Robust $H_\infty$ control for uncertain discrete-time stochastic bilinear systems with Markovian switching

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SUMMARY

This paper is concerned with the problems of robust stochastic stabilization and robust $H_\infty$ control for uncertain discrete-time stochastic bilinear systems with Markovian switching. The parameter uncertainties are time-varying norm-bounded. For the robust stochastic stabilization problem, the purpose is the design of a state feedback controller which ensures the robust stochastic stability of the closed-loop system irrespective of all admissible parameter uncertainties; while for the robust $H_\infty$ control problem, in addition to the robust stochastic stability requirement, a prescribed level of disturbance attenuation is required to be achieved. Sufficient conditions for the solvability of these problems are obtained in terms of linear matrix inequalities (LMIs). When these LMIs are feasible, explicit expressions of the desired state feedback controllers are also given. An illustrative example is provided to show the effectiveness of the proposed approach. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: discrete-time systems; $H_\infty$ control; linear matrix inequality; stabilization; stochastic bilinear systems; stochastic stability; uncertain systems

1. INTRODUCTION

In the past decades much attention has been focused on the study of stochastic bilinear systems due to the extensive applications of these systems to population dynamics, macroeconomics, chemical reactor control, time-sharing and random round-off errors in computer operation, and other areas [1–3]. Stochastic bilinear systems are also referred to as systems with random parametric excitation or linear stochastic systems with multiplicative noise [4, 5]. Many issues for these systems, such as stability analysis, optimal control, observer design, and so on, have

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been investigated by many researchers [6–9]. Recently, the $H_\infty$ control problem for stochastic bilinear systems has been studied and some results on this topic have been reported in the literature [10, 11]. The goal of this problem is to design a controller to stochastically stabilize a given system while satisfying a prescribed level of disturbance attenuation. By developing a bounded real lemma for stochastic systems, necessary and sufficient conditions for the solvability of the $H_\infty$ control problem were obtained in terms of coupled nonlinear matrix inequalities in Reference [11] for continuous stochastic systems. The corresponding results for discrete stochastic bilinear systems can be found in Reference [10] for infinite horizon case and [12] for finite horizon case, respectively.

On the other hand, systems with Markovian jumping parameters have been extensively studied. This class of systems is modelled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. A great number of estimation and control issues concerning these systems have been studied; see e.g. References [13–15], and the references therein. It is noted that the results on $H_\infty$ control for Markovian jump systems were also reported in References [16, 17] for continuous and discrete cases, respectively; these results were further generalized to Markovian jump systems with state delays in [18]. When Markovian switching parameters appear in a stochastic bilinear system, the stability problem was investigated in Reference [19], in which several stability results were proposed. Furthermore, Wang et al. [20] discussed the stability of nonlinear stochastic systems with Markovian switching. It is worth mentioning that both the results in References [19, 20] were obtained in the continuous-time context. When discrete-time stochastic bilinear systems with Markovian switching is concerned, to the authors’ knowledge, the problems of robust stabilization and robust $H_\infty$ control have not been fully investigated in the literature.

In this paper, we consider the problems of robust stochastic stabilization and robust $H_\infty$ control for discrete-time stochastic bilinear systems with Markovian switching and parameter uncertainties. The parameter uncertainties are assumed to be time-varying norm-bounded and appear in both the state and input matrices. The purpose of the robust stochastic stabilization problem is the design of a linear state feedback controller such that the resulting closed-loop system is robustly stochastically stable for all admissible uncertainties; while for the robust $H_\infty$ control problem, in addition to the robust stochastic stability requirement, a prescribed level of disturbance attenuation is required to be achieved. An LMI approach is developed to solve these problems, and sufficient conditions for the solvability are derived. The desired state feedback controllers can be constructed through a convex optimization problem that can be efficiently handled by using standard numerical algorithms [21].

**Notation:** Throughout this paper, for symmetric matrices $X$ and $Y$, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); $I$ is the identity matrix with appropriate dimension; $\mathbb{N}$ is the set of natural numbers; $M^T$ represents the transpose of the matrix $M$; $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$; $l_2[0, \infty)$ refers to the space of square summable infinite vector sequences over $[0, \infty)$; The notation $(\Omega, \mathcal{F}, \mathcal{P})$ represents the probability space with $\Omega$ the sample space and $\mathcal{F}$ the $\sigma$-algebra of subsets of the sample space; $|\cdot|$ is the Euclidean vector norm; $\|\cdot\|_2$ stands for the usual $l_2[0, \infty)$ norm, while $\|\cdot\|_{\ell_2}$ denotes the norm in $l_2(\Omega, \mathcal{F}, \mathcal{P}, [0, \infty))$, i.e.

$$
\|f\|_{\ell_2}^2 = \mathbb{E}\left\{ \sum_{k \in \mathbb{N}} |f(k)|^2 \right\} = \sum_{k \in \mathbb{N}} \mathbb{E}\{|f(k)|^2\} < \infty
$$
where \( f \in l_2((\Omega, \mathcal{F}, \mathcal{P}), [0, \infty)) \) and \( \mathbb{N} \) is the set of natural numbers; \( \lambda_{\text{min}}(\cdot) \) and \( \lambda_{\text{max}}(\cdot) \) are used to denote, respectively, the minimum and maximum eigenvalues of a symmetric matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. DEFINITIONS AND PROBLEM FORMULATION

Consider a discrete-time stochastic bilinear system with Markovian switching and parameter uncertainties described by the following model:

\[
\begin{align*}
(\Sigma) : \quad & x(k + 1) = A(r_k, k)x(k) + B(r_k, k)u(k) + C(r_k, k)v(k) \\
& \quad + [A_0(r_k, k)x(k) + B_0(r_k, k)u(k) + C_0(r_k, k)v(k)]o(k) \quad (1) \\
& z(k) = D(r_k)x(k) + E(r_k)u(k) \quad (2)
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n \) is the state, \( u(k) \in \mathbb{R}^p \) is the control input, \( v(k) \in \mathbb{R}^q \) is the exogenous disturbance input which belongs to \( l_2[0, \infty) \), \( z(k) \in \mathbb{R}^r \) is the controlled output, the variable \( \{o(k)\} \) is a standard random scalar sequence satisfying

\[
\delta \{o(k)\} = 0, \quad \delta \{o(k)^2\} = 1 \quad (3)
\]

and \( o(0), o(1), \ldots \), are independent. The parameter \( r_k \) represents a discrete-time, discrete-state Markovian chain taking values in a finite set \( \mathcal{S} = \{1, 2, \ldots, s\} \) with transition probabilities

\[
\Pr \{r_{k+1} = j | r_k = i \} = \pi_{ij} \quad (4)
\]

where \( \pi_{ij} \geq 0 \) and for any \( i \in \mathcal{S} \),

\[
\sum_{j=1}^{s} \pi_{ij} = 1 \quad (5)
\]

We assume that the Markovian chain \( r_k \) is independent of \( o(k) \) for any \( k = 0, 1, \ldots \). In the system \( (\Sigma) \), for each \( r_k \in \mathcal{S} \),

\[
\begin{align*}
A(r_k, k) &= A(r_k) + \Delta A(r_k, k), \quad B(r_k, k) = B(r_k) + \Delta B(r_k, k) \\
C(r_k, k) &= C(r_k) + \Delta C(r_k, k) \\
A_0(r_k, k) &= A_0(r_k) + \Delta A_0(r_k, k), \quad B_0(r_k, k) = B_0(r_k) + \Delta B_0(r_k, k) \\
C_0(r_k, k) &= C_0(r_k) + \Delta C_0(r_k, k)
\end{align*}
\]

\[
\begin{align*}
\Delta A(r_k, k), \Delta B(r_k, k), \Delta C(r_k, k), \Delta A_0(r_k, k), \Delta B_0(r_k, k), \Delta C_0(r_k, k), \Delta D(r_k), \Delta E(r_k) \text{ are appropriately dimensioned real valued matrix functions of } r_k.
\end{align*}
\]

For the sake of simplicity, in the sequel, for each possible \( r_k = i, i \in \mathcal{S} \), we shall denote the matrices associated with the \( i \)-th mode by

\[
\begin{align*}
A_i(k) &:= A(r_k, k), \quad A_i(k) := A_0(r_k, k), \quad \Delta A_i(k) := \Delta A(r_k, k) \\
A_0(k) &:= A_0(r_k, k), \quad A_0(k) := A_0(r_k, k), \quad \Delta A_0(k) := \Delta A_0(r_k, k) \\
B_i(k) &:= B(r_k, k), \quad B_i(k) := B(r_k, k), \quad \Delta B_i(k) := \Delta B(r_k, k)
\end{align*}
\]
\[ B_0(k) \triangleq B_0(r_k, k), \quad B_0i \triangleq B_0(r_k), \quad \Delta B_0(k) \triangleq \Delta B_0(r_k, k) \]
\[ C_i(k) \triangleq C(r_k, k), \quad C_i \triangleq C(r_k), \quad \Delta C_i(k) \triangleq \Delta C(r_k, k) \]
\[ C_0i \triangleq C_0(r_k), \quad \Delta C_0(k) \triangleq \Delta C_0(r_k) \]
\[ D_i \triangleq D(r_k), \quad E_i \triangleq E(r_k) \]

where \( A_i, A_0i, B_i, B_0i, C_i, C_0i, D_i \) and \( E_i \) are known constant matrices representing the nominal system for each \( i \in \mathcal{I} \), and \( \Delta A_i(k), \Delta A_0i(k), \Delta B_i(k), \Delta B_0i(k), \Delta C_i(k) \) and \( \Delta C_0i(k) \) are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the following form

\[
[\Delta A_i(k) \quad \Delta A_0i(k) \quad \Delta B_i(k) \quad \Delta B_0i(k) \quad \Delta C_i(k) \quad \Delta C_0i(k) ]
\]

\[
= M_i F_i(k) [N_{i1} \quad N_{2i} \quad N_{3i} \quad N_{4i} \quad N_{5i} \quad N_{6i}] \tag{6}
\]

where for each \( i \in \mathcal{I}, N_{1i}, N_{2i}, N_{3i}, N_{4i}, N_{5i} \) and \( N_{6i} \), are known constant matrices, and \( F_i(k) \) are the uncertain time-varying matrices satisfying

\[
F_i(k)^T F_i(k) \leq I, \quad \forall i \in \mathcal{I} \tag{7}
\]

The uncertain matrices \( \Delta A_i(k), \Delta A_0i(k), \Delta B_i(k), \Delta B_0i(k), \Delta C_i(k) \) and \( \Delta C_0i(k) \), for each \( i \in \mathcal{I} \), are said to be admissible if both (6) and (7) hold.

Throughout this paper, we use the following definitions.

**Definition 1**

The uncertain stochastic bilinear system (1) is said to be robustly stochastically stable if there exists a scalar \( \bar{M}(x_0, r_0) > 0 \) such that

\[
\lim_{N \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{N} |x(k)|^2 \left| x_0, r_0 \right\} \right. \leq \frac{\bar{M}(x_0, r_0)}{2} \tag{8}
\]

for all admissible uncertainties \( \Delta A_i(k) \) and \( \Delta A_0i(k) \) when \( u(k) = 0 \) and \( v(k) = 0 \). System (1) is said to be robustly stochastically stabilizable if there exists a linear state feedback

\[
u(k) = K_r x(k) \tag{9}
\]

such that the resulting closed-loop system is robustly stochastically stable, where \( K(r_k)|_{r_k = i} = K_i, i \in \mathcal{I} \), are constant matrices.

**Definition 2**

The uncertain stochastic bilinear system (\( \Sigma \)) is said to be robustly stochastically stable with disturbance attenuation level \( \gamma \) if it is robustly stochastically stable and

\[
\|e\|_{l_2} < \gamma \|v\|_{l_2} \tag{10}
\]

for all non-zero \( v \in l_2[0, \infty) \) subject to zero initial conditions and \( u(k) = 0 \), where \( \gamma > 0 \) is a given scalar. The uncertain stochastic bilinear system (\( \Sigma \)) is said to be robustly stochastically stabilizable with disturbance attenuation level \( \gamma \) if there exists a linear state feedback (9) such that the resulting closed-loop system is robustly stochastically stable with disturbance attenuation level \( \gamma \).
The problem to be addressed in this paper is to develop techniques of robust stabilization and robust $H_{\infty}$ control for uncertain discrete-time stochastic bilinear systems with Markovian switching. More specifically, we are concerned with the development of conditions which ensure either robust stochastic stabilization in the sense of Definition 1 or robust stochastic stabilization with disturbance attenuation level $\gamma$ in the sense of Definition 2 for system ($\Sigma$).

Before concluding this section, we present a preliminary result which will be used in the proof of our main results in the following section.

**Proposition 1** ([Wang et al. [22]])

Let $A$, $D$, $M$, $W$ and $F$ be real matrices of appropriate dimensions such that $W > 0$ and $F^TF \leq I$. Then we have the following:

1. For scalar $e > 0$
   \[
   DF + (DF)^T \preceq eD^TD + e^{-1}M^TM
   \]

2. For any scalar $e > 0$ such that $W - eD^TD > 0$
   \[
   (A + DF)^T(W - eD^TD)^{-1}(A + DF) \preceq (A + DF)^T(W - eD^TD)^{-1}(A + DF^T)^{-1} + e^{-1}M^TM
   \]

### 3. ROBUST STOCHASTIC STABILIZATION

In this section, we shall present a sufficient condition for the uncertain stochastic system (1) to be robustly stochastically stabilizable in terms of LMIs. First, we provide a sufficient condition for robust stochastic stability, which is given in the following theorem.

**Theorem 1**

The uncertain stochastic bilinear system (1) is robustly stochastically stable if there exist matrices $X_i > 0$, $X_2 > 0, \ldots, X_s > 0$ and scalars $\varepsilon_{1i} > 0$ and $\varepsilon_{2i} > 0$, $i = 1, 2, \ldots, s$, such that the following LMIs hold for $i = 1, 2, \ldots, s$,

\[
\begin{bmatrix}
-X_i & X_iA_i^T\Omega_i & X_iA_i^T\Omega_i & X_iN_{1i}^T & X_iN_{2i}^T \\
\Omega_i^T A_iX_i & \varepsilon_{1i}\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 & 0 \\
\Omega_i^T A_0X_i & 0 & \varepsilon_{1i}\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 \\
N_{1i}X_i & 0 & 0 & -\varepsilon_{1i}I & 0 \\
N_{2i}X_i & 0 & 0 & 0 & -\varepsilon_{2i}I
\end{bmatrix} < 0
\]  \hspace{1cm} (11)

where

\[
\Omega_i = [\sqrt{\pi_i}I \sqrt{\pi_i}I \cdots \sqrt{\pi_i}I], \quad i = 1, 2, \ldots, s
\]  \hspace{1cm} (12)

\[
\Psi = \text{diag}(X_1, X_2, \ldots, X_s)
\]  \hspace{1cm} (13)

**Proof**

See the Appendix.
In the case of one mode operation, i.e. there are no jumps in the system in (1), we have LMI techniques, and no tuning of parameters are involved [21].

With Markovian switching. These LMIs can be solved efficiently numerically by using standard stochastic stability and robust stochastic stabilization for uncertain stochastic bilinear systems.

Theorems 1 and 2 provide sufficient conditions in terms of LMIs for, respectively, robust stochastically stable by following the same lines as in the proof of Theorem 1.

Remark 1

Under the condition of the theorem, we can verify that the stochastic system in (16) is robustly stochastically stable by following the same lines as in the proof of Theorem 1.

Proof

Applying the controller in (15) into the uncertain stochastic system (1) and setting \( v(k) = 0 \), we obtain the closed-loop system as

\[
x(k + 1) = (A(r_k, k) + B(r_k, k)Y(r_k)X(r_k)^{-1})x(k) \\
+ (A_0(r_k, k) + B_0(r_k, k)Y(r_k)X(r_k)^{-1})\zeta(k)
\]

Under the condition of the theorem, we can verify that the stochastic system in (16) is robustly stochastically stable by following the same lines as in the proof of Theorem 1.

Remark 1

Theorems 1 and 2 provide sufficient conditions in terms of LMIs for, respectively, robust stochastic stability and robust stochastic stabilization for uncertain stochastic bilinear systems with Markovian switching. These LMIs can be solved efficiently numerically by using standard LMI techniques, and no tuning of parameters are involved [21].

In the case of one mode operation, i.e. there are no jumps in the system in (1), we have \( \pi_i = 1, i \in \mathcal{F} = \{1\} \), and system (1) reduces to the following uncertain discrete-time stochastic bilinear system with no jumping parameters:

\[
(\Sigma_1) : \quad x(k + 1) = (A + \Delta A(k))x(k) + (B + \Delta B(k))u(k) + (C + \Delta C(k))\zeta(k) \\
+ [(A_0 + \Delta A_0(k))x(k) + (B_0 + \Delta B_0(k))u(k) \\
+ (C_0 + \Delta C_0(k))\zeta(k)]\zeta(k) \quad (17)
\]

\[
z(k) = Dx(k) + Eu(k) \quad (18)
\]
where $\Delta A(k), \Delta A_0(k), \Delta B(k), \Delta B_0(k), \Delta C(k)$ and $\Delta C_0(k)$ are unknown matrices, and are assumed to be of the form

$$
[\Delta A(k) \Delta A_0(k) \Delta B(k) \Delta B_0(k) \Delta C(k) \Delta C_0(k)] = MF(k)[N_1 N_2 N_3 N_4 N_5 N_6]
$$

where $M, N_1, N_2, N_3, N_4, N_5$ and $N_6$ are known constant matrices, and $F(k)$ is the uncertain time-varying matrices satisfying

$$
F(k)^TF(k) \leq I
$$

Then, Theorem 2 reduces to the following result.

**Corollary 1**

The uncertain discrete-time stochastic bilinear system (17) is robustly stochastically stabilizable if there exist matrices $X \succ 0, Y$ and scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that the following LMI holds:

$$
\begin{bmatrix}
-X & XA^T + Y^TB^T & XA_0^T + Y^TB_0^T & XN_1^T + Y^TN_3^T & XN_2^T + Y^TN_4^T \\
AX + BY & \epsilon_1MM^T - X & 0 & 0 & 0 \\
A_0X + B_0Y & 0 & \epsilon_1MM^T - X & 0 & 0 \\
N_1X + N_3Y & 0 & 0 & -\epsilon_1I & 0 \\
N_2X + N_4Y & 0 & 0 & 0 & -\epsilon_2I
\end{bmatrix} < 0
$$

In this case, a stochastically stabilizing controller can be chosen by

$$
u(k) = YX^{-1}x(k)
$$

**Remark 2**

Corollary 1 provides a sufficient condition for robust stochastic stabilization of uncertain discrete-time stochastic bilinear systems, which extends the results in Reference [21] where the parameter uncertainty was not considered.

### 4. ROBUST $H_\infty$ DISTURBANCE ATTENUATION

In this section we will consider the robust $H_\infty$ control problem formulated in Section 2. In order to solve this problem, we first give a solution to the $H_\infty$ performance analysis problem for the system ($\Sigma$) with $u(k) = 0$.

**Theorem 3**

Given a scalar $\gamma > 0$, the uncertain stochastic bilinear system ($\Sigma$) is robustly stochastically stable with disturbance attenuation level $\gamma$ if there exist matrices $X_1 > 0, X_2 > 0, \ldots, X_s > 0$ and scalars $\epsilon_{1i} > 0$ and $\epsilon_{2i} > 0$, $i = 1, 2, \ldots, s$, such that the following LMIs hold
for \( i = 1, 2, \ldots, s \),

\[
\begin{bmatrix}
-X_i & 0 & X_iA_i^T\Omega_i & X_iA_i^0\Omega_i & X_iN_{1i}^T & X_iN_{2i}^T & X_iD_i^T \\
0 & -\gamma^2 I & C_i^T\Omega_i & C_i^0\Omega_i & N_{1i}^T & N_{2i}^T & 0 \\
\Omega_i^TA_iX_i & \Omega_i^TC_i & \varepsilon_1\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 & 0 & 0 \\
\Omega_i^TA_0X_i & \Omega_i^TC_0i & 0 & \varepsilon_1\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 & 0 \\
N_{1i}X_i & N_{2i} & 0 & 0 & -\varepsilon_{1i}I & 0 & 0 \\
N_{2i}X_i & N_{0i} & 0 & 0 & 0 & -\varepsilon_{2i}I & 0 \\
D_iX_i & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \quad (19)
\]

were \( \Omega_i, i = 1, 2, \ldots, s \), and \( \Psi \) are given in (12) and (13), respectively.

**Proof**

From (19), it is easy to see that the following LMIs hold for \( i = 1, 2, \ldots, s \),

\[
\begin{bmatrix}
-X_i & X_iA_i^T\Omega_i & X_iA_i^0\Omega_i & X_iN_{1i}^T & X_iN_{2i}^T \\
\Omega_i^TA_iX_i & \varepsilon_1\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 & 0 \\
\Omega_i^TA_0X_i & \varepsilon_1\Omega_i^TM_iM_i^T\Omega_i - \Psi & 0 & 0 & 0 \\
N_{1i}X_i & 0 & 0 & -\varepsilon_{1i}I & 0 \\
N_{2i}X_i & 0 & 0 & 0 & -\varepsilon_{2i}I \\
D_iX_i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0
\]

This, by Theorem 1, implies that the uncertain stochastic system \((\Sigma)\) is robustly stochastically stable. Next, we shall show

\[
\|z\|_{\infty} < \gamma\|v\|_2 \quad (20)
\]

for all non-zero \( v \in l_2[0, \infty) \). To this end, we consider the system \((\Sigma)\) with \( u(k) = 0 \), i.e.

\[
(\Sigma) : \quad x(k+1) = A(r_k,k)x(k) + C(r_k,k)v(k) + [A_0(r_k,k)x(k) + C_0(r_k,k)v(k)]e_0(k) \quad (21)
\]

\[
z(k) = D(r_k)x(k) \quad (22)
\]

In the following we assume zero initial conditions and define

\[
J_N = \delta \left\{ \sum_{k=0}^{N} (|z(k)|^2 - \gamma^2|v(k)|^2) \right\} \quad (23)
\]

where the scalar \( N \in \mathbb{N} \). It then can be shown that

\[
J_N = \delta \left\{ \sum_{k=0}^{N} (|z(k)|^2 + \delta \{ V_{k+1}(x(k+1),r_{k+1})|x_k,r_k| - V_k(x(k),r_k) - \gamma^2|v(k)|^2 \} \right\} \quad (24)
\]

\[
- \delta \{ V_{N+1}(x(N+1),r_{N+1}) \}
\]
$$
\leq \varepsilon \left\{ \sum_{k=0}^{N} (\|z(k)\|^2 + \varepsilon \{ V_{k+1}(x(k + 1), r_{k+1})|x_k, r_k \} - V_k(x(k), r_k) - \gamma^2 \|v(k)\|^2) \right\} \\
= \varepsilon \left\{ \sum_{k=0}^{N} \eta(k)^T \Phi(r_k, k) \eta(k) \right\} \\
$$

where

$$V_k(x(k), r_k) = x(k)^T P(r_k) x(k)$$

$$\eta(k) = [x(k)^T \ v(k)^T]^T,$$  (25)

$$\Phi(r_k, k) = \begin{bmatrix}
D(r_k)^T D(r_k) - P(r_k) & 0 \\
0 & -\gamma^2 I
\end{bmatrix} + \begin{bmatrix}
A(r_k, k) \\
C(r_k, k)
\end{bmatrix} \begin{bmatrix}
A(r_k, k)^T \\
C(r_k, k)
\end{bmatrix}^T + \begin{bmatrix}
A_0(r_k, k) \\
C_0(r_k, k)
\end{bmatrix} \begin{bmatrix}
A_0(r_k, k)^T \\
C_0(r_k, k)
\end{bmatrix}^T$$  (26)

where

$$Q(r_k) = \sum_{j=1}^{s} \pi_{r_k j} P_j$$  (27)

On the other hand, by the Schur complement formula, we can verify that (19) is equivalent to

$$\begin{bmatrix}
D_i^T D_i - P_i & A_i^T \Omega_i & A_T^0 \Omega_i \\
A_i \Omega_i & C_i^T \Omega_i & C_i^T \Omega_i \\
0 & -\gamma^2 I & C_i^T \Omega_i \\
\Omega_i^T A_i & \Omega_i^T C_i & 0 & \Gamma - \Gamma
\end{bmatrix} \\
+ \begin{bmatrix}
N_{1i}^T \\
N_{2i}^T \\
N_{5i}^T \\
0
\end{bmatrix} \begin{bmatrix}
N_{1i}^T \\
N_{2i}^T \\
N_{5i}^T \\
0
\end{bmatrix}^T + \begin{bmatrix}
N_{6i}^T \\
N_{6i}^T \\
N_{6i}^T \\
0
\end{bmatrix} \begin{bmatrix}
N_{6i}^T \\
N_{6i}^T \\
N_{6i}^T \\
0
\end{bmatrix}^T < 0$$  (28)

where

$$P_i = X_i^{-1}, \quad i = 1, 2, \ldots, s$$

$$\Gamma = \text{diag}(P_1^{-1}, P_2^{-1}, \ldots, P_s^{-1})$$
By Proposition 1, we have

\[
\begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix}
F_i(k) \begin{bmatrix} N_{1i} & N_{5i} & 0 & 0 \end{bmatrix} + \begin{bmatrix} N_{1i}^T \\
N_{5i}^T \\
0 \\
0 \end{bmatrix}
F_i(k)^T \begin{bmatrix} 0 & 0 & M_i^T \Omega_i & 0 \end{bmatrix}
\]

\[
\leq e_{1i} \begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix}^T \begin{bmatrix} N_{1i}^T \\
N_{5i}^T \\
0 \\
0 \end{bmatrix} + e_{1i}^{-1} \begin{bmatrix} N_{1i}^T \\
N_{5i}^T \\
0 \\
0 \end{bmatrix}^T
\]

(29)

and

\[
\begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix}
F_i(k) \begin{bmatrix} N_{2i} & N_{6i} & 0 & 0 \end{bmatrix} + \begin{bmatrix} N_{2i}^T \\
N_{6i}^T \\
0 \\
0 \end{bmatrix}
F_i(k)^T \begin{bmatrix} 0 & 0 & M_i^T \Omega_i \end{bmatrix}
\]

\[
\leq e_{2i} \begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
\Omega_i^T M_i \\
0
\end{bmatrix}^T \begin{bmatrix} N_{2i}^T \\
N_{6i}^T \\
0 \\
0 \end{bmatrix} + e_{2i}^{-1} \begin{bmatrix} N_{2i}^T \\
N_{6i}^T \\
0 \\
0 \end{bmatrix}^T
\]

(30)

where \(e_{1i} > 0, e_{2i} > 0, i = 1, 2, \ldots, s\), are used. From inequalities (28)–(30), it can be shown that

\[
\begin{bmatrix}
D_i^T D_i - P_i & 0 & A_i^T(k) \Omega_i & A_i^T(k) \Omega_i \\
0 & -\gamma^2 I & C_i^T(k) \Omega_i & C_i^T(k) \Omega_i \\
\Omega_i^T A_i(k) & \Omega_i^T C_i(k) & -\Gamma & 0 \\
\Omega_i^T A_0(k) & \Omega_i^T C_0(k) & 0 & -\Gamma
\end{bmatrix}
= \begin{bmatrix}
D_i^T D_i - P_i & 0 & A_i^T \Omega_i & A_i^T \Omega_i \\
0 & -\gamma^2 I & C_i^T \Omega_i & C_i^T \Omega_i \\
\Omega_i^T A_i & \Omega_i^T C_i & -\Gamma & 0 \\
\Omega_i^T A_0 & \Omega_i^T C_0 & 0 & -\Gamma
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & N_{1i}^T F_i(k)^T M_i^T \Omega_i & N_{1i}^T F_i(k)^T M_i^T \Omega_i \\
0 & 0 & N_{5i}^T F_i(k)^T M_i^T \Omega_i & N_{5i}^T F_i(k)^T M_i^T \Omega_i \\
\Omega_i^T M_i F_i(k) N_{1i} & \Omega_i^T M_i F_i(k) N_{5i} & 0 & 0 \\
\Omega_i^T M_i F_i(k) N_{2i} & \Omega_i^T M_i F_i(k) N_{6i} & 0 & 0
\end{bmatrix} < 0
\]
which, by the Schur complement formula, implies
\[
\begin{bmatrix}
I - D_i^T D_i - P_i & 0 \\
0 & -\gamma^2 I
\end{bmatrix} + \begin{bmatrix}
A_i^T(k) \Omega_i & \Omega_i^T C_i(k) \\
C_i^T(k) \Omega_i & I
\end{bmatrix} + \begin{bmatrix}
A_{0i}^T(k) \Omega_i & \Omega_i^T C_{0i}(k) \\
C_{0i}^T(k) \Omega_i & I
\end{bmatrix} < 0
\]

Using this together with (26) and noting
\[
\Omega_i \Gamma^{-1} \Omega_i^T = Q_i
\]
we have
\[
\Phi(\gamma, k) < 0
\]

It follows from this and (24) that \(J_N < 0\), which implies that inequality (20) is satisfied for all non-zero \(v \in l_2[0, \infty)\). This completes the proof. \(\square\)

We now present the solution to the robust \(H_\infty\) control problem for the system (\(\Sigma\)).

**Theorem 4**

Given a scalar \(\gamma > 0\), the uncertain stochastic bilinear system (\(\Sigma\)) is robustly stochastically stabilizable with disturbance attenuation level \(\gamma\) if there exist matrices \(X_i > 0, X_0 > 0, \ldots, X_s > 0\), \(Y_i\) and scalars \(\varepsilon_{1i} > 0\) and \(\varepsilon_{2i} > 0\), \(i = 1, 2, \ldots, s\), such that the following LMIs hold for \(i = 1, 2, \ldots, s\):

\[
\begin{bmatrix}
-X_i & 0 & (X_i A_i^T + Y_i^T B_i^T) \Omega_i & (X_i A_{0i}^T + Y_i^T B_{0i}^T) \Omega_i \\
0 & -\gamma^2 I & C_i^T \Omega_i & C_{0i}^T \Omega_i \\
\Omega_i^T (A_i X_i + B_i Y_i) & \Omega_i^T C_i & \varepsilon_{1i} \Omega_i^T M_i M_i^T \Omega_i - \Psi & 0 \\
\Omega_i^T (A_{0i} X_i + B_{0i} Y_i) & \Omega_i^T C_{0i} & 0 & \varepsilon_{2i} \Omega_i^T M_i M_i^T \Omega_i - \Psi \\
N_{1i} X_i + N_{3i} Y_i & N_{5i} & 0 & 0 \\
N_{2i} X_i + N_{4i} Y_i & N_{6i} & 0 & 0 \\
D_i X_i + E_i Y_i & 0 & 0 & 0 \\
X_i N_{1i}^T + Y_i^T N_{3i}^T & X_i N_{2i}^T + Y_i^T N_{4i}^T & X_i D_i^T + Y_i^T E_i^T & \begin{bmatrix}
N_{5i}^T \\
N_{6i}^T \\
0 \\
0 \\
-\varepsilon_{1i} I \\
0 \\
-\varepsilon_{2i} I \\
0
\end{bmatrix} < 0
\]

\(\square\)

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where $\Omega_i$, $i = 1, 2, \ldots, s$, and $\Psi$ are given in (12) and (13), respectively. In this case, a stochastically stabilizing controller can be chosen by

$$u(k) = Y_i X_i^{-1} x(k), \quad i = 1, 2, \ldots, s$$

**Proof**

The proof of the theorem can be carried out by applying Theorem 3 and following the same argument as in the proof of Theorem 2.

In the case of one mode operation, i.e., the system $(\Sigma)$ is specialized to $(\Sigma_i)$ which is presented in Equations (17) and (18), Theorem 4 then reduces to the following result.

**Corollary 2**

Given a scalar $\gamma > 0$, the uncertain stochastic bilinear system $(\Sigma_i)$ is robustly stochastically stabilizable with disturbance attenuation level $\gamma$ if there exist matrices $X > 0$, $Y$ and scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following LMI holds:

$$
\begin{bmatrix}
-X & 0 & XA_1^T + YB_1^T & XA_2^T + YB_2^T & XN_1^T + YN_2^T & XN_1^T + YN_3^T & XD_1^T + Y E \ 
0 & -\gamma^2 I & C_1^T & C_2^T & N_5^T & N_6^T & 0 \\
AX + BY & C & \epsilon_1 MM^T - X & 0 & 0 & 0 & 0 \\
A_0 X + B_0 Y & C_0 & 0 & \epsilon_1 MM^T - X & 0 & 0 & 0 \\
N_1 X + N_3 Y & N_5 & 0 & 0 & -\epsilon_1 I & 0 & 0 \\
N_2 X + N_4 Y & N_6 & 0 & 0 & 0 & -\epsilon_2 I & 0 \\
D_1 X + E Y & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0
$$

In this case, a stochastically stabilizing controller can be chosen by

$$u(k) = Y X^{-1} x(k)$$

**Remark 3**

Corollary 2 provides a methodology for the design of robust $H_\infty$ controllers for discrete-time stochastic bilinear systems with parameter uncertainties. This extends the results in Reference [10] where the systems under consideration involve no parameter uncertainties.

5. AN ILLUSTRATIVE EXAMPLE

In this section, we present an illustrative example to demonstrate the applicability of the proposed solutions.

Consider a discrete-time stochastic bilinear system with Markovian switching and parameter uncertainties in the form of (1) and (2). We suppose the system involves two modes. For mode 1, the system matrices are given by

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0.2 & 1.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$$
Therefore, by Theorem 4, it can be seen that the robust and the suitable state feedback can be chosen as the solution as follows:

To this end, we resort to the Matlab LMI Control Toolbox to solve the LMIs in (31), and obtain the closed-loop system is robustly stochastically stable with disturbance attenuation level $\gamma = 1.5$. The purpose of this example is to design a linear state feedback controller such that the resulting closed-loop system is robustly stochastically stable with disturbance attenuation level $\gamma = 1.5$. The transition probabilities are assumed to be $\pi_{11} = 0.7$, $\pi_{12} = 0.3$, $\pi_{21} = 0.6$, $\pi_{22} = 0.4$.

For mode 2, the system matrices are given by

$$
A_2 = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}
$$

$$
A_{02} = \begin{bmatrix} 0.6 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} -0.3 & 0 \\ 0.5 & 1 \end{bmatrix}, \quad C_{02} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}
$$

$$
D_2 = [0.5 \quad 0.1], \quad E_2 = [0 \quad -0.2], \quad N_{12} = [0.3 \quad 0], \quad N_{22} = [0.1 \quad 0.2]
$$

$$
N_{32} = [-0.1 \quad 0.1], \quad N_{42} = [0 \quad 0.2], \quad N_{52} = -0.3, \quad N_{62} = 0.2
$$

The transition probabilities are assumed to be $\pi_{11} = 0.7$, $\pi_{12} = 0.3$, $\pi_{21} = 0.6$, $\pi_{22} = 0.4$.

The purpose of this example is to design a linear state feedback controller such that the resulting closed-loop system is robustly stochastically stable with disturbance attenuation level $\gamma = 1.5$. To this end, we resort to the Matlab LMI Control Toolbox to solve the LMIs in (31), and obtain the solution as follows:

$$
X_1 = \begin{bmatrix} 5.5318 & -7.1286 \\ -7.1286 & 25.5052 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2.3830 & -0.1403 \\ -0.1403 & 5.8787 \end{bmatrix}
$$

$$
Y_1 = \begin{bmatrix} 6.4203 & -21.1850 \\ 0.7008 & -7.8739 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 4.4376 & 4.2670 \\ -0.0392 & -0.4822 \end{bmatrix}
$$

$$
\varepsilon_{11} = 14.0857, \quad \varepsilon_{12} = 12.1169, \quad \varepsilon_{21} = 12.1059, \quad \varepsilon_{22} = 13.0648
$$

Therefore, by Theorem 4, it can be seen that the robust $H_\infty$ problem for this example is solvable, and the suitable state feedback can be chosen as

$$
u(k) = K_i x(k), \quad i = 1, 2 \quad (32)$$

with

$$
K_1 = \begin{bmatrix} 0.1410 & -0.7912 \\ -0.4238 & -0.4272 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.9076 & 0.7714 \\ -0.0213 & -0.0825 \end{bmatrix}
$$

Then, via the state feedback controller in (32), the closed-loop system will be robustly stochastically stable with disturbance attenuation level $\gamma = 1.5$ for all admissible uncertainties. Certainly, this fact holds for the system without parameter uncertainties, which can also be verified via computer simulations.
6. CONCLUSIONS

In this paper, we have studied the problems of robust stochastic stabilization and robust $H_\infty$ control for uncertain discrete-time stochastic bilinear systems with Markovian switching. Sufficient conditions for the solvability of these problems have been presented and an LMI approach has been developed. An example has been presented to demonstrate the effectiveness of the proposed approach.

APPENDIX A: PROOF OF THEOREM 1

Let

$$P_i = X_i^{-1}, \quad i = 1, 2, \ldots, s$$

Pre- and post-multiplying (11) by $\text{diag}(P_i, I, I, I, I)$ result in

$$
\begin{bmatrix}
-P_i & A_i^T \Omega_i & A_i^T \Omega_i & N_{1i}^T & N_{2i}^T \\
\Omega_i^T A_i & e_1 \Omega_i^T M_i M_i^T \Omega_i - \Gamma & 0 & 0 & 0 \\
\Omega_i^T A_{0i} & 0 & e_2 \Omega_i^T M_i M_i^T \Omega_i - \Gamma & 0 & 0 \\
N_{1i} & 0 & 0 & -e_1 I & 0 \\
N_{2i} & 0 & 0 & 0 & -e_2 I
\end{bmatrix} < 0
$$

(A1)

where

$$\Gamma = \text{diag}(P_1^{-1}, P_2^{-1}, \ldots, P_s^{-1})$$

By the Schur complement formula, it follows from (A1) that there exists a scalar $\delta > 0$ such that

$$A_i^T \Omega_i (\Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i)^{-1} \Omega_i^T A_i + A_{0i}^T \Omega_i (\Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i)^{-1} \Omega_i^T A_{0i}$$

$$+ e_1^{-1} N_{1i}^T N_{1i} + e_2^{-1} N_{2i}^T N_{2i} - P_i + \delta I < 0$$

(A2)

and

$$\Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i > 0, \quad \Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i > 0$$

(A3)

Noting (A3) and applying Proposition 1, we obtain

$$
\sum_{j=1}^s \pi_j A_i(k)^T P_j A_i(k) = [\Omega_i^T A_i + \Omega_i^T M_i F_i(k) N_{1i}]^T \Gamma^{-1} [\Omega_i^T A_i + \Omega_i^T M_i F_i(k) N_{1i}]
$$

$$\leq A_i^T \Omega_i (\Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i)^{-1} \Omega_i^T A_i + e_1^{-1} N_{1i}^T N_{1i}$$

(A4)

and

$$
\sum_{j=1}^s \pi_j A_{0i}(k)^T P_j A_{0i}(k) = [\Omega_i^T A_{0i} + \Omega_i^T M_i F_i(k) N_{2i}]^T \Gamma^{-1} [\Omega_i^T A_{0i} + \Omega_i^T M_i F_i(k) N_{2i}]
$$

$$\leq A_{0i}^T \Omega_i (\Gamma - \delta \Omega_i^T M_i M_i^T \Omega_i)^{-1} \Omega_i^T A_{0i} + e_2^{-1} N_{2i}^T N_{2i}$$

(A5)
Then, from (A3)–(A5), it is easy to see that for $i = 1, 2, \ldots, s$

$$A_i(k)^T Q_i A_i(k) + A_0(k)^T Q_i A_0(k) - P_i + \delta I < 0$$

(A6)

where

$$Q_i = \sum_{j=1}^{s} \pi_{ij} P_j$$

Now, consider the uncertain stochastic system (1) with $u(k) = 0$ and $\nu(k) = 0$, i.e.

$$x(k + 1) = A(r_k, k)x(k) + A_0(r_k, k)x(k)\omega(k)$$

(A7)

Define the following stochastic Lyapunov function candidate for the system in (A7)

$$V_k(x(k), r_k) = x(k)^T P(r_k)x(k)$$

where $P(r_k) = P$, when $r_k = i$ for $i = 1, 2, \ldots, s$. From (A6) we have that for $x(k) \neq 0$ and $i = 1, 2, \ldots, s$

$$\mathbb{E}\{V_{k+1}(x(k+1), r_{k+1})|x_k, r_k = i\} - V_k(x(k), r_k) = \sum_{j=1}^{s} \Pr[r_{k+1} = j|r(k) = i] x(k)^T (A_i(k)^T P_j A_i(k) + A_0(k)^T P_j A_0(k)) x(k) - x(k)^T P_i x(k)$$

$$= x(k)^T [A_i(k)^T Q_i A_i(k) + A_0(k)^T Q_i A_0(k) - P_i] x(k)$$

$$< - \delta |x(k)|^2$$

Therefore,

$$\frac{\mathbb{E}\{V_{k+1}(x(k+1), r_{k+1})|x_k, r_k\} - V_k(x(k), r_k)}{V_k(x(k), r_k)} < - \min_{r_k \in \mathcal{R}} \left( \frac{\delta}{\lambda_{\text{max}}(P(r_k))} \right) = \varepsilon - 1$$

(A8)

where

$$\varepsilon = 1 - \min_{r_k \in \mathcal{R}} \left( \frac{\delta}{\lambda_{\text{max}}(P(r_k))} \right)$$

Note

$$\frac{\mathbb{E}\{V_{k+1}(x(k+1), r_{k+1})|x_k, r_k\} - V_k(x(k), r_k)}{V_k(x(k), r_k)} = \frac{\mathbb{E}\{V_{k+1}(x(k+1), r_{k+1})|x_k, r_k\} - \mathbb{E}\{V_k(x(k), r_k)\}}{V_k(x(k), r_k)}$$

This together with (A8) implies that $\varepsilon > 0$, thus

$$0 < \varepsilon < 1$$

By (A8), we obtain

$$\mathbb{E}\{V_{k+1}(x(k+1), r_{k+1})|x_k, r_k\} < \varepsilon V_k(x(k), r_k)$$

Using this relationship iteratively gives

$$\mathbb{E}\{V_k(x(k), r_k)|x_0, r_0\} < \varepsilon^k V_0(x_0, r_0)$$
and hence
\[
\mathcal{E}\left\{ \sum_{k=0}^{N} V_k(x(k), r_k) \bigg| x_0, r_0 \right\} < (1 + \alpha + \cdots + \alpha^N) V_0(x_0, r_0) < \frac{1}{1 - \alpha} V_0(x_0, r_0)
\]
which implies
\[
\mathcal{E}\left\{ \sum_{k=0}^{N} |x(k)|^2 \bigg| x_0, r_0 \right\} \leq \frac{\min_{i\in\mathbb{I}} \lambda_\min(P_i)}{1 - \alpha} V_0(x_0, r_0) \tag{A9}
\]
By setting
\[
\tilde{M}(x_0, r_0) = \frac{\min_{i\in\mathbb{I}} \lambda_\min(P_i)}{1 - \alpha} V_0(x_0, r_0)
\]
and taking into account (A9), we have
\[
\lim_{N \to \infty} \mathcal{E}\left\{ \sum_{k=0}^{N} |x(k)|^2 \bigg| x_0, r_0 \right\} \leq \tilde{M}(x_0, r_0)
\]
This completes the proof. \hfill \Box

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