Frequency-domain parameter estimation of general multi-rate systems

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Abstract

This paper studies the parameter estimation of a general multi-input, multi-output multi-rate system in the frequency-domain. Two methods, named dividing to subsystems and input extension, are introduced for dealing with multi-rate systems and the later method is easily used to convert a multi-input, multi-output multi-rate system to several sub-problems with fast input updating and slow output sampling. Then all the frequency-domain parameter estimation methods can be applied. Here a least-square parameter estimation method is generalized for parameter estimation in the multi-input, multi-output case. Several examples, including one with real industrial data, are given to show the effectiveness of the methods proposed.

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1. Introduction

In a conventional sampled-data control system, the plant input updating and output sampling are at the same rate. However, it is not always possible to update the control input and sample the output at the same rate due to various limitations such as the cost of fast-rate sensors and actuators. Also sometimes the plant dynamics are such that it is not economical and also not useful to sample the different plant signals at the same rate. As a result, a multi-rate sampling scheme should be considered for such cases. Of course this multi-rate sampling scheme introduces the complication of mixed time steps.

Fig. 1 shows a general multi-input, multi-output (MIMO), multi-rate system, where every input has its own updating rate and every output is sampled at its own rate. Continuous arrows are used for continuous signals and dotted arrows for discrete signals. Here, $P_c$ is a continuous-time plant, $H$ is a multi-rate zero-order hold, and $S$ is a multi-rate output sampling device which can be defined as follows:

\begin{equation}
\mathbb{H} = \begin{pmatrix}
H_{p_1, h} & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
H_{p_n, h} & \cdots & \cdot \\
\end{pmatrix}, \quad \mathbb{S} = \begin{pmatrix}
S_{q_1, h} & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
S_{q_m, h} & \cdots & \cdot \\
\end{pmatrix}
\end{equation}

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These correspond to holding \( m \) input channels of \( u \) with periods \( p_i h, i = 1, \ldots, m \), and sampling \( n \) channels of output \( y_c \) with periods \( q_j h, j = 1, \ldots, n \), respectively. Here, \( p_i \) and \( q_j \) are different integers and \( h \) is a real number called the base period. If we partition the signals accordingly

\[
y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix},
\]

then

\[
u_{ij}(t) = u_i(k), \quad k p_i h \leq t < (k + 1) p_i h, \quad i = 1, \ldots, m,
\]

\[
y_{ij}(k) = y_{c_i}(k q_j h), \quad j = 1, \ldots, n.
\]  
We use \( u \) to denote the fictitious case that all inputs are at the fast-rate, that is \( p_i = 1, i = 1, \ldots, m \). Similarly, \( y \) denotes the output when all outputs are sampled at the fast-rate, or \( q_j = 1, j = 1, \ldots, n \).

Such systems especially arise in the chemical process industry. For example, in polymer reactors (Oshima, Hashimoto, Takeda, Yoneyama, & Goto, 1992), the composition and density measurements are typically obtained after several minutes of analysis, whereas the control inputs can be applied at relatively fast-rate. For another example, we can consider an industrial bleaching process (Han, Shah, Pakpahan, Patwardhan, & Robson, 2004) that is a chemical process applied to cellulose material to increase their brightness and usefulness; in this process, some output variables, like brightness, need laboratory analysis and are in slow rate and are irregularly sampled, while inputs can be applied at relatively fast-rate. One of the problems for such a system is to find the estimation of system parameters and also the output at those time instants when the measurements are not available.

One motivation for this work is output monitoring at fast-rate. Another interesting application is the use of output estimates to run an inferential control scheme, as most inferential control algorithms need the parameters of fast single-rate models, which are not usually available. Some work has been done in this area. The existing multi-rate identification methods can be divided into two main categories: state-space identification and frequency-domain identification. Li, Shah, and Chen (2001) studied the identification of a multi-rate sampled-data system consisting of a continuous-time process with or without time delay, a sampler with period \( n T \) and a zero-order hold with period \( m T (m < n) \) and the problem of identifying a fast-rate model with sampling period \( m T \); the method used is state-space based, employing the lifting technique. Their work is continued by Wang, Chen, and Huang (2004) where a fast-rate model with sampling period \( T \) is extracted. Lu and Fisher (1988, 1989, 1996) and Lu, Fisher, and Shah (1990) studied the parameter and output estimation of dual-rate systems in the frequency-domain; they proposed least-square and projection based algorithms for dual-rate noise-free systems. Ding and Chen (2003) studied the problem for the dual-rate case for stochastic systems.

In this paper, we introduce two simple methods for dealing with general multi-rate systems. These methods are named dividing to subsystems and input extension and are useful in the frequency-domain. In the first method, a multi-rate system can be divided into some dual-rate subsystems and existing estimation methods can be used for parameter estimation of each subsystem; then, parameters of the original system can be extracted. In the second method, a multi-rate system can easily be converted to a dual-rate system with all input updating at fast-rate. A least-square parameter estimation algorithm is derived for such systems.

When system parameters are estimated, we can use it for different applications like inter-sample output estimation as shown in Fig. 2. Here, the process input(s) and sampled output(s) are fed to a parameter estimation engine that produces estimates of the parameters of the assumed model at the fast-rate. Based on estimated parameters and known inputs, output can be estimated at the fast-rate. In Fig. 2, \( \theta \) and \( \hat{\theta} \) are used to show the estimates of parameters and \( y \).

This paper is organized as follows. In Section 2, we consider the problem of transforming a general multi-rate MIMO system to some dual-rate MISO (multi-input, single-output) systems, where each subsystem has all inputs at the fast-rate, and study the parameter estimation of these sub-systems in the frequency-domain. In Section 3, we study the method of extracting a fast-rate model from estimated parameters. Some examples for both SISO and MIMO cases, including a real MIMO example with industrial data, are presented in Section 4. Finally, concluding remarks are given in Section 5.
2. Problem transformation

To estimate a fast-rate model of the system, we assume a model structure for the system and try to estimate the parameters. This model is transformed to some multi-input, single-output, dual-rate subsystems and their parameter estimation is studied in the frequency-domain.

Consider that we have a MIMO multi-rate system as in Fig. 1 and a series of input and output values are given. Then we assume a fast-rate frequency-domain model (transfer function) for this system and want to estimate the model parameters based on these multi-rate data.

Now, suppose that the fast-rate system model with period $h$ in the frequency-domain is $P(z)$, that is

$$y = P(z)u,$$

where $P_{ij}(z) = b_{ij}(z)a_{ij}(z)$

with

$$a_{ij}(z) = 1 + a_{1ij}z^{-1} + a_{2ij}z^{-2} + \cdots + a_{nij}z^{-n},$$

$$b_{ij}(z) = b_{0ij} + b_{1ij}z^{-1} + b_{2ij}z^{-2} + \cdots + b_{nij}z^{-n}.$$ (8)

To deal with this MIMO problem, we can divide it to $n$ MISO subsystems. For each one, we can write

$$y_i(k) = P_{i1}u_1(k) + P_{i2}u_2(k) + \cdots + P_{im}u_m(k).$$ (9)

Without loss of generality, we can assume that all $a_{ij}(z)$ are equal to $a(z)$ with

$$a(z) = 1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_nz^{-n}.$$ (10)

so

$$y_i(k) = \frac{b_{11}(z)}{a(z)}u_1(k) + \frac{b_{12}(z)}{a(z)}u_2(k) + \cdots + \frac{b_{1m}(z)}{a(z)}u_m(k),$$ (11)

or

$$a(z)y_i(k) = b_{11}(z)u_1(k) + b_{12}(z)u_2(k) + \cdots + b_{1m}(z)u_m(k).$$ (12)
This relation, even if available, does not help in the multi-rate problem, because if we expand it in the time domain, we have

\[ y_i(k) = -a_1 y_i(k - 1) - \cdots - a_{i,n} y_i(k - n) + b_{1,1} u_1(k) + b_{1,1} u_1(k - 1) + \cdots + b_{1,1} u_1(k - n) \\
+ \cdots + b_{1,n} u_n(k) + b_{1,n} u_n(k - 1) + \cdots + b_{1,n} u_n(k - n). \quad (13) \]

But, there is no information of \( y_i(k - j), j \neq h_i \), supposing \( k \) is an integer multiple of \( q_i \). To obtain a recursive equation using directly the available multi-rate data, Eq. (13) needs to be transformed into a form so that the \( a(t) \) is a polynomial in \( z^{-1} \) instead of \( z^{-1} \) and \( b(y) \) is a polynomial in \( z^{-1} \). Using properties of zero-order holds and by the method suggested later, \( b(y) \) can still be polynomials in \( z^{-1} \).

For a general discussion, let the roots of \( a(t) \) be \( \lambda_i \) to get

\[ a(t) = \prod_{i=1}^{\hat{\mu}} (1 - \lambda_i z^{-1}). \quad (14) \]

Define

\[ \phi_{y_i}(z) = \prod_{i=1}^{m} \left( 1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{n_i - 1} z^{-n_i+1} \right), \quad (15) \]

then

\[ a(t) \phi_{y_i}(z) = \prod_{i=1}^{\hat{\mu}} (1 - \lambda_i z^{-1}) \prod_{i=1}^{m} \left( 1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{n_i - 1} z^{-n_i+1} \right) \\
= \prod_{i=1}^{\hat{\mu}} (1 - \lambda_i z^{-1}) \prod_{i=1}^{m} \left( 1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{n_i - 1} z^{-n_i+1} \right) \\
= \prod_{i=1}^{\hat{\mu}} (1 - \lambda_i^m z^{-m}) - \lambda_i^{n_i} z^{-n_i+1} + \cdots - \lambda_i^{m-1} z^{-m} \lambda_i^{n_i} z^{-n_i+1} \\
= 1 - (\lambda_i^m + \lambda_i^{m-1} z^{-1} + \cdots + \lambda_i z^{-m} + \lambda_i^{n_i} z^{-n_i+1}) = 1 + a_{i,0} z^{-1} + a_{i,1} z^{-2} + \cdots + a_{i,n_i} z^{-n_i}. \quad (16) \]

But in multiplication of \( b(y) \phi_{y_i}(z) \), generally all the coefficients are nonzero:

\[ b(y) \phi_{y_i}(z) = (b_{1,1} + b_{1,1} z^{-1} + \cdots + b_{1,1}^{n_i} z^{-n_i}) \prod_{i=1}^{\hat{\mu}} (1 + \lambda_i z^{-1} + \lambda_i^2 z^{-2} + \cdots + \lambda_i^{n_i - 1} z^{-n_i+1}) \\
= b_y^{(1)} + b_y^{(2)} z^{-1} + \cdots + b_y^{(n_i)} z^{-n_i}. \quad (17) \]

So, multiplying the numerator and denominator of \( P_{y}(z) \) by \( \phi_{y_i}(z) \) transforms the denominator into the desired form where the denominator is a polynomial in \( z^{-1} \):

\[ P_{y}(z) = b_y(z) \phi_{y_i}(z) \]

This way, Eq. (13) can be written as

\[ y_i(k) = -\sum_{j=1}^{m} \alpha_{i,j} y_i(k - j) + \sum_{j=1}^{m} \sum_{l=0}^{n_i} b_{i,j,l} u_i(k - l). \quad (19) \]

For the input data, we can consider a zero-order hold property. Fig. 3 shows that using the zero-order hold, we have input information to the plant at every \( h \) instant, because the output of zero-order hold remains the same till next update. Using this property, we propose two methods for dealing with multi-rate systems: dividing the multi-rate data to \( \rho \) subsets and extending the input such that input updating rate becomes \( h \).
2.1. Dividing to subsystems

This method can be used to convert a multi-rate system to some dual-rate subsystems. After estimating the parameters of subsystems, parameters of the original multi-rate system can be extracted by simple least-squares methods.

To discuss this method, consider a second-order SISO system with transfer function \( P(z) \):

\[
P(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}
\]

with input updating period as \( 2h \) or \( p = 2 \) and the output sampling period as \( 3h \) or \( q = 3 \). Using the method just discussed, we can find a polynomial \( \phi(z) \) such that:

\[
\phi(z) \equiv \frac{y(k)}{u(k)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_6 z^{-6}}{a_1 z^{-2} + a_2 z^{-3} + a_3 z^{-4} + a_4 z^{-5} + a_5 z^{-6} + a_6 z^{-7}}
\]

Expanding Eq. (21) in the time domain gives:

\[
y(k) = -y(k-3) - y(k-6) + a_0 u(k) + a_1 u(k-1) + \cdots + a_6 u(k-6).
\]

Suppose that a series of input values \( \{u(k), u(k-3), \ldots\} \) and output values \( \{y(k), y(k-4), \ldots\} \) are given. So we can not simply use Eq. (23) as the input values are not given at every time instant, but considering the zero-order hold property we can write:

\[
\begin{align*}
    u(lpq - 1) &= u(lpq - 2), & u(lpq - 3) &= u(lpq - 4), & u(lpq - 5) &= u(lpq - 6), \\
    u(lpq + q) &= u(lpq + q - 1), & u(lpq + q - 2) &= u(lpq + q - 3), & u(lpq + q - 4) &= u(lpq + q - 5).
\end{align*}
\]

So, we can divide the system to two (in general \( p \)) subsystems and write Eq. (23) for each one; one for \( k = lpq + q \) with \( l \) a positive integer. Thus, for subsystem 1 we can write:

\[
y(kpq) + a_1 y(kpq - q) + a_2 y(kpq - 2q)
\]

\[
= b_0 u(kpq) + (b_1 + b_2 u(kpq - 1)) + (b_3 + b_4 u(kpq - 3)) + (b_5 + b_6 u(kpq - 5))
\]

\[
= b_0 u(kpq) + b_{1+1} u(kpq - 1) + b_{1+2} u(kpq - 3) + b_{1+3} u(kpq - 5).
\]

For subsystem 2:

\[
y(kpq + q) + a_1 y(kpq + q - q) + a_2 y(kpq + q - 2q)
\]

\[
= (b_0 + b_1 u(kpq + q)) + (b_2 + b_2 u(kpq + q - 2)) + (b_3 + b_3 u(kpq + q - 4)) + b_6 u(kpq + q - 6)
\]

\[
= b_0 u(kpq + q) + b_{1+1} u(kpq + q - 2) + b_{1+2} u(kpq + q - 4) + b_{1+3} u(kpq + q - 6).
\]
For each of the two subsystems, we can use any parameter estimation method to find an estimation of $a_1$, $a_2$ and $\beta_{i0}$, $\beta_{i1}$, $\ldots$, $\beta_{i2}$. Then using simple least-square method, we can find the estimates of $a_i$ and $\beta_i$.

### 2.2. Input extension

The simpler method given here can easily be used to convert a multi-rate MIMO system to a multi-rate MIMO system with all input updating at the fast rate. Consider Fig. 3 where it shows that we have information of the input at every instant, by simply taking the same input till next update. So if we name slow rate input, updating every $p_i h$ instants as $u_i^s$ and fast-rate input updating every $k$ instants as $u_i^f$, the following relation is held:

$$u_i(k) = u_i^f(i p_i)$$

for $k = p_j, p_j + 1, \ldots, i p_j + p_j - 1$.  

### 2.3. Parameter estimation

When the multi-rate models are transformed properly, any frequency-domain identification method can be used for parameter estimation. Here, we use the dual-rate least-squares method suggested in (Ding & Chen, 2003) and adapt it for MISO systems.

2.3.1. Estimation of $\alpha_i$ and $\beta_i$

To initialize the algorithm, we take

$$\hat{\theta}_i(0) = [a_{i1} \ a_{i2} \ \cdots \ a_{i\alpha} \ \beta_{i1}^T \ \beta_{i2}^T \ \cdots \ \beta_{i\alpha}^T \ \beta_{i\alpha+1}^T \ \cdots \ \beta_{i\beta}^T]$$

and information vector $\psi_i(0)$ are defined by

$$\psi_i(k) = [\psi_{i1}^{T}(k) \ \psi_{i2}^{T}(k) \ \cdots \ \psi_{i\alpha}^{T}(k) \ \psi_{i\beta+1}^{T}(k) \ \cdots \ \psi_{i\beta\alpha}^{T}(k)].$$

Notice that $\hat{\theta}_i$ contains all parameters in the model in (9) to be estimated, and if $k$ is an integer multiple of $q_i$, then $\psi_i(k)$ contains only available data which are the past output measurements (slow-rate) and past and current inputs (fast-rate).

Let $\hat{h}_i(k_0)$ be the estimate of $h_i$ at time $k_0$. The following recursive least-squares algorithm is proposed for estimating the parameter vector $\hat{h}_i$ of the dual-rate system:

$$\hat{h}_i(k_0) = \hat{h}_i(k_0 - q_i) + P_i(k_0)\psi_i(k_0) - \psi_i^{T}(k_0)\hat{h}_i(k_0 - q_i),$$

$$\hat{h}_i(k_0 + l) = \hat{h}_i(k_0), \quad l = 0, 1, \ldots, q_i - 1,$$

$$P_i^{-1}(k_0) = P_i^{-1}(k_0 - q_i) + \psi_i(k_0)\psi_i^{T}(k_0), \quad P_i(0) = P_i^0(0).$$

To initialize the algorithm, we take $P_i(0) = \rho_0 I$ with $\rho_0$, normally a large positive number, and $\hat{h}_i(0) = \hat{h}_i$, some real vector. Notice that the parameter estimate $\hat{h}$ is updated every $q_i$ samples, namely, at the slow rate; so is the matrix $P_i$; in between the slow samples, we simply hold $\hat{h}$ unchanged. It is easy to see that by defining

$$L_i(k_0) := P_i(k_0)\psi_i(k_0) = \frac{P_i(k_0) - \psi_i(k_0)\psi_i^{T}(k_0)}{1 + \psi_i^{T}(k_0)P_i(k_0)\psi_i(k_0)},$$

the covariance matrix $P_i$ can be updated as follows:

$$P_i(k_0) = P_i(k_0 - q_i) - P_i(k_0 - q_i)\psi_i(k_0)\psi_i^{T}(k_0)P_i(k_0 - q_i)$$

$$\quad \cdot \frac{1 + \psi_i^T(k_0)P_i(k_0 - q_i)\psi_i(k_0)}{1 + \psi_i^T(k_0)P_i(k_0)\psi_i(k_0)} = [I - L_i(k_0)\psi_i^T(k_0)P_i(k_0 - q_i)].$$

### 3. Model reconciliation

When a parameter estimation method such as the one in Section 2.3 is applied, $a$ and $\beta$ parameters can be estimated and normally, a fast-rate model as in Eq. (5) is desired. Here we study the problem of extracting $a$ and $\beta$ parameters from estimated $a$ and $\beta$ parameters obtained from $n$ MISO subsystems.

One solution to this problem is using the concept of model order reduction (Glover, 1984), as we want to reduce the order of the model represented by $a$ and $\beta$ parameters. But the model order reduction methods can not guarantee the convergence of parameters.
Actually, for practical cases, when there is no real model of the system and only input–output data are available, we may use the concept of model order reduction to find a reduced order fast-rate model.

Another solution to the problem can be constructed by using Eq. (18). If we denote the estimate of parameters by \( \hat{\beta} \), we can write:
\[
\hat{b}_{ij}(z) \hat{a}(z) = \hat{\beta}_{ij}(z) \hat{\alpha}_i(z) \quad (38)
\]
or
\[
\hat{b}_0^{ij} + \hat{b}_1^{ij} z^{-1} + \cdots + \hat{b}_n^{ij} z^{-n} = \frac{\hat{\beta}_0^{ij} + \hat{\beta}_1^{ij} z^{-1} + \cdots + \hat{\beta}_n^{ij} z^{-n}}{1 + \hat{a}_1^{iz} z^{-1} + \cdots + \hat{a}_n^{iz} z^{-n}} \quad (39)
\]
where \( \hat{\alpha}_i \) and \( \hat{\beta}_{ij} \) are given by the estimation algorithm. By multiplying the polynomials in the nominator and denominator of Eq. (39), we can convert Eq. (38) to set of polynomial equations and the parameters \( \hat{\beta}_i \) and \( \hat{\alpha}_{ij} \) can be extracted. Suppose that the vector of parameters is defined as
\[
\hat{\theta} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b_1^{11} & \cdots & b_1^{nn} & \cdots & b_n^{11} & \cdots & b_n^{nn} \end{bmatrix}^T, \quad (40)
\]
then we can write:
\[
S \hat{\theta} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \Delta = \begin{bmatrix} \hat{\beta}_{11} \\ \hat{\beta}_{12} \\ \vdots \\ \hat{\beta}_{nn} \end{bmatrix} = \rho, \quad (41)
\]
where
\[
S_{ij} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ y_1^{ij} & 1 & \vdots & \vdots \\ y_2^{ij} & y_1^{ij} & 1 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{ij} & y_{n-1}^{ij} & \cdots & 1 \\ 0 & y_n^{ij} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & y_0^{ij} \end{bmatrix}, \quad S_{ji} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (42)
\]
\[
\gamma^{ij} = \begin{cases} 0 & \text{if } i = j, \\ \alpha_k & \text{if } i = j-k \ (k = 1, 2, \ldots, n), \\ 0 & \text{else} \end{cases} \quad (43)
\]
\[
\rho_{ij} = \begin{bmatrix} \hat{\beta}_{11}^{ij} & \hat{\beta}_{12}^{ij} & \cdots & \hat{\beta}_{nn}^{ij} & 0 & \cdots & 0 \end{bmatrix}^T. \quad (44)
\]
The least-squares solution of Eq. (41) is given by
\[
\hat{\theta} = (S^T S)^{-1} S^T \rho. \quad (45)
\]

4. Examples

To show the applicability of the proposed method, three illustrative examples are given, one for the SISO case and two for MIMO systems, including one with real industrial data.

Example 1. Consider a system with the fast-rate transfer function \( P(z) \):
\[
P(z) = \frac{b_2 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{0.412 z^{-1} + 0.309 z^{-2}}{1 - 1.6 z^{-1} + 0.8 z^{-2}} \quad (46)
\]
We consider several sampling cases for this system:

- single-rate case \((p = q = 1)\),
- dual-rate case with output sampling at \(q = 2\),
- multi-rate case with \(p = 2\) and \(q = 3\).

So, for the dual-rate case, we use \(\phi(z) = 1 - a_1z^{-1} + a_2z^{-2}\) to get

\[
P(z) = \frac{b(z)\phi(z)}{\phi(z)} = \frac{0.4120z^{-1} + 0.9682z^{-2} + 0.8240z^{-3} + 0.2472z^{-4}}{1 - 0.96z^{-1} + 0.64z^{-2}}
\]

(47)

We can also use \(\phi(z) = 1 - a_1z^{-1} + (a_1^2 - a_2)z^{-2} - a_1a_2z^{-3} + a_2^2z^{-4}\) for multi-rate case to get

\[
P(z) = \frac{0.412z^{-1} + 0.968z^{-2} - 1.2195z^{-3} - 1.071z^{-4} + 0.659z^{-5} + 0.1978z^{-6}}{1 - 0.2560z^{-1} + 0.5120z^{-2}}
\]

(48)

For dealing with the multi-rate case, both suggested methods are used. To have a fair comparison, we maintain the total data points of input–output data the same for all cases. The relative parameter estimation error (PEE) measured in the Euclidean norm is defined as

\[
P_{\text{EE}}(k) = \frac{\|\hat{\theta}(k) - \theta\|}{\|\theta\|}
\]

(49)

As it seems natural, we obtain a lower PEE for the single-rate case and a higher for the multi-rate case, as there is lack of information in between sampling instants compared to the single-rate case. As the noise is different for different cases, and in order to reduce its effects on comparison, simulations are done several times for each case (100 times in this example) and the mean value of PEE is extracted. To run the simulations, a persistent excitation input sequence, a random binary sequence (RBS) in the frequency range of \(0\) to \(\pi/2\), is applied as an input and an additive white noise with zero mean and variance one is considered at the output. The total number of input–output data is considered to be 2000. The following tables show the results, where SR stands for single-rate, DR for dual-rate, MR1 for the multi-rate case with input extension and MR2 for the multi-rate case of dividing into subsystems.
Table 1: Estimated parameters and PEE% values for Example 2.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>PEE%</th>
</tr>
</thead>
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<td>SR</td>
<td>-1.5077</td>
<td>0.7114</td>
<td>0.4222</td>
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<td>8.3</td>
</tr>
<tr>
<td>DR</td>
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<td>0.6819</td>
<td>0.3971</td>
<td>0.2601</td>
<td>9.5</td>
</tr>
<tr>
<td>MR1</td>
<td>-1.5500</td>
<td>0.7612</td>
<td>0.6530</td>
<td>0.0486</td>
<td>19.33</td>
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<tr>
<td>MR2</td>
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<td>0.5609</td>
<td>0.4257</td>
<td>0.3688</td>
<td>19.59</td>
</tr>
<tr>
<td>True value</td>
<td>-1.6</td>
<td>0.8</td>
<td>0.412</td>
<td>0.309</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 2: Estimated parameters and PEE% values for Example 2.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>PEE%</th>
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<td>0.1657</td>
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<tr>
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<td>0.64</td>
<td>0.4120</td>
<td>0.9682</td>
<td>0.8240</td>
<td>0.2472</td>
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</tr>
</tbody>
</table>

Table 3: Estimated parameters and PEE% values for Example 2.

<table>
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<tr>
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<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
<th>$b_6$</th>
<th>PEE%</th>
</tr>
</thead>
<tbody>
<tr>
<td>MR1</td>
<td>-0.1092</td>
<td>0.4601</td>
<td>0.5712</td>
<td>0.8963</td>
<td>1.2240</td>
<td>1.1271</td>
<td>0.6560</td>
<td>0.1539</td>
<td>18.43</td>
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<tr>
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<td>0.8724</td>
<td>1.3622</td>
<td>1.1358</td>
<td>0.8916</td>
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<td>0.4120</td>
<td>0.9682</td>
<td>1.2195</td>
<td>1.0712</td>
<td>0.6592</td>
<td>0.1978</td>
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</tr>
</tbody>
</table>

Fig. 6. Step responses of the system and the estimated model for Example 2 (for $y_1$).

Fig. 7. Step responses of the system and the estimated model for Example 2 (for $y_2$).
can easily be applied to MIMO systems. Fig. 4 shows the step response of the system and the estimated model using the dividing to

Consider a two-input, two-output system as shown in Fig. 5. Input 1 and output 1 are sampled every 3h period, while input 2 and output 2 are sampled every 2h period, respectively. The following fast-rate model is used for the plant $P$:

$$P = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \left( b_1 z^{-1} + b_1' z^{-2} \right) \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \left( b_1' z^{-1} + b_1 z^{-2} \right)$$

$$= \frac{1}{1 - 1.6z^{-1} + 0.8z^{-2}} \left( 0.412 z^{-1} + 0.309 z^{-2} z^{-2} 0.1 z^{-1} + 0.3 z^{-2} ight)$$

This system is transformed to two two-input, one-output subsystems as follows:

$$y_1 = \frac{b_1 z^{-1} + b_1' z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_1 + \frac{b_1' z^{-1} + b_1 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_2 = \frac{b_1(z)u_1 + b_1'(z)u_2}{a(z)}$$

$$y_2 = \frac{b_2(z)u_1 + b_2'(z)u_2}{a(z)}$$

Now we can use $\phi_1(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - a_1 a_2 z^{-3} + a_2^2 z^{-4}$ for subsystem 1 and $\phi_2(z) = 1 - a_1 z^{-1} + a_2 z^{-2}$ for subsystem 2 to get:

$$y_1 = \frac{b_1(z)\phi_1(z)u_1 + b_1'(z)\phi_1(z)u_2}{a(z)\phi_1(z)} = \frac{\beta_1(z)u_1 + \beta_1'(z)u_2}{a(z)}$$

$$y_2 = \frac{b_2(z)\phi_2(z)u_1 + b_2'(z)\phi_2(z)u_2}{a(z)\phi_2(z)} = \frac{\beta_2(z)u_1 + \beta_2'(z)u_2}{a(z)}$$

Example 2. Consider a two-input, two-output system as shown in Fig. 5. Input $u_1$ is updated every 2h period, while input $u_2$ is updated at the fast-rate. Output $y_1$ and output $y_2$ are sampled every 3h and 2h, respectively. The following fast-rate model is used for the plant $P$:

It can be seen that the two suggested methods for multi-rate systems have very similar results, while the input extension method can easily be applied to MIMO systems. Fig. 4 shows the step response of the system and the estimated model using the dividing to subsystems method.

Fig. 8. Millar Western Bleaching Process (Han et al., 2003).

Now, the suggested least-squares algorithm is applied for each one. To run the simulations, persistent excitation input sequences, random binary sequences (RBS) in the frequency range of $0$ and $1/2\pi$, are applied as inputs and additive white noise with zero mean and variance one is considered at the outputs. Then the average estimation error for 100 simulations for all of the parameters is calculated as $\text{PEE} = 22.81$. Figs. 6 and 7 show the step response of the model and the actual system.

Example 3. The results obtained in this paper can be extended to MIMO cases with irregularly sampled systems. Here, we study a real industrial system shown in Fig. 8. This is an industrial bleaching process at the Millar Western, Alberta. Pulp bleaching is a chemical process applied to cellulose material to increase their brightness, which also increases the capacity of paper for accepting printed or written images and so increases its usefulness. The bleaching process at Millar Western uses hydrogen peroxide as a bleaching agent. In the bleaching process of the pulp mill, the cleaned and filtered pulp is squeezed in presses and heated before entering the bleaching tower $P_1$, where it sits for about one and half hour in a hydrogen peroxide bleach solution. The resulting semi-bleached pulp is de-watered in another press and additional hydrogen peroxide is added in a mixer. This stage takes about three and half to five hours. The pulp is washed and pressed to extract bleach solution (Han et al., 2003). There are nine different inputs consisting peroxide and caustic add rate at $P_1$ and $P_2$, $P_2$ discharge temperature and correction, $\text{Na}_2\text{SO}_3$ and caustic add on chips and PQM freeness. Among the outputs, we use the most important one which is the brightness, just to show the effectiveness of the proposed method. Information of the inputs are ready every 10 min, while the output is sampled completely irregular. To overcome the problem, we use the first-order interpolation to estimate the output in between samples and then re-sample it to obtain a regularly sampled output. Also, as the inputs are constant or with low changing rates, we re-sample the inputs to have a lower input–output
rate ratio. Using this methods, we obtained a nine-input, one-output system with input updating period of 50 min and the output sampling period at 100 min. As the inputs are with slow (some with no) changes and we are in almost steady-state conditions, there is not enough input excitation to find a dynamic model of the system. So only a steady state model (zero-order model) is extracted.

We use the data from April 1st till June 17, 2001 for estimation and data from July 5 to July 30, 2001 for validation. Figs. 9 and 10 show the results for the real output and estimated one for the two different time intervals.

As we do not have the real model of the system, we can not use the PEE as defined earlier as an evaluation benchmark. So to evaluate the model quality, we use the cross-correlation coefficient (CC) and mean squared error (MSE). Both of the benchmarks are based on the measured output ($y$) and the estimated one ($\hat{y}$) and are defined in the following way:

$$CC(y, \hat{y}) = \frac{\text{cov}(y, \hat{y})}{\sqrt{\text{var}(y) \text{var}(\hat{y})}}$$

(53)

$$MSE(y, \hat{y}) = \frac{1}{N} \sum_{i=1}^{N} (y(i) - \hat{y}(i))^2$$

(54)

Using these benchmarks we get $CC = 0.9464$ and $MSE = 0.9484$ for the data in the evaluation time interval. Given the difficulties with the problem (irregularity in output samples and lack of input excitation), we conclude that the model obtained works well.

5. Conclusion

In this paper, parameter estimation methods for general multi-input, multi-output multi-rate systems in the frequency-domain were studied. Two methods for dealing with multi-rate systems were proposed and a least-squares estimation was derived. Simulation examples showed the applicability of the proposed methods for both SISO and MIMO systems.

Acknowledgement

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References


