A Mixture Gamma Distribution to Model the SNR of Wireless Channels

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***Abstract*— Composite fading (i.e., multipath fading and shadowing together) has increasingly been analyzed by means of the K channel and related models. Nevertheless, these models do have computational and analytical difficulties. Motivated by this context, we propose a mixture gamma (MG) distribution for the signal-to-noise ratio (SNR) of wireless channels. Not only is it a more accurate model for composite fading, but is also a versatile approximation for any fading SNR. As this distribution consists of N (≥ 1) component gamma distributions, we show how its parameters can be determined by using probability density function (PDF) or moment generating function (MGF) matching. We demonstrate the accuracy of the MG model by computing the mean square error (MSE) or the Kullback-Leibler (KL) divergence or by comparing the moments. With this model, performance metrics such as the average channel capacity, the outage probability, the symbol error rate (SER), and the detection capability of an energy detector are readily derived.**

***Index Terms*— Fading channels, mixture of gamma distributions, signal-to-noise ratio (SNR).**

I. Introduction

Fundamental wireless propagation effects include macroscopic (large scale or slow) fading and microscopic (small scale or fast) fading. Macroscopic fading results from the shadowing effect by buildings, foliage and other objects. Microscopic fading results from multipath, which occurs in indoor environments, and also both macrocellular and microcellular outdoor environments [2]. Shadowing can significantly impact satellite channels, point-to-point long distance microwave links and macrocellular outdoor environments [3]. Both microscopic and macroscopic fading together are modeled by composite shadowing/fading distributions, of which Rayleigh-lognormal (RL) and Nakagami-lognormal (NL) are the two most common models [2]. But the probability density function (PDF) of these two composite models are not in closed form, making performance analysis of some applications difficult or intractable. Hence, several other composite models have been developed including the Suzuki distribution, the K and generalized-K (K_G) distributions, the $\mathcal{G}$- distribution, and the Gamma distribution [4]–[8]. Note that these models are **approximations** of the RL and NL models.

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A. Performance Analysis Based on K and K_G Models

The use of K and K_G models for performance analysis has greatly increased recently (see [9]–[19], among many others). For instance, the statistics of signal-to-noise ratio (SNR), the average channel capacity and the bit error rate (BER) are analyzed in [9], [10]. The outage performance, the average BER, and the channel capacity of several adaptive schemes are derived in [11]. The average output SNR, amount of fading and outage probability of different diversity receivers are derived in [12]. The closed-form BER is derived for (post-detection) equal gain combining (EGC) in [13]. The performance of dual-hop non-regenerative relays and multihop regenerative relays is analyzed in [14]–[16]. The average BER of orthogonal frequency-division multiplexing (OFDM) systems is evaluated in [17]. The ergodic capacity of multiple-input multiple-output (MIMO) systems is investigated in [18]. In [19], the performance of an energy detector is analyzed. These studies and others show the importance of K and K_G models.

B. Limitations of K and K_G Models

While RL and NL do not have closed-form PDFs, the K, K_G and $\mathcal{G}$- models do. Nevertheless, their PDFs include special functions (e.g., modified Bessel functions). Consequently, mathematical complications arise in the evaluation of wireless performance metrics. For instance, the cumulative distribution function (CDF) of the K_G model is derived in [9] by using generalized hypergeometric functions. The computation of such functions can be difficult as their series expressions may give rise to numerical issues. Asymptotic expansions may hence be required for certain ranges of the parameters and the variables. Moreover, the PDF of a sum of SNRs (required in maximal ratio combining [MRC]) is intractable. As well, even numerical methods for MRC by using the characteristic function approach is quite difficult due to the Whittaker function [8]. To avoid these difficulties, K_G random variables (RVs) are approximated by Gamma RVs in [8], and the PDF of the sum of independent K_G RVs is further approximated by PDF of another K_G RV [20]. These approximations are based on moment matching.

C. Mixture Gamma Distribution

While performance evaluation over composite channels is highly important, the use of K, K_G and $\mathcal{G}$ models is not without analytical and/or computational difficulties. Hence, we develop an alternative approach by using the mixture gamma (MG) distribution [21], [22].

This distribution avoids the problems mentioned above due to several reasons. First, since it is a linearly weighted sum of gamma distributions, it inherits several advantages of the gamma distribution. For example, the moment generating function (MGF) and CDF, which are required in wireless system analysis, have mathematically tractable expressions. Second, this distribution can approximate not only composite fading channels, but also any small-scale fading channels. Third, high accuracy is possible by adjusting the parameters. Overall, by using the MG model, performance of any wireless systems over a variety of fading channels can be analyzed in a unified framework.

D. Contributions

Our main contributions are summarized as follows.

- We propose an MG distribution to model the SNR of wireless channels. Although primarily intended to model composite shadowing/fading channels, this distribution is nevertheless effective for many other existing small-scale fading channels as well. The MG distribution is discussed in the statistics literature [21], [22]. The use of such a distribution to model wireless channels is new as far as we know.
- In the existing literature [6]–[8], K_G, $\mathcal{G}$ and Gamma distributions are used to approximate the composite shadowing/fading models. The MG distribution is more accurate than all those. The accuracy is measured by the mean square error (MSE), the Kullback-Leibler (KL) divergence, or by a comparison of the moments or the PDFs.
- The PDFs of K, K_G, η-μ, Nakagami-q (Hoyt), κ-μ, or Nakagami-n (Rician) distributions contain special functions, and thus, performance analysis is complicated or intractable. The MG model, a linear combination of gamma distributions, offers a solution. Thus, the distributions mentioned above can be approximated by the MG model, facilitating performance analysis.
- Another advantage of the MG distribution is the simplicity of performance analysis. Specifically, its CDF, MGF and moments are readily mathematically tractable. Typical performance analysis scenarios are derived under a unified framework. Therefore, case-by-case analysis of different channel models is unnecessary. Moreover, performance metrics, such as the average capacity, the outage probability, the symbol error rate (SER), and particularly, performance of an energy detector (essential for future cognitive radio networks), are derived by using the unified framework.

The rest of the paper is organized as follows. The MG distribution is described in Section II. Several common wireless channels are represented by using the MG model in Section III. In Section IV, the accuracy of the MG representation of composite fading channels and small-scale fading channels is examined. Performance analysis and the numerical results from the unified framework are shown in Section V. The concluding remarks are in Section VI.

II. The MG Wireless Channel Model

We start with the SNR distribution, which is required for analysis of wireless communication systems. The instantaneous received SNR and the average SNR are denoted by γ and $\bar{\gamma}$, respectively.

A. Probability Density Function (PDF)

In [23], it is shown that any function $f(x)$, where $x \in (0,\infty)$ and $\lim_{x\to+\infty} f(x) \to 0$, can be given as $f(x) = \lim_{u\to+\infty} S_u(x)$ where $S_u(x) := e^{-ux}\sum_{k=0}^{\infty}\frac{(ux)^k}{k!}f\left(\frac{k}{u}\right)$, $u > 0$. Thus, an arbitrarily close approximation to $f(x)$ can be obtained by increasing the number of terms in the mixture [24]. Note that $S_u(x)$ is a weighted sum of gamma PDFs. This result provides the motivation for using the MGF distribution to represent any wireless SNR models.

Therefore, we propose to use the following MG distribution to approximate the PDF of γ as

$$f_\gamma(x) = \sum_{i=1}^{N} w_i f_i(x) = \sum_{i=1}^{N}\alpha_i x^{\beta_i-1}e^{-\zeta_i x}, \qquad x \geq 0 \qquad (1)$$

where $f_i(x) = \frac{\zeta_i^{\beta_i}x^{\beta_i-1}e^{-\zeta_i x}}{\Gamma(\beta_i)}$ is a standard Gamma distribution, $\Gamma(\cdot)$ is the gamma function, $w_i = \frac{\alpha_i\Gamma(\beta_i)}{\zeta_i^{\beta_i}}$, N is the number of terms, and α_i, β_i and ζ_i are the parameters of the ith Gamma component. Further, $\sum_{i=1}^{N} w_i = 1$ as $\int_0^\infty f_\gamma(x)dx = 1$. The special case $N = 1$ reverts to Rayleigh and Nakagami-m fading. Discussion of how to choose N is provided in Section IV. Note that formula (1) can approximate the PDF $f(x)$ of any positive random variable.

B. Cumulative Distribution Function (CDF)

The CDF of the MG distribution can be evaluated as $F_\gamma(x) = \int_0^x f_\gamma(t)dt$ to yield

$$F_\gamma(x) = \sum_{i=1}^{N}\alpha_i\zeta_i^{-\beta_i}\gamma\left(\beta_i, \zeta_i x\right) \qquad (2)$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function defined as $\gamma(a,\rho) \triangleq \int_0^\rho t^{a-1}e^{-t}dt$ [25, eq. (8.350.1)]. This function is readily available in mathematical software packages.

C. Moment Generating Function (MGF)

The MGF of MG distribution, $\mathcal{M}_\gamma(s)$, can be evaluated as $\mathcal{M}_\gamma(s) = \mathbb{E}(e^{-sx})$ where $\mathbb{E}(\cdot)$ denotes the expectation operator. Thus, $\mathcal{M}_\gamma(s) = \int_0^\infty e^{-sx}f_\gamma(x)dx$ can be derived as

$$\mathcal{M}_\gamma(s) = \sum_{i=1}^{N}\frac{\alpha_i\Gamma(\beta_i)}{(s+\zeta_i)^{\beta_i}}. \qquad (3)$$

D. Moments

The r^{th} moment associated with the MG distribution, $m_\gamma(r)$, can be calculated as $m_\gamma(r) = \mathbb{E}(\gamma^r)$, to yield

$$m_\gamma(r) = \sum_{i=1}^{N}\alpha_i\Gamma(\beta_i+r)\zeta_i^{-(\beta_i+r)}. \qquad (4)$$

The mathematically tractable expressions (1)-(4) demonstrate the major benefit of the MG distribution. The performance metrics have convenient expressions, i.e., no complicated special functions are required. This fact can facilitate the performance studies enormously. If a given wireless channel can be represented as an MG distribution, the common performance metrics such as error rates, outage and others are immediately derived, with details given in Section V.

III. MG Distribution for Typical Wireless Channels

This section shows how to represent the SNR PDF of the NL, K, K_G, κ-μ, Nakagami-q (Hoyt), η-μ and Nakagami-n (Rician) channel models in the form of an MG model, as in (1).

A. NL Composite Channel

The SNR distribution of the NL channel is a gamma-lognormal (GL) distribution, given as [26]

$$f_\gamma(x) = \int_0^\infty \frac{x^{m-1}e^{-\frac{mx}{\rho y}}}{\Gamma(m)}\left(\frac{m}{\rho y}\right)^m \frac{e^{-\frac{(\ln y-\mu)^2}{2\lambda^2}}}{\sqrt{2\pi}\lambda y}dy \quad (5)$$

where m is the fading parameter in Nakagami-m fading, ρ is the unfaded SNR, and μ and λ are the mean and the standard deviation of the lognormal distribution, respectively. When $m = 1$, expression (5) is the SNR distribution of the RL distribution. The fading and shadowing effects diminish for larger m and smaller λ, respectively. A closed-form expression of the composite GL SNR distribution is not available in the literature.

By using substitution $t = \frac{\ln y-\mu}{\sqrt{2}\lambda}$, expression (5) can be written as

$$f_\gamma(x) = \frac{x^{m-1}}{\sqrt{\pi}\,\Gamma(m)}\left(\frac{m}{\rho}\right)^m \int_{-\infty}^{\infty} e^{-t^2}g(t)dt \quad (6)$$

where $g(t) = e^{-m(\sqrt{2}\lambda t+\mu)}e^{-\frac{m}{\rho}e^{-(\sqrt{2}\lambda t+\mu)}x}$. The integration in (6), $I = \int_{-\infty}^{\infty} e^{-t^2}g(t)dt$, is a Gaussian-Hermite integration which can be approximated as $I \approx \sum_{i=1}^N w_i g(t_i)$ where t_i and w_i are the abscissas and weight factors for the Gaussian-Hermite integration [27]. Therefore, we can express $f_\gamma(x)$ in (6) as the MG distribution given in (1). After normalization of $\int_0^\infty f_\gamma(x)dx = 1$, we find

$$\alpha_i = \psi(\theta_i,\beta_i,\zeta_i),\ \beta_i = m,\ \zeta_i = \frac{m}{\rho}e^{-(\sqrt{2}\lambda t_i+\mu)},\quad \theta_i = \left(\frac{m}{\rho}\right)^m \frac{w_i e^{-m(\sqrt{2}\lambda t_i+\mu)}}{\sqrt{\pi}\Gamma(m)} \quad (7)$$

where $\psi(\theta_i,\beta_i,\zeta_i) = \frac{\theta_i}{\sum_{i=1}^N \theta_i\Gamma(\beta_i)\zeta_i^{-\beta_i}}$. Function $\psi(\theta_i,\beta_i,\zeta_i)$ is also used for subsequent cases.

B. K and K_G Channels

The SNR distribution of the K_G channel has a closed-form expression with the nth-order modified Bessel function of the second kind [9]. With some mathematical simplifications, the SNR distribution of the K_G channel, which is a gamma-gamma distribution, can be rewritten in an integral form as

$$f_\gamma(x) = \frac{\lambda^m x^{m-1}}{\Gamma(m)\Gamma(k)}\int_0^\infty e^{-t}g(t)dt \quad (8)$$

where $g(t) = t^{\alpha-1}e^{-\frac{\lambda x}{t}}$, $\lambda = \frac{km}{\bar{\gamma}}$ and $\alpha = k - m$. Here k and m are the distribution shaping parameters, which represent the multipath fading and shadowing effects of the wireless channel, respectively. The integral in (8), $I = \int_0^\infty e^{-t}g(t)dt$, can be approximated as a Gaussian-Laguerre quadrature sum as $I \approx \sum_{i=1}^N w_i g(t_i)$ where t_i and w_i are the abscissas and weight factors for the Gaussian- Laguerre integration [27]. Thus, (8) can be written as the MG distribution with parameters

$$\alpha_i = \psi(\theta_i,\beta_i,\zeta_i),\ \beta_i = m,\ \zeta_i = \frac{\lambda}{t_i},\ \theta_i = \frac{\lambda^m w_i t_i^{\alpha-1}}{\Gamma(m)\Gamma(k)}. \quad (9)$$

C. η-μ Channel

The η-μ channel model is a generalized form to model the non-line of sight small-scale fading of a wireless channel [28]. The Rayleigh, Nakagami-m and Hoyt distributions are special cases of the η-μ channel model. The η-μ SNR distribution is given as [29]

$$f_\gamma(x) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^\mu x^{\mu-\frac{1}{2}}e^{\frac{-2\mu hx}{\bar{\gamma}}}}{\Gamma(\mu)H^{\mu-\frac{1}{2}}\bar{\gamma}^{\mu+\frac{1}{2}}}I_{\mu-\frac{1}{2}}\left(\frac{2\mu Hx}{\bar{\gamma}}\right) \quad (10)$$

where the parameter $\mu = \frac{(1+H^2/h^2)\mathbb{E}^2(\gamma)}{2Var(\gamma)}$ $(\mu > 0)$ represents the number of multipath clusters ($Var(\cdot)$ represents the variance), and $I_v(\cdot)$ is the vth-order modified Bessel function of the first kind. Parameters h and H are to be explained in the following. The η-μ channel includes two fading formats, Format 1 and Format 2, for two different physical representations. In Format 1, the independent in-phase and quadrature components of the fading signal have different powers, and η $(0 < \eta < \infty)$ is the power ratio of the in-phase component to the quadrature component. Two parameters h and H are defined as $h = \frac{2+\eta^{-1}+\eta}{4}$ and $H = \frac{\eta^{-1}-\eta}{4}$, respectively. In Format 2, the in-phase and quadrature components of the fading signal are correlated and have identical powers. η $(-1 < \eta < 1)$ is the correlation coefficient between the in-phase and quadrature components. Two parameters h and H are defined as $h = \frac{1}{1-\eta^2}$ and $H = \frac{\eta}{1-\eta^2}$, respectively [28].

Only a few performance studies for the η-μ channel have been published in the literature, probably because the modified Bessel function of the first kind in (10) leads to mathematical complexity [29]–[33]. In the following, the η-μ SNR distribution is approximated by using the MG distribution.

For a real number v, the function $I_v(z)$ can be computed using [25]

$$I_v(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(v+k+1)}\left(\frac{z}{2}\right)^{2k+v}. \quad (11)$$

Therefore, the η-μ SNR distribution (10) can be given in an alternative form as

$$f_\gamma(x) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}} h^\mu e^{\frac{-2\mu h x}{\bar{\gamma}}}}{\Gamma(\mu) H^{\mu-\frac{1}{2}} \bar{\gamma}^{\mu+\frac{1}{2}}} \sum_{i=1}^{\infty} \frac{\left(\frac{\mu H}{\bar{\gamma}}\right)^{2i+\mu-\frac{5}{2}} x^{2\mu-3+2i}}{(i-1)!\Gamma(\mu+i-\frac{1}{2})}. \tag{12}$$

The required accuracy[1] to approximate the exact $f_\gamma(x)$ can be achieved by summing a finite number, N, of terms in (12). By matching the two PDFs given in (10) and (12), the parameters of the MG distribution can be evaluated as

$$\begin{aligned} \alpha_i =& \psi(\theta_i, \beta_i, \zeta_i),\ \beta_i = 2(\mu - 1 + i),\ \zeta_i = \frac{2\mu h}{\bar{\gamma}}, \\ \theta_i =& \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}} h^\mu}{\Gamma(\mu) H^{\mu-\frac{1}{2}} \bar{\gamma}^{\mu+\frac{1}{2}}} \frac{\left(\frac{\mu H}{\bar{\gamma}}\right)^{2i+\mu-\frac{5}{2}}}{(i-1)!\Gamma(\mu+i-\frac{1}{2})}. \end{aligned} \tag{13}$$

Alternatively, the vth-order modified Bessel function of the first kind, $I_v(z)$, can be approximated by using the integral representation [25, eq. (8.431.5)]

$$I_v(z) = \int_0^\pi \frac{e^{z\cos\vartheta}\cos(v\vartheta)d\vartheta}{\pi} - \int_0^\infty \frac{\sin(v\pi)e^{-z\cosh t - vt}dt}{\pi}. \tag{14}$$

With $\vartheta = \frac{u\pi}{2} + \frac{\pi}{2}$ and $vt = p$, $I_v(z)$ can be further written as $I_v(z) = I_1 - I_2$, where $I_1 = \int_{-1}^{1} g_1(u)du$ is a Gaussian-Legendre integration, and $I_2 = \int_0^\infty e^{-p} g_2(p)dp$ is a Gaussian-Laguerre integration where $g_1(u) = \frac{1}{2}e^{-z\sin(\frac{\pi u}{2})}\cos\left((u+1)\frac{\pi v}{2}\right)$ and $g_2(p) = \sin(\pi v)e^{-z\cosh(\frac{p}{v})}/(\pi v)$. Similar to Section III-A with Gaussian-Hermite integration, the SNR distribution of the η-μ channel can be approximated by the MG model.

D. Nakagami-q (Hoyt) Channel

Satellite links with strong ionospheric scintillation can be modeled with the Nakagami-q distribution, and the SNR distribution of the Nakagami-q channel is given as [26]

$$f_\gamma(x) = \frac{1+q^2}{2q\bar{\gamma}} e^{-\frac{(1+q^2)^2}{4q^2\bar{\gamma}}x} I_0\left(\frac{1-q^4}{4q^2\bar{\gamma}}x\right) \tag{15}$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind. The fading parameter q varies from 0 to 1, where $q = 0$ and $q = 1$ represent the one-sided Gaussian and Rayleigh distributions, respectively. Further, this distribution is a special case of the η-μ distribution when $\mu = \frac{1}{2}$ and $\eta = q^2$. Using Format 1 of the η-μ distribution, the parameters of the MG distribution for the Nakagami-q channel can be derived from (13) to yield

$$\begin{aligned} \alpha_i =& \psi(\theta_i, \beta_i, \zeta_i),\quad \beta_i = 2i - 1,\ \zeta_i = \frac{(1+q^2)^2}{4q^2\bar{\gamma}}, \\ \theta_i =& \frac{(1+q^2)}{2q\bar{\gamma}\Gamma(i)(i-1)!}\left(\frac{1-q^4}{8q^2\bar{\gamma}}\right)^{2i-2}. \end{aligned} \tag{16}$$

[1]The required accuracy can be defined in terms of the mean-square error (MSE) between the exact and approximated expressions or by matching the first r moments.

E. κ-μ Channel

The κ-μ distribution fits well with channels having line-of-sight components. Nakagami-n (Rician) and Nakagami-m channels are special cases of the κ-μ channel. The κ-μ SNR distribution is [29]

$$\begin{aligned} f_\gamma(x) =& \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}} e^{\mu\kappa} \bar{\gamma}^{\frac{\mu+1}{2}}} \\ & \cdot x^{\frac{\mu-1}{2}} e^{-\frac{\mu(1+\kappa)}{\bar{\gamma}}x} I_{\mu-1}\left(2\mu\sqrt{\frac{\kappa(1+\kappa)}{\bar{\gamma}}x}\right) \end{aligned} \tag{17}$$

where κ ($\kappa > 0$) is the power ratio of the dominant components to the scattered components of the signal, and μ ($\mu > 0$) is defined as $\mu = \frac{(1+2\kappa)\mathbb{E}^2(\gamma)}{(1+\kappa)^2 Var(\gamma)}$. Since $f_\gamma(x)$ includes the modified Bessel function of the first kind with the square root of the random parameter x, it is difficult to obtain the MG form with one of the Gaussian integration methods, as discussed in previous subsections. To address this difficulty, the κ-μ SNR distribution given in (17) can be written using (11) as

$$\begin{aligned} f_\gamma(x) =& \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}} e^{\mu\kappa} \bar{\gamma}^{\frac{\mu+1}{2}}} \sum_{i=1}^{\infty}\left[\frac{\mu^{2i+\mu-3}}{\Gamma(\mu-1+i)(i-1)!}\right. \\ & \left.\cdot\left(\frac{\kappa(1+\kappa)}{\bar{\gamma}}\right)^{\frac{2i+\mu-3}{2}} x^{\mu+i-2} e^{-\frac{\mu(1+\kappa)}{\bar{\gamma}}x}\right]. \end{aligned} \tag{18}$$

The required accuracy for approximating the exact $f_\gamma(x)$ can be achieved by summing a finite number, N, of terms in (18). By matching the two PDFs given in (17) and (18), the parameters of the MG distribution can be evaluated as

$$\begin{aligned} \alpha_i =& \psi(\theta_i, \beta_i, \zeta_i),\ \beta_i = \mu + i - 1,\ \zeta_i = \frac{\mu(1+\kappa)}{\bar{\gamma}}, \\ \theta_i =& \frac{\mu(1+\kappa)^{\frac{\mu+1}{2}}}{\kappa^{\frac{\mu-1}{2}} e^{\mu\kappa} \bar{\gamma}^{\frac{\mu+1}{2}}} \frac{\mu^{2i+\mu-3}\left(\frac{\kappa(1+\kappa)}{\bar{\gamma}}\right)^{\frac{2i+\mu-3}{2}}}{\Gamma(\mu-1+i)(i-1)!}. \end{aligned} \tag{19}$$

Alternatively, one can use a different approach in which the MGF of SNR under $\kappa - \mu$ distribution can be matched with the MGF of SNR under the MG distribution given in (3). Using the power series expansion of the exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, the MGF of the κ-μ SNR distribution given in [30] can be re-written as an infinite form. By matching the two MGFs, the parameters of (1) can be evaluated. Details are omitted due to space limit.

F. Nakagami-n (Rician) Channel

The Nakagami-n or Rician channel model fits well with channels having a strong line-of-sight component. The corresponding SNR distribution is given as [26]

$$f_\gamma(x) = \frac{(1+n^2)e^{-n^2}}{\bar{\gamma}} e^{-\frac{(1+n^2)}{\bar{\gamma}}x} I_0\left(2n\sqrt{\frac{(1+n^2)}{\bar{\gamma}}x}\right) \tag{20}$$

where n is the fading parameter ($0 \leq n < \infty$), and the Rician factor K is given as $K = n^2$. The Nakagami-n distribution is a special case of the κ-μ distribution when $\mu = 1$ and $\kappa = n^2$.

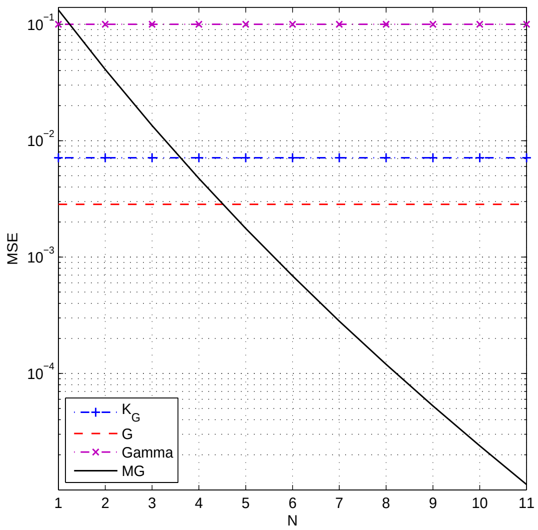

Fig. 1: The MSE versus N when the GL distribution is approximated by K_G, $\mathcal{G}$, Gamma and MG for $m = 2.7$, $\lambda = 1$, $\mu = 0$, and the average SNR = 0 dB.

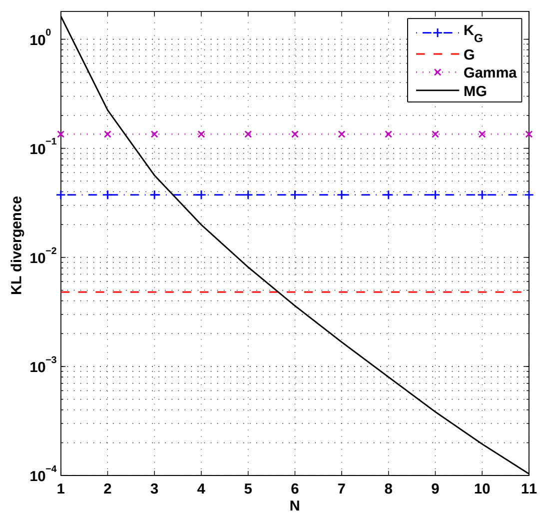

Fig. 2: The KL divergence ($\mathcal{D}_{KL}$) versus N when the GL distribution is approximated by K_G, $\mathcal{G}$, Gamma and MG for $m = 2.7$, $\lambda = 1$, $\mu = 0$, and the average SNR = 0 dB.

Therefore, the parameters of the MG distribution given in (1) can be evaluated as

$$\begin{aligned}\alpha_i =&\psi(\theta_i, \beta_i, \zeta_i), \quad \beta_i = i, \quad \zeta_i = \frac{(1+n^2)}{\bar{\gamma}},\\ \theta_i =&\frac{(1+n^2)}{e^{n^2}[(i-1)!]^2\bar{\gamma}}\left(\frac{n^2(1+n^2)}{\bar{\gamma}}\right)^{i-1}.\end{aligned} \tag{21}$$

G. Rayleigh and Nakagami-m Channels

The SNR distributions of the Rayleigh and Nakagami-m channels are exponential and gamma distributions, respectively [26, eq. (2.7) and (2.21)]. The two distributions are special cases of the MG distribution. When the Rayleigh distribution is written in the MG form given in (1), the corresponding parameters are $N = 1$, $\alpha_1 = \frac{1}{\bar{\gamma}}$, $\beta_1 = 1$ and $\zeta_1 = \frac{1}{\bar{\gamma}}$. For the Nakagami-$m$ distribution, the corresponding parameters are $N = 1$, $\alpha_1 = \frac{m^m}{\Gamma(m)\bar{\gamma}^m}$, $\beta_1 = m$ and $\zeta_1 = \frac{m}{\bar{\gamma}}$.

IV. Determination of the Number N in the MG Distribution

For the MG distribution to approximate other channel models, the number of components N needs to be determined. This can be selected as the minimum value such that (i) the mean-square error (MSE) or KullbackLeibler (KL) divergence between the target distribution and the MG distribution is below a threshold; or (ii) the first r moments of the two distributions match.

A. Accuracy of MG Distribution to Approximate Wireless Channel SNR

In the literature, the composite GL model has been approximated by K_G, $\mathcal{G}$ and Gamma models. Here, we compare the accuracy of the MG approximation with that of those approximations. One of the possible measures of accuracy is the MSE between two PDFs: the approximate PDF $f_{App}(x)$ and the exact PDF $f_{Ext}(x)$. It is defined as MSE $= \mathbb{E}\left[(f_{Ext}(x) - f_{App}(x))^2\right]$. Another possible measure of accuracy is the KL divergence ($\mathcal{D}_{KL}$) between $f_{App}(x)$ and $f_{Ext}(x)$, which is defined as $\mathcal{D}_{KL} = \int_{-\infty}^{\infty} f_{Ext}(x)\log\frac{f_{Ext}(x)}{f_{App}(x)}dx$.[2]

We select K_G, $\mathcal{G}$ and Gamma distributions for the comparison. The Gamma approximation is obtained by approximating K_G PDF by a Gamma PDF [8]. The MSEs and $\mathcal{D}_{KL}$s between the GL and its MG approximation (eq. (7)), GL and K_G, GL and $\mathcal{G}$, and GL and Gamma can be calculated numerically, as shown in Fig. 1 and Fig. 2, respectively, for $m = 2.7$, $\lambda = 1$, $\mu = 0$, and the average SNR = 0 dB. The parameters of K_G, $\mathcal{G}$ and Gamma distributions to match the target GL distribution are obtained from [5], [7], [8]. The MSE and $\mathcal{D}_{KL}$ between GL and MG distributions are less than 10^{-3} when the number of components $N \geq 6$ and $N \geq 8$, respectively. Based on MSE, MG model is better than Gamma, K_G and $\mathcal{G}$ models when $N \geq 2, 4$ and 5, respectively. Based on KL divergence, MG model is better than Gamma, K_G and $\mathcal{G}$ models when $N \geq 3, 4$ and 6, respectively. It can be seen that MSE and KL divergence give similar results for the minimum value of N that makes the MG model more accurate than Gamma, K_G or $\mathcal{G}$ model. Further, these MSE and $\mathcal{D}_{KL}$ with MG model decrease significantly as N slightly increases.

This fact is also evidenced by Fig. 3, which shows the CDFs of the GL and its approximations K_G, $\mathcal{G}$, Gamma and

[2]Although both the mean-square error (MSE) and the Kullback-Leibler (KL) divergence are measures of the difference between two PDFs, they give different measurements. Nevertheless, they give similar results for the minimum value of N that makes the MG model more accurate than Gamma, K_G or $\mathcal{G}$ model, as shown subsequently.

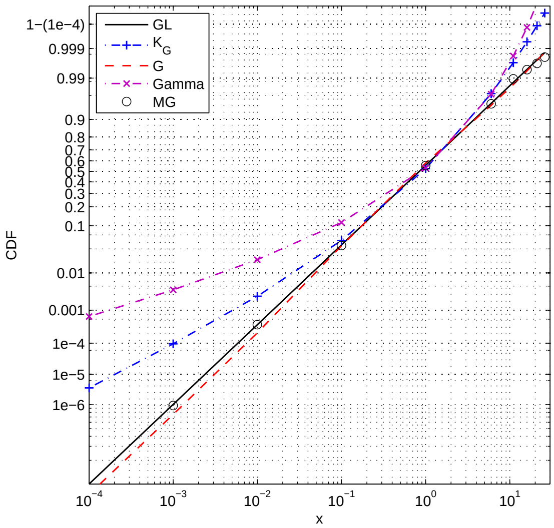

Fig. 3: The exact CDF of GL distribution and the CDFs of the K_G, $\mathcal{G}$, Gamma and MG approximations. The parameter values are $m = 2.7$, $\lambda = 1$, $\mu = 0$, and the average SNR = 0 dB. The number of components in the MG model is $N = 5$.

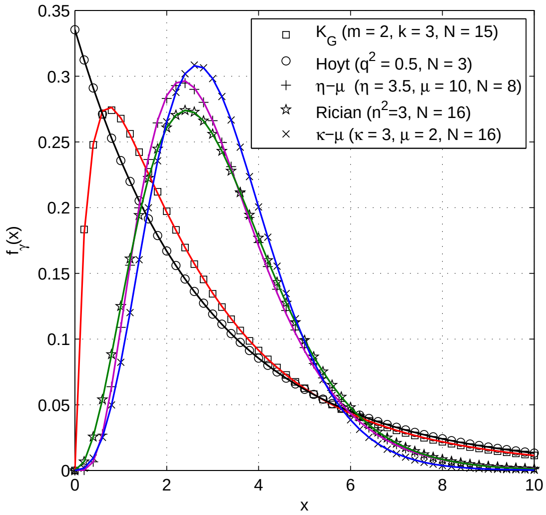

Fig. 4: Exact SNR distributions of K_G, Nakagami-q (Hoyt), η-μ (Format 1), Rician and κ-μ channel models and their MG approximations.

MG. These curves are plotted on a GL paper. The ordinate of the GL paper is obtained using the transformation $F_{\text{GL}}^{-1}(t)$ where $F_{\text{GL}}(x)$ is the CDF of GL distribution. Thus the GL distribution is a straight line on the GL paper, and others are not. The inverse function is numerically calculated using MATLAB. The following observations are made:

1) The exact GL CDF (solid line) matches perfectly with the MG approximation (small circles) for all x. Just $N = 5$ terms have been used in this case. Even better accuracy is possible by slightly increasing N.
2) The K_G approximation [6] deviates significantly in the lower tail ($x < 0.09$) and also in the upper tail ($x > 5$).
3) The $\mathcal{G}$ approximation [7] deviates in the lower tail ($x < 0.1$).
4) The Gamma approximation [8] deviates significantly in both lower tail ($x < 1$) and the upper tail ($x > 5$).

Clearly, the MG distribution is a more accurate representation of composite fading channels.

Similarly, Fig. 4 shows the SNR distributions of the K_G, Nakagami-q (Hoyt), η-μ (Format 1), Rician and κ-μ channel models and their corresponding approximations in the MG form, when the value of N in the MG distribution is selected as the minimum value that satisfies MSE $\leq 10^{-6}$. Excellent match is also observed in all curves. Note that in Fig. 4 and subsequent figures, the continuous lines and discrete markers show the curves corresponding to the exact distribution and the approximated MG distribution, respectively.

B. Moment Matching

The parameters of α_i, β_i and ζ_i in the MG distribution can be determined based on matching the MGFs of the exact distribution and the MG distribution. For brevity, we determine the value of N as the minimum value such that the first r moments of the two distributions have the same nearest integer values. Table I shows the selected N value when $r = 3$ for κ-μ distribution. The exact distribution and the approximated MG distribution have the same nearest integer values for the first 3 moments, which are also shown in Table I. In Table I, the (3, 1) and (7, 1) columns are corresponding to Nakagami-n distribution (Rician) with Rician factors $K = 3$ and $K = 7$, respectively.

	(κ, μ)					
	(3, 0.5)	(3, 1)	(3, 2)	(7, 0.5)	(7, 1)	(7, 2)
1st moment	3	3	3	3	3	3
2nd moment	19	14	12	15	12	11
3rd moment	154	83	55	87	57	43
N	8	11	16	12	20	26

TABLE I: The selected value of N and the nearest integer values of the first three moments of both exact and approximated SNR distributions of κ-μ channel.

To determine the parameters of the MG distribution to approximate other channel models, the exact SNR moment expressions ($m_\gamma(r) = \mathbb{E}(\gamma^r)$) for the Nakagami-lognormal, K_G, Nakagami-q and Nakagami-n are available in the literature [9], [26], [34], and the exact SNR moment expressions of η-μ and κ-μ distributions can be derived from their moments of the envelop distribution given in [28, eqs. (5), (43), and (46)].

C. Complexity of Determining N

The number of terms N in the MG model may be determined iteratively. N can be increased until the MSE or KL divergence requirements are met. In our numerical result shown in Figs. 1 and 2, approximately $N = 8$ can meet the accuracy requirement of 10^{-3}. To achieve 10^{-6} accuracy (although we may not need that level of accuracy), approximately $N = 15$ is needed.

Each iteration (corresponding to a particular N) requires $3N$ parameters α_i, β_i and ζ_i ($i = 1, \cdots, N$). If Gaussian integration methods (Gaussian-Legendre, Gaussian- Laguerre, or Gaussian-Hermite) are used, their abscissas and weight factors are already tabulated (e.g., in [27]), or can be generated efficiently by using simple MATLAB codes. Note that special functions are not involved in the calculations of the parameters.

V. Performance Analysis based on the MG Channel Model

Performance analysis of wireless technologies such as MIMO, cooperative communications, cognitive radio and ultra-wideband (UWB) radio has become important recently. The MG distribution helps to provide a unified performance analysis framework, because of the mathematical tractability of its CDF, MGF and moments (Section II) and because of its versatility (Section III). To this end, this section shows how the MG distribution allows the derivation of typical performance metrics such as error rate, outage, and others.

A. Diversity Order and Array Gain

The diversity order is the magnitude of the slope of the error probability versus SNR curve (log-log scale) in the high SNR region. The array gain measures the shift of the error probability curve to the left. The diversity order and the array gain relate to the asymptotic value of the MGF near the infinity, i.e., if the MGF, $\mathcal{M}_\gamma(t)$, can be written in the form

$$|\mathcal{M}_\gamma(t)| = b|t|^{-d} + \mathcal{O}(|t|^{-(d+1)}) \quad \text{as} \quad t \to \infty$$

then b and d define the array gain and diversity order, respectively [35]. Clearly, using the binomial series expansion, (3) can be rewritten as

$$|\mathcal{M}_\gamma(s)| = \sum_{i=1}^{N} \alpha_i \Gamma(\beta_i) \left(s^{-\beta_i} + \sum_{k=1}^{\infty} \binom{-\beta_i}{k} \zeta_i^k s^{-(\beta_i+k)} \right). \tag{22}$$

Therefore, the array gain is $b \approx \alpha_n \Gamma(\beta_n)$ and the diversity order is $d = \beta_n$, where n is the index of the first nonzero α_i, i.e., $\alpha_i = 0 \; \forall i < n$, and $\alpha_n \neq 0$. Accordingly, the diversity orders of NL, K, K_G, η-μ, Hoyt, κ-μ, Rician, Rayleigh and Nakagami-m fading channels are m, 1, m, 2μ, 1, μ, 1, 1, and m , respectively.

B. Average Channel Capacity

By using Shannon's theorem, the average channel capacity of a single-input single-output (SISO) channel, C, can be calculated by averaging the instantaneous channel capacity over the SNR distribution as $C = \int_0^\infty B \log_2(1+x) f_\gamma(x)dx$, where B is the signal transmission bandwidth. If β_i is an integer, the average channel capacity over the MG distribution, C, can be calculated by using results in [36], as

$$C = \frac{B}{\ln 2} \sum_{i=1}^{N} \alpha_i (\beta_i - 1)! e^{\zeta_i} \sum_{k=1}^{\beta_i} \frac{\Gamma(k - \beta_i, \zeta_i)}{\zeta_i^k} \tag{23}$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function defined as $\Gamma(a, \rho) \triangleq \int_\rho^\infty t^{a-1} e^{-t} dt$ [25, eq. (8.350.2)]. Next we provide a method to calculate C for any value of β_i. By replacing $\log_2(1 + x)$ with the Meijer's G-function [37, eq. (01.04.26.0003.01)], C can be evaluated in closed-form, which is valid for any β_i, as

$$C = \frac{B}{\ln 2} \sum_{i=1}^{N} \alpha_i \zeta_i^{-\beta_i} G_{3,2}^{1,3} \left[\zeta_i^{-1} \middle| \begin{array}{c} 1 - \beta_i, \; 1, \; 1 \\ 1, \; 0 \end{array} \right]. \tag{24}$$

For integer β_i, both expressions in (23) and (24) are equal numerically.

C. Average Symbol Error Rate (SER)

Since we have a MGF without special functions in the MG channel model, it can be used to evaluate the average SER of M-PSK, M-QAM and M-AM, as follows.

1) M-PSK: The average SER for M-PSK, P_e^{psk}, is given in [26, eq. (9.15)] for some channel models. With the MGF given in (3), the average SER for M-PSK over the MG distribution P_e^{psk} can be evaluated as

$$P_e^{psk} = \sum_{i=1}^{N} \frac{\alpha_i \Gamma(\beta_i)}{\pi \zeta_i^{\beta_i}} \int_0^{\frac{(M-1)\pi}{M}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{g_{psk}}{\zeta_i}} \right)^{\beta_i} d\theta \tag{25}$$

where $g_{psk} = \sin^2(\frac{\pi}{M})$. Therefore, the average SER of the M-PSK modulation can be evaluated in closed-form for any value of β_i with the aid of [38, eq. (10)].

2) M-QAM: Square M-QAM signals with a constellation size $M = 2^k$ with even k values are considered. The average SER for M-QAM, P_e^{qam}, is given in [26, eq. (9.21)] for some channel models. When the MG distribution is used, P_e^{qam} can be evaluated as

$$\begin{aligned} P_e^{qam} = \sum_{i=1}^{N} \frac{K \alpha_i \Gamma(\beta_i)}{\zeta_i^{\beta_i}} \Bigg[\int_0^{\frac{\pi}{2}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{g_{qam}}{\zeta_i}} \right)^{\beta_i} d\theta \\ - \frac{\sqrt{M} - 1}{\sqrt{M}} \int_0^{\frac{\pi}{4}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{g_{qam}}{\zeta_i}} \right)^{\beta_i} d\theta \Bigg] \end{aligned} \tag{26}$$

where $g_{qam} = \frac{3}{2(M-1)}$ and $K = \frac{4}{\pi}(1 - \frac{1}{\sqrt{M}})$. P_e^{qam} in (26) can be evaluated in closed-form for any value of β_i with the aid of [38, eq. (12)].

3) M-AM: Similarly, the average SER for M-AM, P_e^{am}, is given in [26, eq. (9.19)] for some channels. If it is evaluated based on the MG distribution, we have

$$P_e^{am} = \frac{2(M-1)}{\pi M} \sum_{i=1}^{N} \frac{\alpha_i \Gamma(\beta_i)}{\pi \zeta_i^{\beta_i}} \int_0^{\frac{\pi}{2}} \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{g_{am}}{\zeta_i}} \right)^{\beta_i} d\theta \tag{27}$$

where $g_{am} = \frac{3}{(M^2-1)}$. With the aid of [26, eq. (5A.1)] or [38, eq. (5)], P_e^{am} can be evaluated in closed-form for any value of β_i.

Similarly, the SER analysis for other modulation schemes over different digital communication systems, for example, as given in [39], [40], can be performed using the MG distribution.

D. Outage Probability

The outage probability, which is the probability that the received SNR is below a given threshold γ_{th}, can easily be calculated as $P_{out} = F_\gamma(\gamma_{th})$, where $F_\gamma(x)$ is given in (2).

E. Energy Detection in Cognitive Radio

In cognitive radio networks, energy detection of a primary signal is essential. The detection performance is typically illustrated by using the receiver operating characteristic (ROC) curve. Although such performance analysis for the Rayleigh, Nakagami-m, Rician and η-μ fading channels is available in [41], [42], the results for the Rician channel are limited. This is due to the detection probability being expressed by the generalized Marcum-Q function with limited analytical results. This problem can be circumvented by using the MG model, as we will illustrate next.

1) Average Detection Probability: When a primary signal exists, the detection probability is defined as the probability that the received energy is higher than a pre-defined threshold λ. The probability of detection (P_d) is expressed as $P_d = Q_u(\sqrt{2\gamma}, \sqrt{\lambda})$, where $Q_u(\cdot,\cdot)$ is the generalized Marcum-Q function. The number of samples, u, is an integer value [41]. The average detection probability, $\overline{P_d} = \int_0^\infty Q_u(\sqrt{2x}, \sqrt{\lambda}) f_\gamma(x)dx$, can further be re-written by replacing the Marcum-Q function by its circular contour integral representation [31] to yield

$$\overline{P_d} = \frac{e^{-\frac{\lambda}{2}}}{j2\pi} \oint_\Gamma \mathcal{M}_\gamma\left(1 - \frac{1}{z}\right) \frac{e^{\frac{\lambda}{2}z}}{z^u(1-z)} dz \tag{28}$$

where Γ is a circular contour with radius $r \in [0, 1)$. After substituting the MGF given in (3), the average detection probability can be re-written as

$$\overline{P_d} = e^{-\frac{\lambda}{2}} \sum_{i=1}^N \frac{\alpha_i \Gamma(\beta_i)}{(\zeta_i + 1)^{\beta_i}} \frac{1}{j2\pi} \oint_\Gamma g(z)dz \tag{29}$$

where

$$g(z) = \frac{e^{\frac{\lambda}{2}z}}{z^{u-\beta_i}(1-z)\left(z - \frac{1}{1+\zeta_i}\right)^{\beta_i}}.$$

We can solve the contour integral by applying the Residue Theorem assuming integer values for β_i. There are two possible scenarios, $u > \beta_i$ and $u \le \beta_i$.

When $u > \beta_i$: There are $(u - \beta_i)$ poles at $z = 0$ and β_i poles at $z = \frac{1}{1+\zeta_i}$. Therefore, $\overline{P_d}$ can be calculated as

$$\overline{P_d} = e^{-\frac{\lambda}{2}} \sum_{i=1}^N \frac{\alpha_i \Gamma(\beta_i)}{(\zeta_i + 1)^{\beta_i}} \left[\text{Res}\,(g; 0) + \text{Res}\left(g; \frac{1}{1+\zeta_i}\right)\right] \tag{30}$$

where $\text{Res}\,(g; 0)$ and $\text{Res}\left(g; \frac{1}{1+\zeta_i}\right)$ are the residues of $g(z)$ at $z = 0$ and $z = \frac{1}{1+\zeta_i}$, respectively. Further, $\text{Res}\,(g; 0)$ and $\text{Res}\left(g; \frac{1}{1+\zeta_i}\right)$ can be evaluated as

$$\text{Res}\,(g; 0) = \frac{\left[\frac{d^{u-\beta_i-1}}{dz^{u-\beta_i-1}} g(z) z^{u-\beta_i}\right]\Big|_{z=0}}{(u - \beta_i - 1)!}$$

$$\text{Res}\left(g; \frac{1}{1+\zeta_i}\right) = \frac{\left[\frac{d^{\beta_i-1}}{dz^{\beta_i-1}} g(z)(z - \frac{1}{1+\zeta_i})^{\beta_i}\right]\Big|_{z=\frac{1}{1+\zeta_i}}}{(\beta_i - 1)!}.$$

When $u \le \beta_i$: There are β_i poles at $z = \frac{1}{1+\zeta_i}$. Therefore, $\overline{P_d}$ can be calculated as

$$\overline{P_d} = e^{-\frac{\lambda}{2}} \sum_{i=1}^N \frac{\alpha_i \Gamma(\beta_i)}{(\zeta_i + 1)^{\beta_i}} \text{Res}\left(g; \frac{1}{1+\zeta_i}\right). \tag{31}$$

2) Area Under the ROC Curve (AUC): The area under the ROC curve (AUC) is another method to describe the overall energy detector performance, which has been recently introduced to the wireless communication field [43]. The AUC of an energy detector for a specific value of instantaneous SNR γ, $A(\gamma)$, is derived as [43]

$$\begin{aligned} A(\gamma) =& 1 - \sum_{k=0}^{u-1} \frac{1}{2^k\ k!} \gamma^k e^{-\frac{\gamma}{2}} \\ &+ \sum_{k=1-u}^{u-1} \frac{\Gamma(u+k)}{2^{u+k}\Gamma(u)} e^{-\gamma} {}_1\tilde{F}_1\left(u+k; 1+k; \frac{\gamma}{2}\right) \end{aligned} \tag{32}$$

where ${}_1\tilde{F}_1(\cdot;\cdot;\cdot)$ is the regularized confluent hypergeometric function of ${}_1F_1(\cdot;\cdot;\cdot)$ [37]. The average AUC under the MG distribution, $\overline{A}$, can be derived as $\overline{A} = \int_0^\infty A(x) f_\gamma(x)dx$ to yield

$$\begin{aligned} \overline{A} =& 1 - \sum_{k=0}^{u-1} \frac{1}{2^k k!} \sum_{i=1}^N \alpha_i \frac{\Gamma(k+\beta_i)}{(\frac{1}{2}+\zeta_i)^{k+\beta_i}} + \sum_{k=1-u}^{u-1} \frac{\Gamma(u+k)}{2^{u+k}\Gamma(u)} \\ & \sum_{i=1}^N \alpha_i \frac{\Gamma(\beta_i)}{(1+\zeta_i)^{\beta_i}} {}_2\tilde{F}_1\left(\beta_i; u+k; 1+k; \frac{1}{2(1+\zeta_i)}\right) \end{aligned} \tag{33}$$

where ${}_2\tilde{F}_1(\cdot;\cdot;\cdot;\cdot)$ is the regularized confluent hypergeometric function of ${}_2F_1(\cdot;\cdot;\cdot)$ [37]. This derivation is very similar to the derivation given in [43] for Nakagami-m fading.

Note that the performance of cooperative relay channels has been extensively studied over Rayleigh and Nakagami-m fading channels (e.g., [44], [45]). However, analysis may be lacking for lognormal shadowing, Rician, Hoyt, K, η-μ and κ-μ fading channels. The MG distribution may help in this case. For example, in [46], energy detector performance is analyzed for cooperative spectrum sensing in a cognitive radio network over fading and shadowing, which are modeled by using the MG distribution. The SER analysis of an amplify-and-forward relay network [45] and optimal resource allocation [47] are two potential applications of the MG model. Due to the space limitation, these topics are omitted here.

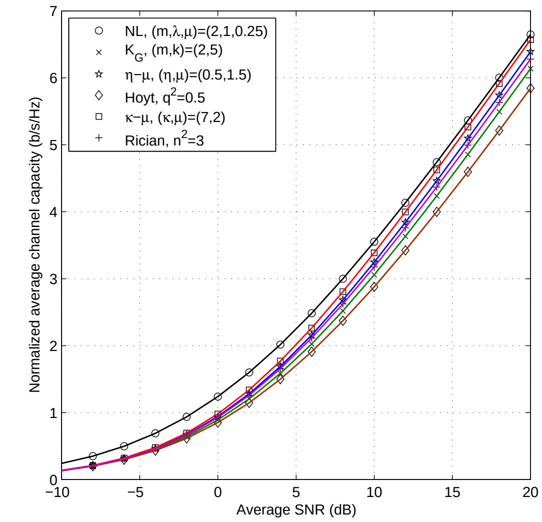

Fig. 5: The average capacity of an SISO channel versus average SNR over different fading channels.

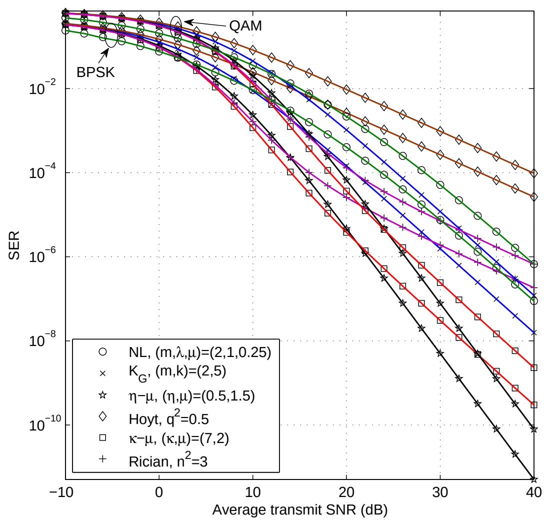

Fig. 7: The SER of an SISO channel versus average SNR over different fading channels for BPSK and QAM.

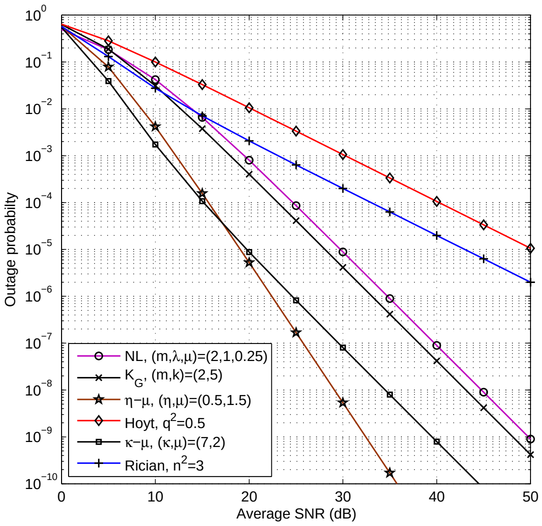

Fig. 6: The outage probability of an SISO channel versus average SNR over different fading channels for $\gamma_{th} = 0$ dB.

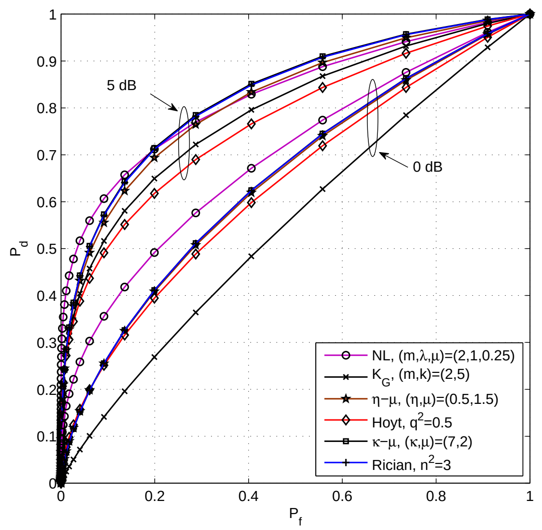

Fig. 8: The receiver operating characteristic (ROC) curves of an energy detector over different fading channels for average SNR $\bar{\gamma} = 0$ dB, 5 dB and $u = 1$.

F. Numerical Results

Two main focuses of this sub-section are (1) to show how the performance analysis based on the MG approximation matches with the exact results, and (2) to compare the performance of different fading channels. We choose typical distribution parameters. The value of N is selected as the minimum value to satisfy MSE $\leq 10^{-6}$. As an example, the parameters and N for the NL, K_G, η-μ (Format 1), Hoyt, κ-μ and Rician channels are chosen as (m, λ, μ, N)=$(2, 1, 0.25, 10)$, (m, k, N)=$(2, 5, 6)$, (η, μ, N)=$(0.5, 1.5, 5)$, (q^2, N)=$(0.5, 3)$, (κ, μ, N)= $(7, 2, 36)$ and (n^2, N)=$(3, 16)$, respectively.

The performance curves for the average channel capacity, the outage probability, average SER for BPSK and QAM, the ROC and the complementary AUC (CAUC = 1-AUC) of an energy detector are plotted in Figs. 5-9, respectively, based on both exact (continuous lines) and approximated (discrete points) MG distributions. All figures show an excellent match.

As discussed in Section V-A, the achievable diversity orders of NL, K, K_G, η-μ, Hoyt, κ-μ, Rician, Rayleigh and Nakagami-m fading channels are m, 1, m, 2μ, 1, μ, 1, 1, and m, respectively. The diversity order can be illustrated by using outage probability (Fig. 6), SER (Fig. 7) or complementary AUC (Fig. 9) versus average SNR plots in the high SNR region. From the figures, NL and K_G models show diversity

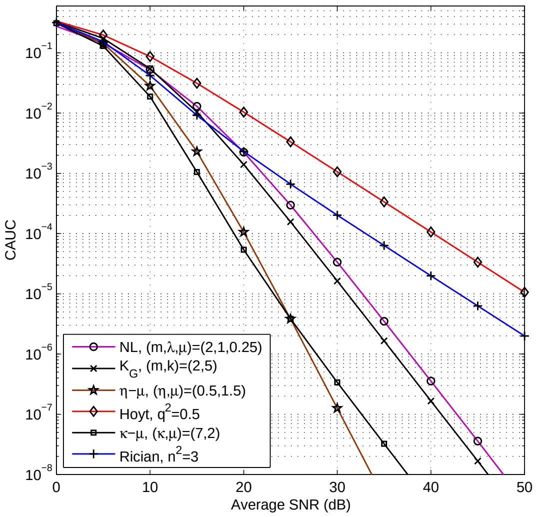

Fig. 9: The average complementary area under the ROC curves ($CAUC = 1 - AUC$) of an energy detector versus average SNR over different fading channels for $u = 1$.

order 2 because their fading parameters are $m = 2$. Hoyt and Rician models always have diversity order 1. Since η-μ and κ-μ models have $\mu = 1.5$ and $\mu = 2$, they have diversity order 3 and 2, respectively. All these confirm accuracy of our analysis in Section V-A. Further, we compare the performance of η-μ and Hoyt channels. Although both channels have same power ratios in our numerical examples (i.e., η=q^2=0.5), the η-μ channel has higher effective multipath clusters, which help to achieve a diversity order of 3 (=2μ) while the Hoyt channel has diversity order of one. Therefore, performance of η-μ channel with (η, μ)=(0.5,1.5) is better than the performance of Hoyt channel with q^2=0.5 in terms of channel capacity, outage, SER, and energy detection capability (Figs. 5-9). Similarly, we can compare performance of κ-μ and Rician channels. For the two channels in our numerical examples, the power ratios of the dominant components to the scattered components of the signal are $\kappa = 7$ and $n^2 = 3$, respectively, and the diversity orders are 2 and 1, respectively. So performance of κ-μ channel with (κ, μ)=(7,2) is better than the performance of Rician channel with n^2=3 in terms of channel capacity, outage, SER, and energy detection capability (Figs. 5-9). Since NL, K_G, η-μ, and κ-μ channel models do not have straightforward relationships among each other, no clear-cut performance comparison can be done.

VI. Conclusions

The MG distribution to model the SNR of the wireless channels has been proposed. Theoretical results [23] [48] show it converges to any PDF over $(0, \infty)$, a justification of this model. It is not only ideal for composite channels, but also effective for small-scale fading channels. The parameters of the MG distribution to match a target distribution can be obtained by approximating with Gaussian quadrature formulas, by matching moments (MGF) or by matching PDFs. We demonstrate that the MG model offers a more accurate representation of composite fading channels than those provided by the K models and other alternatives, which have recently been used in wireless research. Due to its mathematically tractable form and high accuracy, the MG distribution thus allows rapid evaluation of performance metrics such as channel capacity, outage, error rate and others of MIMO systems, cooperative relay channels, cognitive radio, UWB and others. Further research directions include the performance analysis of other wireless systems and model fitting based on measured channel data. We believe that the MG distribution opens up these and other research opportunities.

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