(t))\big) \\ e_{22} &\triangleq q_2(t) - \big(\alpha q_1(t - T_{12}(t)) + (1-\alpha) q_s(t - T_{s2}(t))\big) \\ e_{2s} &\triangleq q_2(t - T_{2s}(t)) - \big(\alpha q_1(t - T_{1s}(t)) + (1-\alpha) q_s(t)\big) \end{aligned} \tag{36}$$

Note that e_{21}, e_{22} and e_{2s} can be seen as e_2 which as measured in master 1, master 2 and the slave, respectively. Lastly,

$$\begin{aligned} e_{s1} &\triangleq q_s(t - T_{s1}(t)) - \big(\alpha q_1(t) + (1-\alpha) q_2(t - T_{21}(t))\big) \\ e_{s2} &\triangleq q_s(t - T_{s2}(t)) - \big(\alpha q_1(t - T_{12}(t)) + (1-\alpha) q_2(t)\big) \\ e_{ss} &\triangleq q_s(t) - \big(\alpha q_1(t - T_{1s}(t)) + (1-\alpha) q_2(t - T_{2s}(t))\big) \end{aligned} \tag{37}$$

where e_{s1}, e_{s2} and e_{ss} can be seen as e_s measured at master 1, master 2 and the slave, respectively. Using the above definitions of e_{ij} $,i,j \in \{1,2,s\}$ in (35)-(37), let us simplify (27)-(29) as

$$M_1\big(q_1(t)\big)\ddot{q}_1(t) + C_1\big(q_1(t),\dot{q}_1(t)\big)\dot{q}_1 = -\tau_{h_1}(t) - B_1\dot{q}_1(t) - k_1\big(e_{11} - \alpha e_{21}\big) \tag{38}$$

$$M_2\big(q_2(t)\big)\ddot{q}_2(t) + C_2\big(q_2(t),\dot{q}_2(t)\big)\dot{q}_2 = -\tau_{h_2}(t) - B_2\dot{q}_2(t) - k_2\big(e_{22} - (1-\alpha) e_{12}\big) \tag{39}$$

$$\begin{aligned} M_s\big(q_s(t)\big)\ddot{q}_s(t) + C_s\big(q_s(t),\dot{q}_s(t)\big)\dot{q}_s = -\tau_e(t) - B_s\dot{q}_s(t) - k_s\big(- \alpha e_{1s} - \\ (1-\alpha) e_{2s}\big) \end{aligned} \tag{40}$$

Considering (38)-(40) and Property P-2, the time derivative of $V_{11}(x_t)$,

$V_{12}(x_t)$, and $V_{1s}(x_t)$ in (31) can be found as

$$\begin{aligned}
\dot{V}_{11}(x_t) &= \Big(\frac{k_s}{k_1}\Big)\dot{q}_1^T(t)\big(-\tau_{h_1}(t) - B_1\dot{q}_1(t) - k_1(e_{11} - \alpha e_{21})\big)\\
\dot{V}_{12}(x_t) &= \Big(\frac{k_s}{k_2}\Big)\dot{q}_2^T(t)\big(-\tau_{h_2}(t) - B_2\dot{q}_2(t) - k_2(e_2 2 - (1-\alpha)e_{12})\big)\\
\dot{V}_{1s}(x_t) &= \dot{q}_s^T(t)\big(-\tau_e(t) - B_s\dot{q}_s(t) - k_s(-\alpha e_{1s} - (1-\alpha)e_{2s})\big)
\end{aligned} \tag{41}$$

Also, the time derivatives of $V_{21}(x_t)$ and $V_{22}(x_t)$ are given by

$$\begin{aligned}
&\dot{V}_{21}(x_t) = k_s\dot{q}_1^T(t)\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big) + k_s\dot{q}_2^T(t)(-(1-\alpha))\Big(q_1(t)\\
&- \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big) + k_s\dot{q}_s^T(t)(-\alpha)\Big(q_1(t) - \big(\alpha q_s(t) + (1-\alpha)q_2(t)\big)\Big)\\
&\dot{V}_{22}(x_t) = k_s\dot{q}_2^T(t)\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big) + k_s\dot{q}_1^T(t)(-\alpha)\Big(q_2(t)-\\
&\big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big) + k_s\dot{q}_s^T(t)(-(1-\alpha))\Big(q_2(t) - \big(\alpha q_1(t) + (1-\alpha)q_s(t)\big)\Big)
\end{aligned} \tag{42}$$

Using definitions of $e_1(t)$ and $e_2(t)$ in (5), it can be easily seen that

$$\begin{aligned}
\dot{V}_{21}(x_t) &= k_s\dot{q}_1^T(t)e_1(t) + k_s\dot{q}_2^T(t)(-(1-\alpha))e_1(t) + k_s\dot{q}_s^T(t)(-\alpha)e_1(t)\\
\dot{V}_{22}(x_t) &= k_s\dot{q}_2^T(t)e_2(t) + k_s\dot{q}_1^T(t)(-\alpha)e_2(t) + k_s\dot{q}_s^T(t)(-(1-\alpha))e_2(t)
\end{aligned} \tag{43}$$

Considering (41) and (43), $\dot{V}_1(x_t) + \dot{V}_2(x_t)$ can be simplified as

$$\begin{aligned}
&\dot{V}_1(x_t) + \dot{V}_2(x_t) = -\Big(\frac{k_s}{k_1}\Big)\dot{q}_1^T(t)\tau_{h_1}(t) - \Big(\frac{k_s}{k_2}\Big)\dot{q}_2^T(t)\tau_{h_2}(t) - \dot{q}_s^T(t)\tau_e(t) -\\
&\Big(\frac{k_s}{k_1}\Big)\dot{q}_1^T(t)B_1\dot{q}_1(t) - \Big(\frac{k_s}{k_2}\Big)\dot{q}_2^T(t)B_2\dot{q}_2(t) - \dot{q}_s^T(t)B_s\dot{q}_s(t) - k_s\dot{q}_1^T(t)(e_{11} - \alpha e_{21})\\
&- k_s\dot{q}_2^T(t)(e_{22} - (1-\alpha)e_{12}) - k_s\dot{q}_s^T(t)(-\alpha e_{1s} - (1-\alpha)e_{2s}) + k_s\dot{q}_1^T(t)(e_1(t)\\
&- \alpha e_2(t)) + k_s\dot{q}_2^T(t)(e_2(t) - (1-\alpha)e_1(t)) + k_s\dot{q}_s^T(t)(-\alpha e_1(t) - (1-\alpha)e_2(t))
\end{aligned} \tag{44}$$

Using (5) and (35)-(37), we can see that

$$
\begin{aligned}
e_1(t) - e_{11} &= -\alpha \int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta) d\zeta - (1-\alpha) \int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta) d\zeta \\
e_2(t) - e_{21} &= \int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta) d\zeta - (1-\alpha) \int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta) d\zeta \\
e_2(t) - e_{22} &= -\alpha \int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta) d\zeta - (1-\alpha) \int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta) d\zeta \\
e_1(t) - e_{12} &= \int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta) d\zeta - \alpha \int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta) d\zeta \\
e_1(t) - e_{1s} &= \int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta) d\zeta - (1-\alpha) \int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta) d\zeta \\
e_2(t) - e_{2s} &= \int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta) d\zeta - \alpha \int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta) d\zeta
\end{aligned} \tag{45}
$$

The time derivative of $V_3(x_t)$ is

$$
\dot{V}_3(x_t) = (\frac{k_s}{k_1})\dot{q}_1(t)\tau_{h_1}(t) + (\frac{k_s}{k_2})\dot{q}_2(t)\tau_{h_2}(t) + \dot{q}_s(t)\tau_e(t) \tag{46}
$$

Applying (45) to the (44) and using (46), it can be seen that

$$
\begin{aligned}
&\dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) = -(\frac{k_s}{k_1})\dot{q}_1^T(t)B_1\dot{q}_1(t) - (\frac{k_s}{k_2})\dot{q}_2^T(t)B_2\dot{q}_2(t) - \\
&\dot{q}_s^T(t)B_s\dot{q}_s(t) + k_s\dot{q}_1^T(t)\Big(\Big(-\alpha \int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta)d\zeta - (1-\alpha)\int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta)d\zeta\Big) - \\
&\alpha\Big(\int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta)d\zeta - (1-\alpha)\int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta)d\zeta\Big)\Big) + k_s\dot{q}_2^T(t)\Big(\Big(-\alpha \\
&\int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta)d\zeta - (1-\alpha)\int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta)d\zeta\Big) - (1-\alpha)\Big(\int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta)d\zeta - \\
&\alpha\int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta)d\zeta\Big)\Big) + k_s\dot{q}_s^T(t)\Big(\Big(-\alpha\int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta)d\zeta - (1-\alpha) \\
&\int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta)d\zeta\Big) - (1-\alpha)\Big(\int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta)d\zeta - \alpha\int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta)d\zeta\Big)\Big)
\end{aligned} \tag{47}
$$

After some simplifications, the above can be simplified to

$$\dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) = -(\frac{k_s}{k_1})\dot{q}_1^T(t)B_1\dot{q}_1(t) - \alpha^2 k_s\dot{q}_1^T(t)\int_{t-T_{s1}(t)}^{t}\dot{q}_s(\zeta)d\zeta -$$
$$k_s\dot{q}_1^T(t)\int_{t-T_{21}(t)}^{t}\dot{q}_2(\zeta)d\zeta - (\frac{k_s}{k_2})\dot{q}_2^T(t)B_2\dot{q}_2(t) - (1-\alpha)^2 k_s\dot{q}_2^T(t)\int_{t-T_{s2}(t)}^{t}\dot{q}_s(\zeta)d\zeta -$$
$$k_s\dot{q}_2^T(t)\int_{t-T_{12}(t)}^{t}\dot{q}_1(\zeta)d\zeta - \dot{q}_s^T(t)B_s\dot{q}_s(t) - \alpha^2 k_s\dot{q}_s^T(t)\int_{t-T_{1s}(t)}^{t}\dot{q}_1(\zeta)d\zeta - (1-\alpha)^2$$
$$k_s\dot{q}_s^T(t)\int_{t-T_{2s}(t)}^{t}\dot{q}_2(\zeta)d\zeta \tag{48}$$

The time derivatives of $V_{41}(x_t)$, $V_{42}(x_t)$ and $V_{43}(x_t)$, after algebraic manipulations, are found to satisfy

$$\dot{V}_{41}(x_t) \le \alpha^2\Big(T_{s1_{max}}\dot{q}_s^T(t)\dot{q}_s(t) - \int_{t-T_{s1}(t)}^{t}\dot{q}_s^T(\zeta)\dot{q}_s(\zeta)d\zeta + T_{1s_{max}}\dot{q}_1^T(t)\dot{q}_1(t)$$
$$-\int_{t-T_{1s}(t)}^{t}\dot{q}_1^T(\zeta)\dot{q}_1(\zeta)d\zeta\Big)$$
$$\dot{V}_{42}(x_t) \le (1-\alpha)^2\Big(T_{s2_{max}}\dot{q}_s^T(t)\dot{q}_s(t) - \int_{t-T_{s2}(t)}^{t}\dot{q}_s^T(\zeta)\dot{q}_s(\zeta)d\zeta + T_{2s_{max}}\dot{q}_2^T(t)\dot{q}_2(t)$$
$$-\int_{t-T_{2s}(t)}^{t}\dot{q}_2^T(\zeta)\dot{q}_2(\zeta)d\zeta\Big)$$
$$\dot{V}_{43}(x_t) \le T_{12_{max}}\dot{q}_1^T(t)\dot{q}_1(t) - \int_{t-T_{12}(t)}^{t}\dot{q}_1^T(\zeta)\dot{q}_1(\zeta)d\zeta + T_{21_{max}}\dot{q}_2^T(t)\dot{q}_2(t)$$
$$-\int_{t-T_{21}(t)}^{t}\dot{q}_2^T(\zeta)\dot{q}_2(\zeta)d\zeta \tag{49}$$

Using the following inequalities which resulted from Lemma 1 in Hua & Liu (2010), we have

$$
\begin{aligned}
&-\dot{q}_1^T(t)\int_{t-T_{s1}(t)}^{t}\dot{q}_s(\zeta)d\zeta-\int_{t-T_{s1}(t)}^{t}\dot{q}_s^T(\zeta)\dot{q}_s(\zeta)d\zeta\le T_{s1_{max}}\dot{q}_1^T(t)\dot{q}_1(t)\\
&-\dot{q}_1^T(t)\int_{t-T_{21}(t)}^{t}\dot{q}_2(\zeta)d\zeta-\int_{t-T_{21}(t)}^{t}\dot{q}_2^T(\zeta)\dot{q}_2(\zeta)d\zeta\le T_{21_{max}}\dot{q}_1^T(t)\dot{q}_1(t)\\
&-\dot{q}_2^T(t)\int_{t-T_{s2}(t)}^{t}\dot{q}_s(\zeta)d\zeta-\int_{t-T_{s2}(t)}^{t}\dot{q}_s^T(\zeta)\dot{q}_s(\zeta)d\zeta\le T_{s2_{max}}\dot{q}_2^T(t)\dot{q}_2(t)\\
&-\dot{q}_2^T(t)\int_{t-T_{12}(t)}^{t}\dot{q}_1(\zeta)d\zeta-\int_{t-T_{12}(t)}^{t}\dot{q}_1^T(\zeta)\dot{q}_1(\zeta)d\zeta\le T_{12_{max}}\dot{q}_2^T(t)\dot{q}_2(t)\\
&-\dot{q}_s^T(t)\int_{t-T_{1s}(t)}^{t}\dot{q}_1(\zeta)d\zeta-\int_{t-T_{1s}(t)}^{t}\dot{q}_1^T(\zeta)\dot{q}_1(\zeta)d\zeta\le T_{1s_{max}}\dot{q}_s^T(t)\dot{q}_s(t)\\
&-\dot{q}_s^T(t)\int_{t-T_{2s}(t)}^{t}\dot{q}_2(\zeta)d\zeta-\int_{t-T_{2s}(t)}^{t}\dot{q}_2^T(\zeta)\dot{q}_2(\zeta)d\zeta\le T_{2s_{max}}\dot{q}_s^T(t)\dot{q}_s(t) \qquad (50)
\end{aligned}
$$

Applying (50) to the sum of (48) and (49) and after some algebraic simplifications, we have

$$
\begin{aligned}
&\dot{V}_1(x_t)+\dot{V}_2(x_t)+\dot{V}_3(x_t)+\dot{V}_4(x_t)\le-(\frac{k_s}{k_1})\dot{q}_1^T(t)B_1\dot{q}_1(t)-(\frac{k_s}{k_2})\dot{q}_2^T(t)B_2\dot{q}_2(t)-\\
&\dot{q}_s^T(t)B_s\dot{q}_s(t)+k_s\Big(\alpha^2(T_{s1_{max}}+T_{1s_{max}})+T_{21_{max}}+T_{12_{max}}\Big)\dot{q}_1^T(t)\dot{q}_1(t)\\
&+k_s\Big((1-\alpha)^2(T_{s2_{max}}+T_{2s_{max}})+T_{21_{max}}+T_{12_{max}}\Big)\dot{q}_2^T(t)\dot{q}_2(t)+k_s\Big(\alpha^2(T_{1s_{max}}\\
&+T_{s1_{max}})+(1-\alpha)^2(T_{2s_{max}}+T_{s2_{max}})\Big)\dot{q}_s^T(t)\dot{q}_s(t) \qquad (51)
\end{aligned}
$$

Therefore, if

$$
\begin{aligned}
&B_1-\Big(k_1\big(\alpha^2(T_{s1_{max}}+T_{1s_{max}})+T_{21_{max}}+T_{12_{max}}\big)\Big)I>0\\
&B_2-\Big(k_2\big((1-\alpha)^2(T_{s2_{max}}+T_{2s_{max}})+T_{21_{max}}+T_{12_{max}}\big)\Big)I>0\\
&B_s-\Big(k_s\big(\alpha^2(T_{1s_{max}}+T_{s1_{max}})+(1-\alpha)^2(T_{2s_{max}}+T_{s2_{max}})\big)\Big)I>0
\end{aligned}
$$

then $\dot{V}(x_t)\le 0$. Sine $V(x_t)\ge 0$ and $\dot{V}(x_t)\le 0$, all elements in $V(t)$ are bounded. Therefore, the velocities $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_s$ and the position errors

$(q_1(t)-(\alpha q_s(t)+(1-\alpha)q_2(t)))$ and $(q_2(t)-(\alpha q_1(t)+(1-\alpha)q_s(t)))$ are bounded. Similar to the analysis at the end of the Theorem I, boundedness of e_1 and e_2 will ensure the boundedness of e_s and proof completed.

■

Theorem IV. In the trilateral system (1)-(3) with controllers (24)-(26), the absolute values of the velocities $|\dot{q}_1(t)|$, $|\dot{q}_2(t)|$ and $|\dot{q}_s(t)|$ and the position errors $q_1(t)-(\alpha q_s(t-T_{s1}(t))+(1-\alpha)q_2(t-T_{21}(t)))$, $q_2(t)-(\alpha q_1(t-T_{12}(t))+(1-\alpha)q_s(t-T_{s2}(t)))$, $q_s(t)-(\alpha q_1(t-T_{1s}(t))+(1-\alpha)q_2(t-T_{2s}(t)))$ tend to zero asymptotically in free motion (i.e., $\tau_{h_1}(t)$,$\tau_{h_2}(t)$,$\tau_e(t) \to 0$) if all conditions in Theorem III are satisfied and $\dot{T}_{12}(t)$, $\dot{T}_{21}(t)$, $\dot{T}_{1s}(t)$, $\dot{T}_{s1}(t)$, $\dot{T}_{2s}(t)$ and $\dot{T}_{s2}(t)$ are also bounded.

Proof of Theorem IV: Integrating both sides of (51), it is possible to see that $\dot{q}_1(t)$, $\dot{q}_2(t)$ and $\dot{q}_s(t) \in L_2$. Based on the result of Theorem III, $V(x_t)$ is a lower-bounded decreasing function. Therefore, $\dot{q}_1(t)$, $\dot{q}_2(t)$ and $\dot{q}_s(t)$ and $q_1(t)-(\alpha q_s(t)+(1-\alpha)q_2(t))$ and $q_2(t)-(\alpha q_1(t)+(1-\alpha)q_s(t)) \in L_\infty$. Given

$$
\begin{aligned}
e_{11} &= \Big(q_1(t) - \big(\alpha q_s(t - T_{s1}(t)) + (1-\alpha)q_2(t - T_{21}(t))\big)\Big) = \Big(q_1(t) - \big(\alpha q_s(t) \\
&\quad + (1-\alpha)q_2(t)\big)\Big) + \alpha \int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta)d\zeta + (1-\alpha)\int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta)d\zeta \\
e_{21} &= \Big(q_2(t - T_{21}(t)) - \big(\alpha q_1(t) + (1-\alpha)q_s(t - T_{s1}(t))\big)\Big) = \Big(q_2(t) - \big(\alpha q_1(t) \\
&\quad + (1-\alpha)q_s(t)\big)\Big) - \int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta)d\zeta + (1-\alpha)\int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta)d\zeta \\
e_{22} &= \Big(q_2(t) - \big(\alpha q_1(t - T_{12}(t)) + (1-\alpha)q_s(t - T_{s2}(t))\big)\Big) = \Big(q_2(t) - \big(\alpha q_1(t) \\
&\quad + (1-\alpha)q_s(t)\big)\Big) + \alpha \int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta)d\zeta + (1-\alpha)\int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta)d\zeta \\
e_{12} &= \Big(q_1(t - T_{12}(t)) - \big(\alpha q_s(t - T_{s2}(t)) + (1-\alpha)q_2(t)\big)\Big) = \Big(q_1(t) - \big(\alpha q_s(t) \\
&\quad + (1-\alpha)q_2(t)\big)\Big) - \int_{t-T_{21}(t)}^{t} \dot{q}_1(\zeta)d\zeta + \alpha\int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta)d\zeta \\
e_{1s} &= \Big(q_1(t - T_{1s}(t)) - \big(\alpha q_s(t) + (1-\alpha)q_2(t - T_{2s}(t))\big)\Big) = \Big(q_1(t) - \big(\alpha q_s(t) \\
&\quad + (1-\alpha)q_2(t)\big)\Big) - \int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta)d\zeta + (1-\alpha)\int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta)d\zeta \\
e_{2s} &= \Big(q_2(t - T_{2s}(t)) - \big(\alpha q_1(t - T_{1s}(t)) + (1-\alpha)q_s(t)\big)\Big) = \Big(q_2(t) - \big(\alpha q_1(t) \\
&\quad + (1-\alpha)q_s(t)\big)\Big) - \int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta)d\zeta + \alpha\int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta)d\zeta
\end{aligned} \tag{52}
$$

and $\int_{t-T_{12}(t)}^{t} \dot{q}_1(\zeta)d\zeta$, $\int_{t-T_{1s}(t)}^{t} \dot{q}_1(\zeta)d\zeta$, $\int_{t-T_{21}(t)}^{t} \dot{q}_2(\zeta)d\zeta$, $\int_{t-T_{2s}(t)}^{t} \dot{q}_2(\zeta)d\zeta$, $\int_{t-T_{s1}(t)}^{t} \dot{q}_s(\zeta)d\zeta$ and $\int_{t-T_{s2}(t)}^{t} \dot{q}_s(\zeta)d\zeta \in L_\infty$, we have e_{11}, e_{12}, e_{1s}, e_{21}, e_{22} and $e_{2s} \in L_\infty$. Given Property P-1 and the boundedness of the gravity terms $G_1(q_1(t))$, $G_2(q_2(t))$ and $G_s(q_s(t))$, it can be seen that $\ddot{q}_1(t)$, $\ddot{q}_2(t)$ and $\ddot{q}_s(t) \in L_\infty$. Because $\dot{q}_1(t) \in L_2$ and $\ddot{q}_1(t) \in L_\infty$, using Barbalats lemma we have that $\dot{q}_1(t) \to 0$. Similarly, via Barbalats lemma, it can be reasoned that $\dot{q}_2(t)$ and $\dot{q}_s(t) \to 0$. Now, if $\ddot{q}_1$, $\ddot{q}_2$ and $\ddot{q}_s$ are continuous in time, or equivalently $\dddot{q}_1(t)$, $\ddot{q}_2(t)$ and $\ddot{q}_s(t) \in L_\infty$, then $\dot{q}_1(t)$, $\dot{q}_2(t)$ and $\dot{q}_s(t) \to 0$ ensures that $\ddot{q}_1(t)$, $\ddot{q}_2(t)$ and $\ddot{q}_s(t) \to 0$. Let us investigate the boundedness of $\dddot{q}_1(t)$ in the following. Using (27),

$$\ddot{q}_1(t) = \Big(M_1(q_1(t))\Big)^{-1}\Big\{C_1(q_1(t),\dot{q}_1(t))\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Bigg(\Big(q_1(t) - \big(\alpha q_s(t-T_{s1}(t)) + (1-\alpha)q_2(t-T_{21}(t))\big)\Big) - \alpha\Big(q_2(t-T_{21}(t)) - (\alpha q_1(t) + (1-\alpha)q_s(t-T_{s1}(t)))\Big)\Bigg)\Big\}$$

Differentiating both sides with respect to time produces $\dddot{q}_1(t)$:

$$\dddot{q}_1(t) = \frac{d}{dt}\Big(M_1(q_1(t))\Big)^{-1}\Big\{C_1(q_1(t),\dot{q}_1(t))\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Bigg(\Big(q_1(t) - \big(\alpha q_s(t-T_{s1}(t)) + (1-\alpha)q_2(t-T_{21}(t))\big)\Big) - \alpha\Big(q_2(t-T_{21}(t)) - \big(\alpha q_1(t) + (1-\alpha)q_s(t-T_{s1}(t))\big)\Big)\Bigg)\Big\} + \Big(M_1(q_1(t))\Big)^{-1}\frac{d}{dt}\Big\{C_1(q_1(t),\dot{q}_1(t))\dot{q}_1 - B_1\dot{q}_1(t) - k_1\Bigg(\Big(q_1(t) - \big(\alpha q_s(t-T_{s1}(t)) + (1-\alpha)q_2(t-T_{21}(t))\big)\Big) - \alpha\Big(q_2(t-T_{21}(t)) - \big(\alpha q_1(t) + (1-\alpha)q_s(t-T_{s1}(t))\big)\Big)\Bigg)\Big\}$$

Using

$$\frac{d}{dt}\big(M_1(q_1(t))\big)^{-1} = -\big(M_1(q_1(t))\big)^{-1}\Big(C_1(q_1(t),\dot{q}_1(t)) + C_1^T\big(q_1(t),\dot{q}_1(t)\big)\Big)M_1\big(q_1(t)\big)$$

and based on Properties P-1 and P-III and given the boundedness of $\dot{q}_1$, it is easy to see that $d((M_1(q_1(t)))^{-1})/dt$ is bounded.

Using properties P-1, P-III and P-IV and the boundedness of $(q_1(t) - (\alpha q_s(t-T_{s1}(t)) + (1-\alpha)q_2(t-T_{21}(t))))$, $(q_2(t-T_{21}(t)) - (\alpha q_1(t) + (1-\alpha)q_s(t-T_{s1}(t))))$, $\dot{q}_1$, $\ddot{q}_1$, $\dot{q}_2$ and $\dot{q}_s$, it can be seen that $\dddot{q}_1$ is bounded. Given that $\dot{q}_1(t) \to 0$ and $\dddot{q}_1(t) \in L_\infty$, using Barbalats lemma we have that $\ddot{q}_1(t) \to 0$. Based on (45) and given $\dot{q}_1(t)$, $\dot{q}_2(t)$, $\dot{q}_s(t) \to 0$, it can be easily seen that e_{11},e_{12},$e_{1s} \to e_1(t)$, and e_{21},e_{22},$e_{2s} \to e_2(t)$. Based on the above results and considering (38), (39) and (40), when $t \to \infty$ it can be easily seen that,

$$
\begin{aligned}
&e_1(t) - \alpha e_2(t) = 0 \\
&e_2(t) - (1-\alpha)e_1(t) = 0 \\
&-\alpha e_1(t) - (1-\alpha)e_2(t) = 0
\end{aligned}
$$

which can be solved to $e_1(t) = e_2(t) = 0$ for any $\alpha \in (0,1)$. Given $e_s(t) = \gamma_1 e_1(t) + \gamma_2 e_2(t)$, we can see that $e_s(t) = 0$ when $t \to \infty$ and the proof is completed.

■

Remark 1: In both Sections III and IV, it was proved that $e_1(t)$, $e_2(t)$ and $e_s(t)$ converge to zero. Therefore, in free motion teleoperation, we have

$$
\begin{pmatrix} I_3 & (\alpha-1)I_3 & -\alpha I_3 \\ -\alpha I_3 & I_3 & (\alpha-1)I_3 \\ -\alpha I_3 & (\alpha-1)I_3 & I_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0
$$

Note that in the above equation, the determinant of the matrix is zero and the rank of the matrix is six. Therefore, there is a nonzero solution $q_1 = q_2 = q_s$ to this equation.

5. Simulation and Experiment Results

In this part, in simulations and experiments, the stability and performance of the proposed controller is studied by studying joint positions of three robots in a trilateral system configuration for different values of the dominance factor α.

In the simulation study, three identical planar, 2-DOF, revolute-joint robots with link length parameters $l_1 = 0.38$ and $l_2 = 0.38$ and link mass parameters $m_1 = 0.3$ and $m_2 = 0.2$ are considered. It is assumed that there are random time delays among the three robots with a uniform distribution between $[0.9, 1]$ seconds. To study the effect of the dominance factor on the robots positions, we fixated master 2 to a fixed position $[q_1, q_2] = [\pi/6, \pi/10]$ and simulated the application of external torques to the master 1 robot by operator 1 in the time interval from 5 to 25 seconds. The initial positions of master 1 and slave were $[q_1, q_2] = [\pi/4, \pi/4]$ and $[q_1, q_2] = [\pi/12, \pi/12]$, respectively. The positions of the first and the second joints of master 1 and the slave for $\alpha = 0.02$, $\alpha = 0.50$ and $\alpha = 0.98$ are shown in Figures 3 and

4, respectively. As seen from these figures, for the large $\alpha = 0.98$, master 1 makes more contribution to the slaves position compared to master 2 robot. For the small $\alpha = 0.02$, the slave tends to have a position similar to the fix position of master 2 robot. For the midrange $\alpha = 0.50$, both master 1 and master 2 have the same influence on the slave robots position, meaning that the slave follows the average of the positions of master 1 and master 2.

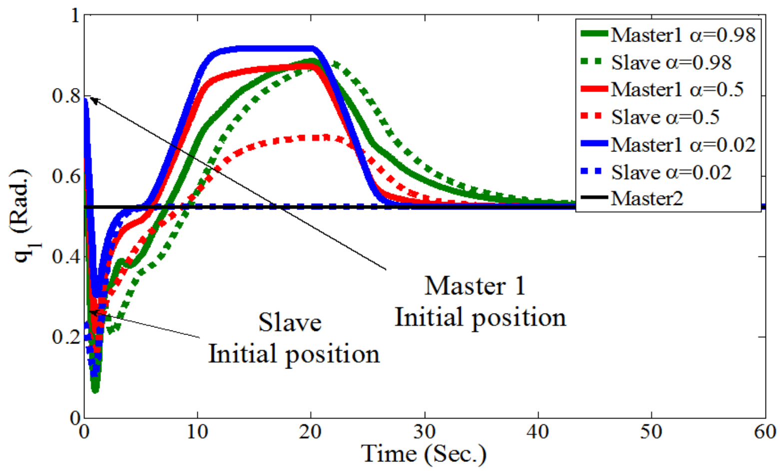

Figure 3: The first joints positions of master 1 and the slave in trilateral teleoperation for different α values. Master 2 has a fixed position.

In Figures 5 and 6, the perceived torques are calculated and shown for the first and second joints of master 1, master 2 and the slave for different values of α. Note that these perceived torques are the result of the torque applied by operator 1 to master 1. Based on these figures, as α decreases, the torque that can be sensed by the environment of the slave robot decreases as operator 1 will have less and less control on the slaves position (which was evident in Figures 3 and 4). The perceived torques at master 1 and master 2 are similar but negative in sign and for smaller α values the similarity between them increases. Note that for small α, the perceived torque at the slave robot (τ_e) are very small and τ_{h_1} will tend to the $-\tau_{h_2}$.

To show the stability of the proposed scheme in a contact scenario, the master 2 robot is left free and operator 1 is grabbing the master 1 robot. It is assumed that there is a wall near the slave robot at $X = 40cm$ which is modeled by a spring with a stiffness coefficient of 100 N/m in X-direction to the slave robot based on the penetration of the slave robot to the environment. All master 1, master 2 and slave robots are 2-DOF rotary-joint planar

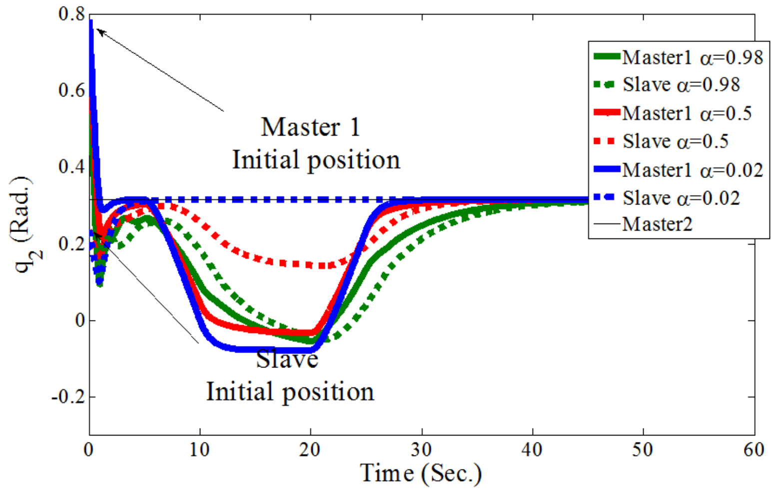

Figure 4: The second joints positions of master 1 and the slave in trilateral teleoperation for different α values. Master 2 has a fixed position.

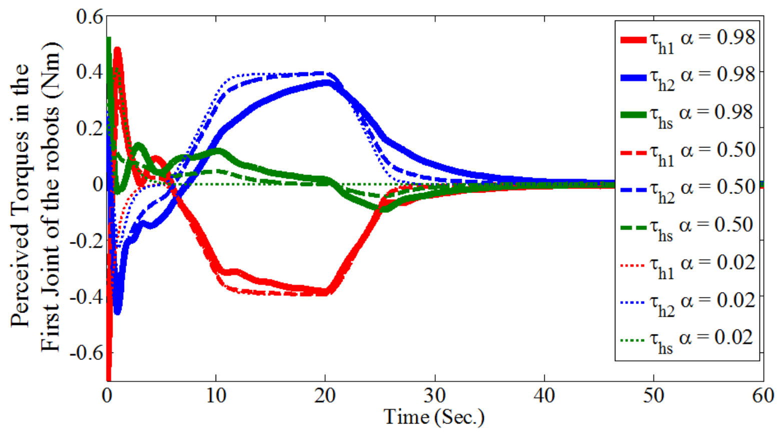

Figure 5: Perceived torques in the first joints of master 1, master 2 and the slave for different values of the dominance factor α.

robots with link lengths $l_1 = l_2 = 38cm$ and with link masses $m_1 = 0.3kg$ and $m_2 = 0.2kg$ and with the same initial joint angles $q_1(0) = 2\pi/4$ and $q_2(0) = -\pi/4$. Also, the three communication channels are modeled by time-varying time delay that is a random variable with a uniform distribution between $[0, 50]ms$. As shown in Figure 7, operator 1 applies constant force $0.5N$ in the time interval from $10s$ to $20s$ in the X direction and reduces a constant force to $0.25N$ in the time interval from $25s$ to $35s$ and to zero in

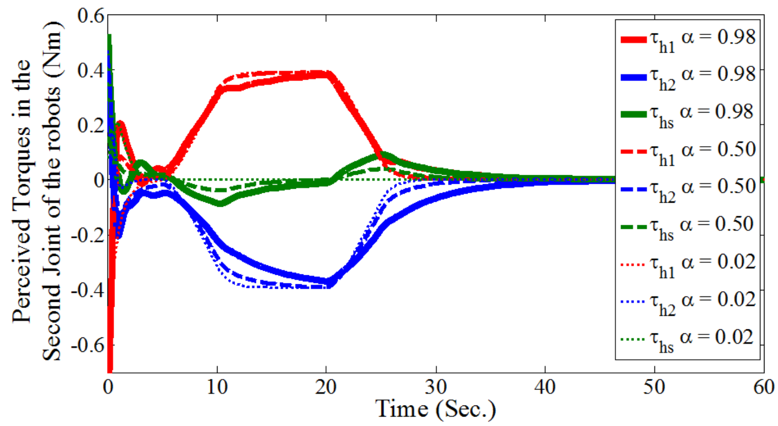

Figure 6: Perceived torques in the second joints of master 1, master 2 and the slave for different values of the dominance factor α

the time interval from $40s$ to $60s$.

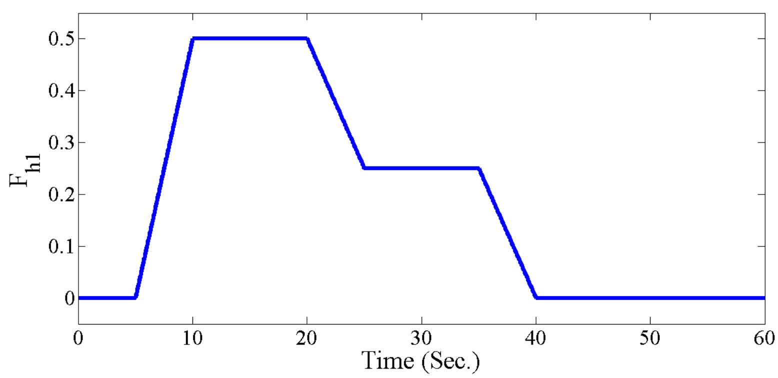

Figure 7: Operator 1 force which is applied to the master 1 robot in a hard contact scenario.

The slave robot moves toward the environment and reaches it at $X = 40cm$. After that due to the continuation of the operator 1 force, the slave robot wants to penetrate the environment. When operator 1 remove the force from the master 1 robot, all robots converge to the same position. In Figures 8-10 the end effector positions of the master 1, master 2 and slave robots are shown for dominance factors $\alpha = 0.1$ and $\alpha = 0.9$.

As seen from Figures 8 and 9, as the slave robot reaches the wall at time $10s$, the master 1 robot moves forward due to continuing applied force from

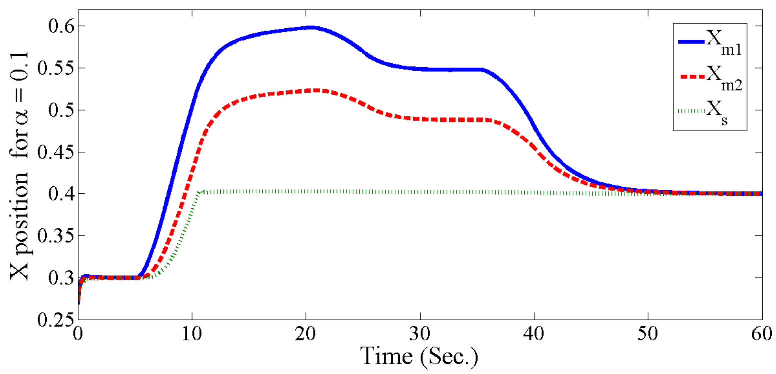

Figure 8: X positions of end-effector of the master 1, master 2 and slave robots for $\alpha = 0.1$.

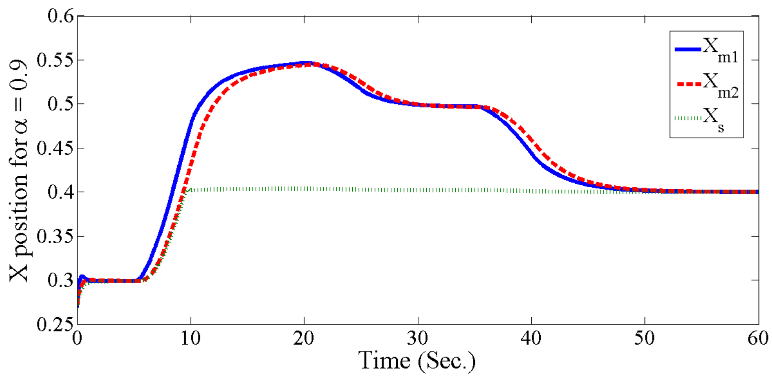

Figure 9: X positions of end-effector of the master 1, master 2 and slave robots for $\alpha = 0.9$.

the operator 1 and makes master 2 robot moves forward. When $\alpha = 0.9$ the effect of master 1 robot to the master 2 robot is much more than when $\alpha = 0.1$. In Figures 11 and 12, the first and second joint positions of the master 1, master 2 and slave robots are shown in contact motion for dominance factors $\alpha = 0.1$ and $\alpha = 0.9$.

As it can be easily seen from Figures 11 and 12, there are joint position errors in contact motion between master 1, master 2 and slave robots and for $\alpha = 0.9$, master 2 robot follow the master 1 robot better than when $\alpha = 0.1$.

In the experiment shown in Figure 13, two 3-DOF PHANToM Premium robots (Master1 and Master2) robots are connected to a planar robot (Slave). The operator is handling Master2 while Master1 is let free and Slave is in contact with the environment involving stiff springs.

Random delays with uniform distributions between $[0, 50]ms$ are consid-

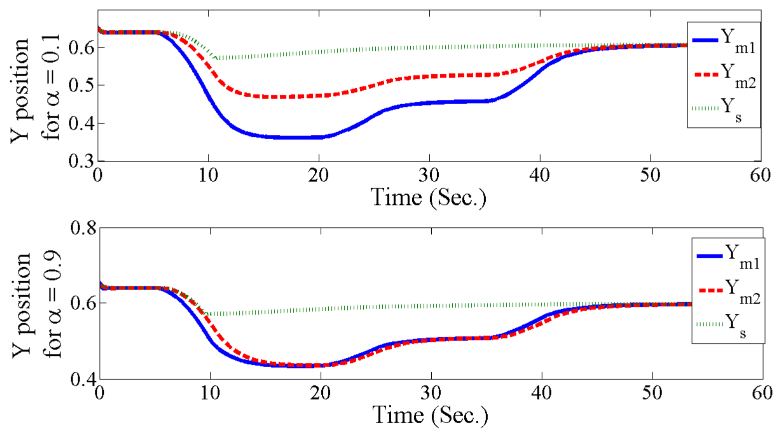

Figure 10: Y positions of end-effector of the master 1, master 2 and slave robots for $\alpha = 0.1$ and $\alpha = 0.9$.

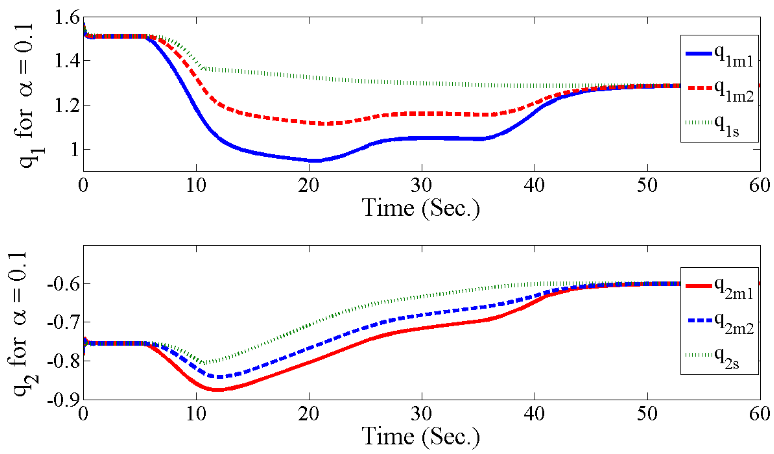

Figure 11: First and second joint positions of the master 1, master 2 and slave robots for $\alpha = 0.1$.

ered in the trilateral channels between Master1, Master2 and Slave. The position tracking of Master1, Master2 and Slave are shown in Figures 14 and 15 for the dominance factor of $\alpha = 0.9$, which means that the operator has low authority. For the dominance factor of $\alpha = 0.1$ where the operator has high authority, the position tracking performances of Master1, Master2 and Slave are shown in Figures 16 and 17.

As it can be easily seen from Figures 14-17, for low and high dominance factors α, the trilateral system is stable. Also, when the operator has high

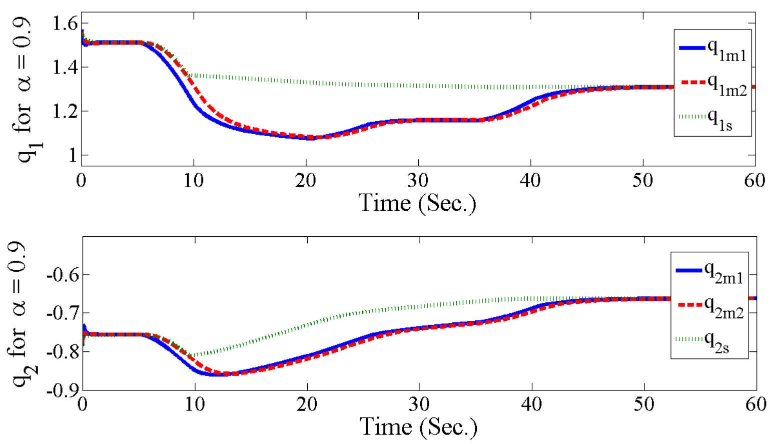

Figure 12: First and second joint positions of the master 1, master 2 and slave robots for $\alpha = 0.9$.

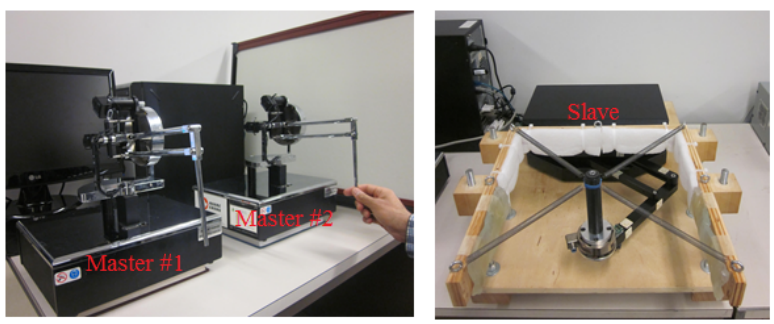

Figure 13: Two PHANToM Premium robots connected to a planar robot in the Trilateral configuration.

authority, Master1 and Slave closely follow Master2 (i.e., the operators position).

In our another experimental study involving two 3-DOF PHANToM Premium 1.5A and one 3-DOF Phantom Omni haptic robots (Geomagic, Morrisville, NC, USA) shown in Figure 18, a human operator is in contact with master 1 while the slave and master 2 are in contact with two springs with stiffness coefficients of 15N/m and 3 N/m, respectively. We will see later why these two springs are chosen to have different stiffness values. Using a JR3 force sensor (JR3, Inc., Woodland, CA, USA) at the end-effector of master 1, the stiffness that the operator senses is measured for different values of α. This introduces a more useful benchmark for the kinesthetic feeling an op-

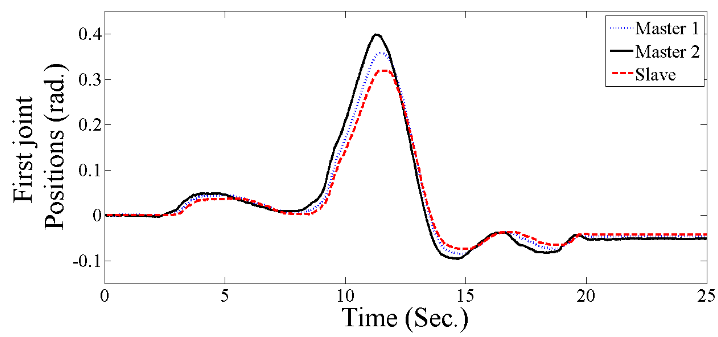

Figure 14: First joint positions of the master1, master2 and slave robots with dominance factor $\alpha = 0.9$.

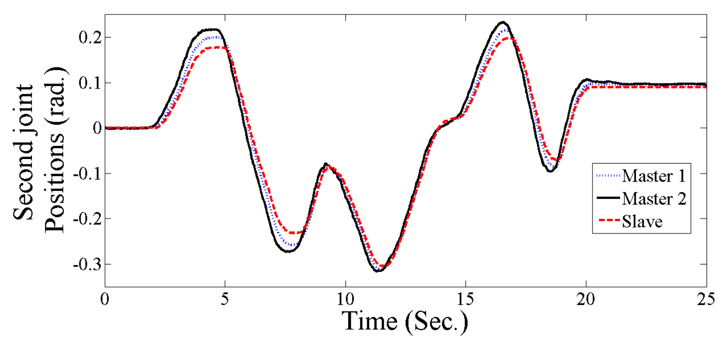

Figure 15: Second joint positions of the master1, master2 and slave robots with dominance factor $\alpha = 0.9$.

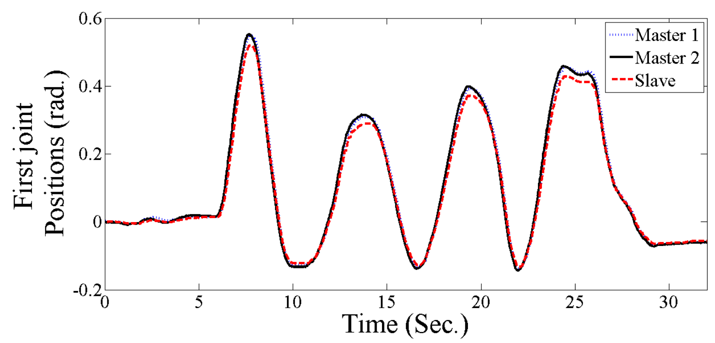

Figure 16: First joint positions of the master1, master2 and slave robots with dominance factor $\alpha = 0.1$.

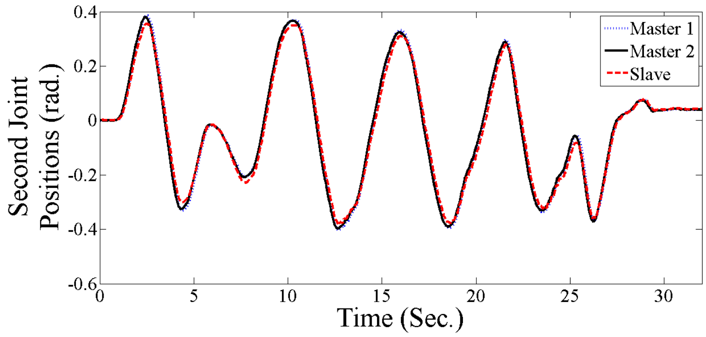

Figure 17: Second joint positions of the master1, master2 and slave robots with dominance factor $\alpha = 0.1$.

erator has during dual-user teleoperation as compared to a mere comparison of the robots positions.

The JR3 force sensor weight affects the dynamics of the haptic systems. To consider this effect, $J^T F$ should be added to the system dynamics where J is the robot Jacobian and F is the weight of the sensor in end-effector coordinate. For the Phantom Premium 1.5A robots (onto which JR3 sensors were mounted), following torques should be added to the system dynamics (Cavusoglu et al. (2002)).

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} l_1 cos(q_2) + l_2 sin(q_3) & 0 & 0 \\ 0 & l_1 cos(q_2 - q_3) & 0 \\ 0 & -l_1 sin(q_2 - q_3) & l_2 \\ 0 & 0 & -1 \\ cos(q_3) & 0 & 0 \\ sin(q_3) & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ mgcos(q_3) \\ mgsin(q_3) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that the proposed controller is a non-model-based but with gravity compensation. In all non-model based controllers that involve the gravity compensation, the model of the gravity forces should be known. The effect of the force sensors weight on the gravity vector was considered when implementing the controller.

To study the effect of the stiffness that the operator can sense in the direction of the first robot joint, the second and third joints of the robots are fixed through high-gain control. Using controllers (24)-(26) and assuming

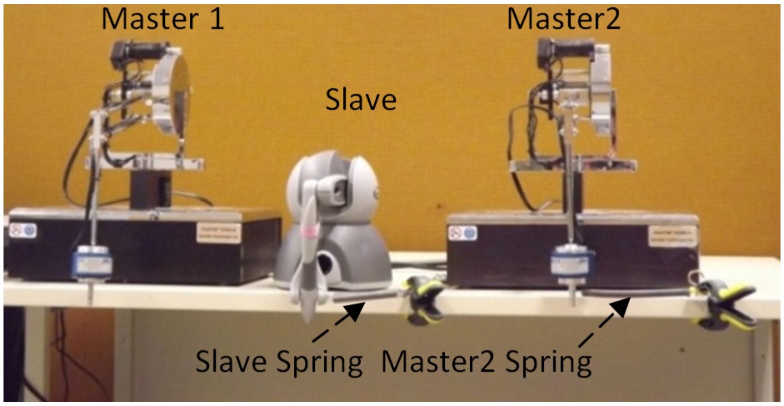

Figure 18: Trilateral teleoperation where master 1 is moved by a human operator while master 2 and slave are connected with springs to stiff walls.

$\tau_{h_{11}}$, $\tau_{h_{21}}$ and τ_{e_1} are the external torques applied by operator1, operator2 and the environment to the first joint of master 1, master 2 and the slave robots, respectively, it is possible to find that at the steady state we have

$$\begin{aligned}
\tau_{h_{11}} &= -k_1\Big((1+\alpha^2)q_{11} - q_{21} - \alpha^2 q_{s1}\Big)\\
\tau_{h_{21}} &= -k_2\Big((1+(1-\alpha)^2)q_{21} - q_{11} - (1-\alpha)^2 q_{s1}\Big)\\
\tau_{e_1} &= -k_s\Big(-\alpha^2 q_{11} - (1-\alpha)^2 q_{21} + \big(\alpha^2+(1-\alpha)^2\big)q_{s1}\Big)
\end{aligned}$$

where q_{11}, q_{21} and q_{s1} are the first joint positions of master 1, master 2 and the slave, respectively. Defining $K_{h_{11}}$, $K_{h_{21}}$ and K_e1 as stiffnesses that can be sensed at the first joint of the robots, i.e.,$\tau_{h_{11}} = K_{h_{11}}q_{11}$, $\tau_{h_{21}} = K_{h_{21}}q_{21}$ and $\tau_{e_1} = K_{e1}q_{s1}$, at the steady state it is easy to find the following simplified relationship:

$$(K_{h_{11}}/k_1 + 1 + \alpha^2)(K_{h_{21}}/k_2 + 1 + (1-\alpha)^2) = 1 + B/A$$

where

$$\begin{aligned}
B &= (K_{h_{11}}/k_1 + 1 + \alpha^2)(1-\alpha)^2 + \alpha^2\\
A &= (K_{h_{11}}/k_1 + 1 + \alpha^2)(K_e1/k_s + \alpha^2 + (1-\alpha)^2) - \alpha^4
\end{aligned}$$

Given different spring stiffnesses $K_{h_{21}}$ and K_{e1} coupled to the first joint of master 2 and the slave, it is possible to study the effect of the dominance factor α on the stiffness $K_{h_{11}}$ that operator 1 can sense in the first joint of master 1. In Figure 19, simulation and experimental results of the perceived stiffness $K_{h_{11}}$ are shown for different values of α between 0.1 and 1 but for the two different springs coupled to master 2 and the slave. It is easy to see from Figure 19 that when α increases, the stiffness $K_{h_{11}}$ that the operator senses at the end-effector of master 1 increases. This is what we expected because of the larger stiffness of the slave robots spring compared to master 2s spring. Conversely, for a small α, the stiffness $K_{h_{11}}$ decreases to reflect more on the small stiffness of the master 2s spring.

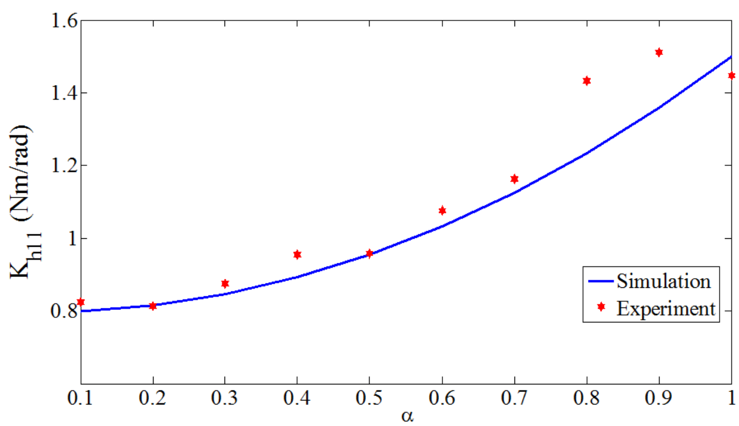

Figure 19: Experimental and simulation results measuring the sensed stiffness K_{h1} in a trilateral teleoperation system for different values of α between 0.1 and 1

6. Conclusion

In this paper, a PD like controller is proposed to cope with the nonlinear dynamics of three n-DOF robots in a trilateral teleoperation configuration and possibly subjected to time-varying delays in all communication channels. Stability in free and contact motion and asymptotic position tracking of the closed-loop system in free motion are proven using a Lyapunov functional and Barbalats lemma both in the absence and the presence of delay. In contact motion, a relationship between the interaction torques is derived. In experiments, master 2 and the slave were connected via two springs with different stiffness coefficients to stiff wall, and the stiffness that the human operator can senses in the end-effector of master 1 is analyzed. Experimental results shows that for large α, the stiffness perceived by the operator increases to display the larger stiffness of the slave-connected spring. Conversely, for small α, it decreases to give the operator a sense of the small stiffness of the master2-connected spring.

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