Robust Adaptive Beamforming: Evolution of Approaches, Analysis and Comparison

Sergiy A. Vorobyov

Department of Electrical and Computer Engineering
University of Alberta

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Outline

1. Introduction
2. Worst-Case Performance Optimization Approach
3. Probabilistically-Constrained Optimization Approach
4. Analysis of Approaches and a New One
5. Robust Adaptive Beamforming Using SQP
6. Comparison
Introduction

- Adaptive Beamforming finds applications in many areas such as **radar, sonar, wireless communications**, etc.

- Conventional beamforming techniques assume
  - *the steering vector of the desired signal is known precisely*
  - *large number of snapshots (training sample size)*
  - *stationary training data set*

- In many practical situations there is mismatch between the presumed steering vector and the actual one!
Signal Model

The output of a narrowband beamformer

\[ y(k) = w^H x(k) \]

where

\[ x(k) = \underbrace{s(k)p}_{\text{signal}} + \underbrace{i(k)}_{\text{interference}} + \underbrace{n(k)}_{\text{noise}} \]
Signal Model

The output of a narrowband beamformer

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where

\[ x(k) = s(k)p + i(k) + n(k) \]

Actual steering vector

\[ a = p + e \]

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Robust Adaptive Beamforming: Evolution of Approaches
Maximum SINR criterion

\[
\max_w \text{SINR}, \quad \text{SINR} = \frac{\sigma_s^2 |w^H(p + e)|^2}{w^H R_{i+n} w}
\]

Interference-plus-noise covariance matrix

\[
R_{i+n} = E \left\{ (i(k) + n(k))(i(k) + n(k))^H \right\}
\]

Note

- In practice, \( R_{i+n} \) is unavailable
- Sample estimate \( \hat{R} \triangleq \frac{1}{N} \sum_{k=1}^{N} x(k)x^H(k) \) is used
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Interference-plus-noise covariance matrix

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The essence of this approach is to:

- **Maintain a distortionless response** towards a continuum of steering vectors that belong to a certain uncertainty set.

- Guarantee that the distortionless response is maintained in the worst case.

- **Model the uncertainty** about the mismatch vector using:
  - spherical uncertainty set [Vorobyov, Gershman, Luo ’03]
  - elliptical uncertainty set [Lorenz and Boyd ’05]
Problem Formulation and Main Result

- The spherical uncertainty set is (for some known $\varepsilon > 0$)
  \[
  \|e\| \leq \varepsilon
  \]

  The robust MVDR beamforming problem is formulated as
  \[
  \min_w w^H \hat{R} w \quad \text{s. t.} \quad |w^H (p + e)| \geq 1, \quad \forall \|e\| \leq \varepsilon
  \]

  **Result 1** [Vorobyov, Gershman, Luo '03]: Infinite number of non-convex constraints
  \[
  |w^H (p + e)| \geq 1, \quad \forall \|e\| \leq \varepsilon
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  is equivalent to a single convex constraint
  \[
  \varepsilon \|w\| \leq w^H p - 1
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  is equivalent to a single convex constraint
  \[ \varepsilon \|w\| \leq w^H p - 1 \]
The robust MVDR beamforming problem is equivalent to:

$$\min_w w^H \hat{R} w \quad \text{s. t.} \quad \varepsilon \| w \| \leq w^H p - 1$$

This is so-called convex second order cone (SOC) programming problem! It can be easily solved!
Problem Formulation and Main Results

- The probabilistically-constrained beamformer guarantees that the distortionless response is maintained with a certain “sufficient” probability

\[
\min_w w^H \hat{R} w \quad \text{s. t.} \quad \Pr\{|w^H (p + e)| \geq 1\} \geq p_0
\]

- Result 2 [Vorobyov, Chen, Gershman ’08]: For Gaussian mismatch

\[e \sim \mathcal{N}_C(0, C_e)\]

the probabilistic constraint is tightly approximated by the deterministic constraint

\[\sqrt{-\ln(1 - p_0)} \| C_e^{1/2} w \| \leq w^H p - 1\]
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$$\sqrt{-\ln(1 - p_0)} \|C_e^{1/2}w\| \leq w^Hp - 1$$
Result 3 [Vorobyov, Chen, Gershman ’08]: For mismatch with the worst-case distribution the probabilistic constraint is tightly approximated by the deterministic constraint

\[
\frac{1}{\sqrt{1 - p_0}} \left\| C_e^{1/2} w \right\| \leq w^H p - 1
\]

Moreover, the worst-case distribution is discrete.

The problem is equivalent to that of the worst-case based robust adaptive beamforming if \( C_e = (\sigma_e^2 / M) I \).

For the worst-case mismatch distribution: \( \varepsilon = \sigma_e \sqrt{\frac{1}{M(1 - p_0)}} \)

For Gaussian mismatch: \( \varepsilon = \sigma_e \sqrt{-\ln(1 - p_0) / M} \)
**Result 3** [Vorobyov, Chen, Gershman ’08]: For mismatch with the worst-case distribution the probabilistic constraint is tightly approximated by the deterministic constraint

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For the worst-case mismatch distribution: \( \varepsilon = \sigma_e \sqrt{\frac{1}{M(1-p_0)}} \)

For Gaussian mismatch: \( \varepsilon = \sigma_e \sqrt{\frac{-\ln(1-p_0)}{M}} \)
Problems with previous approaches

- If mismatch is Gaussian, its norm is Chi-square distributed (not norm bounded)
- Over/under estimation of the parameters, e.g. $\varepsilon$, may lead to degradation in performance

Essence of a new approach

Estimate the mismatch vector and form the beam using the corrected steering vector [Hassanien, Vorobyov, Wong ’08]
Analysis and a New Idea

Problems with previous approaches

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Essence of a new approach

Estimate the mismatch vector and form the beam using the corrected steering vector [Hassanien, Vorobyov, Wong ’08]
Problem Formulation

- First maximize the beamformer output SINR by solving the optimization problem

$$\min_w w^H \hat{R} w \quad \text{subject to} \quad w^H (p + e) = 1$$

- Solution

$$w(e) = \frac{\hat{R}^{-1}(p + e)}{(p + e)^H \hat{R}^{-1}(p + e)}$$

- The beamformer output power

$$P(e) = \frac{1}{(p + e)^H \hat{R}^{-1}(p + e)}$$
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- The beamformer output power

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P(e) = \frac{1}{(p + e)^H \hat{R}^{-1}(p + e)}
\]
Problem Formulation and Difficulties

Estimate the unknown mismatch vector $\mathbf{e}$ by maximizing the beamformer output power

$$\min_{\mathbf{e}} (\mathbf{p} + \mathbf{e})^H \hat{\mathbf{R}}^{-1} (\mathbf{p} + \mathbf{e}) \quad \text{s. t.} \quad \| \mathbf{p} + \mathbf{e} \| = \sqrt{M}$$

Two difficulties

- The corrected vector $\mathbf{p} + \hat{\mathbf{e}}$ might converge to a vector associated with interference
- Non-convex constraint!
Estimate the unknown mismatch vector $\mathbf{e}$ by maximizing the beamformer output power

$$\min_{\mathbf{e}} (\mathbf{p} + \mathbf{e})^H \hat{\mathbf{R}}^{-1} (\mathbf{p} + \mathbf{e}) \quad \text{s. t.} \quad \| \mathbf{p} + \mathbf{e} \| = \sqrt{M}$$

**Two difficulties**

- The corrected vector $\mathbf{p} + \hat{\mathbf{e}}$ might converge to a vector associated with interference
- Non-convex constraint!
To avoid first difficulty, enforce $p + e$ to belong to a subspace that is spanned by the actual steering vector

$$P_p \perp (p + e) = 0$$

$P_p \perp \triangleq I - UU^H$ is a projection onto a subspace that is orthogonal to the actual steering vector.

$U \triangleq [u_1, u_2, \ldots, u_K]$, $\{u_k\}_{k=1}^K$ are $K$ principal eigenvectors of

$$C \triangleq \int_{\Theta} p(\theta)p^H(\theta) \, d\theta$$
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\]
Result 4 [Hassanien, Vorobyov, Wong ’08]: The initial optimization problem is equivalent to the problem

\[
\min_e (p + e)^H \hat{R}^{-1} (p + e)
\]

subject to

\[
P_p^\perp (p + e) = 0
\]

\[
\|p + e\| = \sqrt{M}
\]

How to get rid of non-convexity?
Result 4 [Hassanien, Vorobyov, Wong '08]: The initial optimization problem is equivalent to the problem

\[
\min_{\mathbf{e}} \quad (\mathbf{p} + \mathbf{e})^H \mathbf{R}^{-1} (\mathbf{p} + \mathbf{e})
\]

subject to

\[
\mathbf{P}_{\mathbf{p}}^\perp (\mathbf{p} + \mathbf{e}) = \mathbf{0}
\]

\[
\|\mathbf{p} + \mathbf{e}\| = \sqrt{M}
\]

How to get rid of non-convexity?
Iterative Solution

\[ \min_{e_{\perp}} (p + e_{\perp}) H \hat{R} - 1 (p + e_{\perp}) \]
subject to
\[ P_{\perp} p (p + e_{\perp}) = 0 \]
\[ \| p + e_{\perp} \| \leq \sqrt{M} \]

\[ \| a \| = \| p \| = \sqrt{M} \]
Iterative Solution

\[
\min_{e_{\perp}} \quad (p + e_{\perp})^H \hat{R}^{-1} (p + e_{\perp}) \\
\text{subject to} \quad P_p (p + e) = 0 \\
\|p + e_{\perp}\| \leq \sqrt{M} + \delta \\
p^H e_{\perp} = 0
\]
Iterative Algorithm

Algorithm:

1. Estimate $\mathbf{e}_\perp$ by solving the problem in previous slide
2. If $\|\mathbf{e}_\perp\| = \text{“small”}$, go to Step 5.
3. Update the presumed steering vector $\mathbf{p} = \mathbf{p} + \mathbf{e}_\perp$.
4. Project the updated steering vector back to the sphere $\mathbf{p} = \left(\frac{\sqrt{M}}{\|\mathbf{p}\|}\right)\mathbf{p}$, then go to Step 1.
5. Calculate the robust adaptive beamformer weights

$$ \mathbf{w}_{\text{SQP}} = \frac{\hat{\mathbf{R}}^{-1}\mathbf{p}}{\mathbf{p}^H\hat{\mathbf{R}}^{-1}\mathbf{p}}, $$
Simulation Setup

- $M = 10$ sensors spaced half wavelength apart. $N = 100$ data snapshots.
- Desired signal is assumed to impinge on the array from direction $\theta_p = 5^\circ$
- Two interfering sources with DOAs $-50^\circ$ and $-20^\circ$; INR $= 30$ dB.
- Look direction mismatch: actual DOA is uniformly drawn from $[1^\circ, 9^\circ]$
- Array perturbation:
  Sensors are assumed to be displaced from its original location and the displacement is drawn uniformly from the set $[-0.05, 0.05]$ measured in wavelength.
Simulation Results

![Graph showing performance comparison of different beamformers. The x-axis represents input SNR (dB), and the y-axis represents output SINR (dB). The graph compares SMI Beamformer, LSMI Beamformer, Worst-case Beamformer ($\epsilon=0.3M$), Probability-Constrained Beamformer, Proposed SQP Beamformer, and Optimal SINR. The x-axis ranges from -20 to 20 dB, and the y-axis ranges from -20 to 30 dB.](Image)
References


References (Cont’d)


