REDUCED COMPLEXITY BLIND UNITARY PREWHITENING WITH APPLICATION TO BLIND SOURCE SEPARATION

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ABSTRACT
Eigenvalue decomposition (EVD) of the sample data covariance matrix is, typically, used for calculating the whitening matrix and prewhitening the noisy signals. An important problem here is to reduce the computational complexity of the EVD of the complex-valued sample data covariance matrix. In this paper, we show that the complexity of the prewhitening step for complex-valued signals can be reduced approximately by a factor of four when the real-valued EVD is used instead of the complex-valued. Such complexity reduction can be achieved for any axis-symmetrical array. For such class of arrays it enables real-time implementation of the prewhitening step for complex-valued signals. The performance of the proposed procedure is shown in application to a blind source separation (BSS) problem.

1. INTRODUCTION
The prewhitening step is often a computational bottleneck of many signal processing procedures. For example, the complexity of many blind source separation (BSS) methods [1] essentially depends on the complexity of the prewhitening step. This is especially true for the processing of complex-valued signals. Therefore, it is important to reduce the complexity of the prewhitening step when processing noisy complex-valued signals. Reduced complexity of the prewhitening step is also appealing from a hardware implementation point of view.

In this paper, we propose a simple procedure that enables the reduction of the computational complexity of the prewhitening step for complex-valued signals by a factor of four. Such new procedure makes use of forward-backward averaging [2], and therefore, could be applied only to axis-symmetrical arrays [3]. The latter fact should not be taken as a serious limitation because the most appealing in practice array geometry - uniform linear array (ULA) belongs to the class of axis-symmetrical arrays. No other additional limitations and assumptions are needed for applicability of a new blind unitary prewhitening technique. A derivation of a new prewhitening procedure is based of the well know fact that the centro-Hermitian matrices [4] can be transformed into real-valued matrices which leads usually to computational gain [5], [6].

2. BACKGROUND
We consider an axis-symmetrical array of \( K \) sensors receiving the signals from \( M \) narrowband sources, \( M \leq K \). The \( K \times 1 \) snapshot vector of antenna array outputs can be written as

\[
y(n) = x(n) + v(n) = As(n) + v(n)
\]

where \( x(n) = [x_1(n), \ldots, x_K(n)]^T \in \mathbb{C}^{K \times 1} \) is the vector which contains the noiseless array output sampled at time \( n \), \( A = [a_1, \ldots, a_M] \in \mathbb{C}^{K \times M} \) is the matrix transfer complex operator between sources and sensors, \( a_m = [a_{1,m}, \ldots, a_{K,m}]^T \in \mathbb{C}^{K \times 1} \) is the spatial signature of the \( m \)th source signal, \( s(n) = [s_1(n), \ldots, s_M(n)]^T \in \mathbb{C}^{M \times 1} \) is the vector of the source waveforms, \( v(n) = [v_1(n), \ldots, v_K(n)]^T \in \mathbb{C}^{K \times 1} \) is the vector of additive spatially and temporally white Gaussian noise, and \([\cdot]^T\) denotes the transpose. We also assume that the sources are spatially uncorrelated to each other and to noise.

The basic problem of array signal processing is to estimate the signals \( s_1(n), \ldots, s_M(n), n = 1, N \) using only the array measurements \( y_1(n), \ldots, y_K(n), n = 1, \ldots, N \), where \( N \) is the total number of available snapshots.

The prewhitening step is typically applied to the received signals before using any array signal processing algorithm. The prewhitening is achieved by applying to \( x(n) \) a whitening matrix \( W \), i.e., an \( M \times K \) matrix verifying

\[
E(Wx(n)x^H(n)W^H) = WR_s(0)W^H = WAA^H = I
\]

where

\[
R_s(0) = E[x(n)x^H(n)] = AR_s(0)A^H = AA^H
\]

is the noiseless array output covariance matrix, \( I \) is the identity matrix, and \( E[\cdot] \) and \([\cdot]^H\) stand for expectation and Hermitian transpose operations, respectively. Without any
loss of generality, we assume that the source covariance matrix \( \mathbf{R_s}(0) = \mathbf{I} \), i.e., the source signals have unit variance so that the dynamic range of the sources is accounted for by the magnitude of the corresponding columns of \( \mathbf{A} \) [7]. Equation (1) shows that for any whitening matrix \( \mathbf{W} \), there exists an \( M \times M \) unitary matrix \( \mathbf{U} \) such that \( \mathbf{WA} = \mathbf{U} \). Then matrix \( \mathbf{A} \) can be factored as
\[
\mathbf{A} = \mathbf{W}^\dagger \mathbf{U}
\]
where \([\cdot]^\dagger\) denotes the Moore-Penrose pseudoinverse.

The vector of whitened signals \( \mathbf{z}(n) \) obeys the following linear model
\[
\mathbf{z}(n) = \mathbf{W} \mathbf{y}(n) = \mathbf{U} \mathbf{s}(n) + \mathbf{W} \mathbf{v}(n).
\]

Since the noiseless array output \( \mathbf{x}(n) \) is not known and \( \mathbf{R_s}(0) \) cannot be directly computed, the whitening matrix \( \mathbf{W} \) should be found from the noisy array output covariance matrix
\[
\mathbf{R_y}(0) = \mathbf{E} \{ \mathbf{y}(n) \mathbf{y}^H(n) \} = \mathbf{A} \mathbf{A}^H + \sigma^2 \mathbf{I}
\]
provided that the noise variance \( \sigma^2 \) (or, in general, noise covariance matrix) is known or can be estimated. If the noise variance \( \sigma^2 \) is known, the covariance matrix \( \mathbf{R_y}(0) \) can be computed as
\[
\mathbf{R_y}(0) = \mathbf{R_s}(0) - \mathbf{y}(n) \mathbf{y}^H(n).
\]

Typically, the whitening matrix \( \mathbf{W} \) is obtained using eigenvalue decomposition (EVD) of the sample estimate of the covariance matrix (3) calculated as
\[
\hat{\mathbf{R}}_y(0) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}(n) \mathbf{y}^H(n).
\]

The computation of the EVD of the complex-valued matrix (4) usually requires a significant amount of computations. However, the EVD of the complex-valued sample covariance matrix \( \hat{\mathbf{R}}_y(0) \) can be equivalently substituted by the EVD of a specially constructed real-valued covariance matrix. It leads to significant computational savings, because the real-valued EVD has a reduced computational complexity as compared to the complex-valued one, approximately by a factor of four [8].

### 3. UNITARY PREWHITENING

It is easy to check that the covariance matrix \( \mathbf{R}_y(0) \) given by (3) is a centro-Hermitian matrix because the following property [4]
\[
\mathbf{R}_y(0) = \mathbf{J} \mathbf{R}_y^*(0) \mathbf{J}
\]
is satisfied. Here \([\cdot]^*\) stands for the complex conjugate, and \( \mathbf{J} \) is the exchange matrix.

On the other hand, the sample covariance matrix \( \hat{\mathbf{R}}_y(0) \) given by (4) is used in practice instead of (3) which is not available. The matrix (4) is not necessarily centro-Hermitian because of the finite sample size effect. However, the centro-Hermitian property can be forced by forward-backward (FB) averaging:
\[
\hat{\mathbf{R}}_{FB} = \frac{1}{2} (\hat{\mathbf{R}}_y(0) + \mathbf{J} \hat{\mathbf{R}}_y^*(0) \mathbf{J}).
\]

Another advantage of using FB averaging is that the number of snapshots can be nearly doubled and possibly correlated source pairs can be decorrelated [2].

Next we can introduce the real-valued covariance matrix, which uniquely corresponds to \( \hat{\mathbf{R}}_{FB} \), as
\[
\hat{\mathbf{C}} = \hat{\mathbf{Q}}^H \hat{\mathbf{R}}_{FB} \hat{\mathbf{Q}}
\]
where \( \mathbf{Q} \) is any unitary \( K \times K \) matrix. For example, as it is discussed in [4] and [5], the good choice for \( \mathbf{Q} \) could be the following sparse unitary matrices
\[
\hat{\mathbf{Q}} = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} \mathbf{I} & \mathbf{j} \mathbf{I} \\ \mathbf{J} & -\mathbf{j} \mathbf{J} \end{array} \right]
\]
where \( \mathbf{0} \) denotes the vector which contains only zeros, and \( \mathbf{j} = \sqrt{-1} \). The matrix of even dimension can be chosen for an arrays with an even number of sensors, and the matrix of odd dimension can be taken for an array with an odd number of sensors, respectively.

It is interesting to note that the real-valued covariance matrix (5) can also be obtained via the forward-only covariance matrix \( \hat{\mathbf{R}}_y(0) \) as
\[
\hat{\mathbf{C}} = \frac{1}{2} (\hat{\mathbf{Q}}^H \hat{\mathbf{R}}_y(0) \hat{\mathbf{Q}} + \hat{\mathbf{Q}}^H \mathbf{J} \hat{\mathbf{R}}_y^*(0) \mathbf{J} \hat{\mathbf{Q}})
\]
\[
= \text{Re} \{ \hat{\mathbf{Q}}^H \hat{\mathbf{R}}_y(0) \hat{\mathbf{Q}} \}
\]
where \( \text{Re} \{ \cdot \} \) stands for the real part of complex-valued matrix.

In order to estimate the whitening matrix \( \mathbf{W} \) we propose to use the EVD of the real-valued matrix \( \hat{\mathbf{C}} \) instead of the EVD of the complex-valued matrix \( \hat{\mathbf{R}}_y(0) \).

The EVDs of the matrices \( \hat{\mathbf{C}} \) and \( \hat{\mathbf{R}}_{FB} \) can be written as
\[
\hat{\mathbf{C}} = \hat{\mathbf{E}} \hat{\Lambda}_s \hat{\mathbf{E}}_s^H + \hat{\mathbf{E}} \hat{\Lambda}_n \hat{\mathbf{E}}_n^H
\]
\[
\hat{\mathbf{R}}_{FB} = \hat{\mathbf{L}} \hat{\Lambda}_s \hat{\mathbf{L}}_s^H + \hat{\mathbf{L}} \hat{\Lambda}_n \hat{\mathbf{L}}_n^H
\]
where \( \hat{\mathbf{E}}_s = [\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_M], \hat{\mathbf{E}}_n = [\hat{\mathbf{e}}_{M+1}, \ldots, \hat{\mathbf{e}}_K], \hat{\Lambda}_s = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_M), \) and \( \hat{\Lambda}_n = \text{diag}(\hat{\lambda}_{M+1}, \ldots, \hat{\lambda}_K) \) are the matrices of eigenvectors and eigenvalues, which correspond,
respectively, to the signal and noise subspaces of the real-valued covariance matrix $\hat{C}$, and $\hat{L}_s = [l_1, \ldots, l_M]$, $\hat{L}_n = [l_{M+1}, \ldots, l_{K}]$, $\hat{\Gamma}_s = \text{diag}\{\hat{\gamma}_1, \ldots, \hat{\gamma}_M\}$, and $\hat{\Gamma}_n = \text{diag}\{\hat{\gamma}_{M+1}, \ldots, \hat{\gamma}_K\}$ are the matrices of eigenvectors and eigenvalues, which correspond, respectively, to the signal and noise subspaces of the forward-backward covariance matrix $\hat{R}_{FB}$.

Using the properties that $\hat{E} = Q^H \hat{L}$ and $\hat{A} = \hat{Y}$ the complex-valued whitening matrix can be estimated as

$$\hat{W} = \hat{L}_s (\hat{\Gamma}_s - \hat{\sigma}^2 I)^{-1/2} = Q^{-H} \hat{E}_s (\hat{\Lambda}_s - \hat{\sigma}^2 I)^{-1/2}$$

where

$$\hat{\sigma}^2 = \frac{1}{K - M} \sum_{i=M+1}^{K} \hat{\lambda}_i$$

is the averaging of the $K - M$ smallest eigenvalues of $\hat{C}$, which correspond to the noise variance.

Note that in the last expression of (7) only the matrices of eigenvectors and eigenvalues of the real-valued covariance matrix $\hat{C}$ are used. Finally, the vector of whitened signals can be obtained as

$$z(n) = \hat{W} y(n) = Q^{-H} \hat{E}_s (\hat{\Lambda}_s - \hat{\sigma}^2 I)^{-1/2} y(n).$$

The proposed blind unitary prewhitening procedure gives significant computational savings, and we demonstrate below by the example of the BSS problem, can lead to better performance if the sample size is small and the source pairs are possibly correlated.

4. BLIND SOURCE SEPARATION WITH UNITARY PREWHITENING

We apply the proposed unitary prewhitening scheme to the BSS problem. Particularly, we propose a modification of the second-order based identification (SOBI) procedure by Belouchrani et al. [7]. The SOBI procedure consists of two steps: the prewhitening step and the joint approximate diagonalization step [9]. The precise estimate of the computational complexity of the joint approximate diagonalization is not known and depends on the specific set of matrices which should be jointly diagonalized. However, based on the extensive simulations, it can be estimated as such that is comparable to the complexity of the prewhitening of the complex-valued signals. The computational complexity of the prewhitening step is defined by the complexities of the EVD and inversion operations, where the complexity on each operation can be given as $O(K^3)$. Thus, the prewhitening step is a bottleneck of the SOBI algorithm, and by reducing the computational complexity of this step by a factor of four in case of separation of complex-valued signals, the complexity of the overall SOBI procedure can be reduced at least by a factor of eight.

The SOBI with unitary prewhitening can be given as a sequence of the following steps.

1) Estimate the sample data covariance matrix $\hat{R}_s(0)$ and calculate the real-valued covariance matrix $\hat{C}$ using (6).

2) Estimate the noise variance $\hat{\sigma}^2$ using (8) and calculate the whitening matrix and whitened signals using (7) and (9), respectively.

3) Compute the sample covariance matrices $\hat{R}_s(\tau)$ for a fixed set of time lags $\tau \in \{\tau_i | i = 1, \ldots, I\}$, where $I$ is the total number of time lags.

4) Find the estimate of the unitary matrix $U$ (see (2)) by applying the joint approximate diagonalization procedure [9] to the set $\{\hat{R}_s(\tau_i) | i = 1, \ldots, I\}$.

5) Using (2), estimate the mixing matrix $A$, which we denote as $\hat{A}$.

5. SIMULATIONS

We assume a uniform linear array (ULA) with $K = 10$ omnidirectional sensors spaced half a wavelength apart, and two source signals corrupted by a stationary complex white Gaussian noise. The source signals are narrowband far-field unit variance complex circular Gaussian, and arrive from different directions $\theta_1 = 2^\circ$ and $\theta_2 = 15^\circ$. The source waveforms are generated by filtering a complex circular white Gaussian random processes by the autoregressive (AR) models of first order with the coefficients $a_1 = 0.85 \exp\{j0.5\}$ and $a_2 = 0.85 \exp\{j0.55\}$, respectively. The single-sensor signal-to-noise ratio (SNR) is taken to be equal to 10 dB.

Two BSS methods are compared: the SOBI with conventional prewhitening [7] and the SOBI with the proposed unitary prewhitening. Six time lags are used to compute six covariance matrices $R_s(\tau_i), i = 1, \ldots, 6$ for each aforementioned method.

The performances are compared in terms of the root-mean-square error (RMSE):

$$\text{RMSE} = \sqrt{\frac{1}{LMK} \sum_{l=1}^{L} \| \hat{A}(l) - A \|_F^2}$$

where $L = 100$ is the number of independent simulation runs, $\hat{A}(l)$ is the estimate of $A$ obtained from the $l$th run, and $\| \cdot \|_F^2$ stands for the Frobenius norm of matrix. Permutation and scaling of columns (which are inherent in the BSS problem) are fixed by means of a least-squares ordering and normalization of the columns of $\hat{A}(l)$.

In our first example, the sources are assumed to be uncorrelated. However, the correlation between sources can be
caused in this case by the finite sample size effect. Figure 1 shows the RMSEs of the aforementioned methods versus the number of snapshots.

In the second example, mutually correlated sources are assumed with the correlation coefficient equal to 0.97. Note that the assumption of spatially uncorrelated sources which enables to use the conventional SOBI is violated in this example. Figure 2 shows the RMSEs of the tested methods versus the number of snapshots.

From the figures we observe that the SOBI with unitary prewhitening outperforms the SOBI with conventional prewhitening. Especially it is true if the sample size is small or the source signals are correlated. Moreover, the computational complexity of the prewhitening step for SOBI with unitary prewhitening is reduced by a factor of four, and the overall complexity is reduced by a factor of eight.

### 6. Conclusions

A new prewhitening scheme, in which the complex-valued EVD is equivalently substituted by the real-valued EVD, has been proposed. The computational complexity of the new prewhitening scheme is reduced approximately by a factor of four as compared to the conventional one. The improvement in the performance for the proposed unitary prewhitening scheme is demonstrated in application to the BSS problem.

### 7. References


