Technical communique

Norm invariant discretization for sampled-data fault detection

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Abstract

In this paper, the problem of fault detection in sampled-data systems is studied. It is shown that norms of a sampled system are equal to the corresponding norms of a certain discrete time system. Based on this discretization, the sampled-data fault detection problem can be converted to an equivalent discrete-time problem. A framework that unifies the $H_2$ and $H_\infty$ optimal residual generators in sampled-data systems is then proposed.

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1. Introduction

Practical feedback control systems are widely exposed to faults which can cause undesirable performance or even instability. In order to maintain performance of control systems, faults should be promptly detected and identified so that appropriate remedies can be applied. The problem of fault detection and isolation (FDI) has widely been studied in the past decades and numerous design methods are available in the literature (Chen & Patton, 1999; Frank & Ding, 1997; Gertler, 1998). A suitable FDI system compromises between the sensitivity to faults in order to detect incipient faults and the robustness to unknown inputs in order to avoid false alarms.

Nowadays most controllers are implemented on computers. In these types of systems the actual process is a continuous-time system while the controller and the fault detection system are implemented by digital computers.

Fig. 1 illustrates a typical FDI system in a sampled-data scheme where $u(t)$ is the control signal, $y(t)$ the plant output, $d(t)$ the unknown input (disturbance), and $f(t)$ the fault to be detected. The topic of fault detection in sampled-data systems has received extensive attention during the past decade. Many proposed approaches suggest the indirect design in either

- designing a continuous-time FDI for the continuous-time process and then discretizing the FDI and applying it to the real system, or
- discretizing the continuous-time process, and designing a discrete-time FDI based on that, then applying it to the real system.

Approximations exist in both approaches, hence the fault detection system may not work properly. Recently a direct design approach was introduced for fault detection in sampled-data systems (Zhang, Ding, Wang, & Zhou, 2001, 2003; Zhang, Ding, Wang, Zhou, & Ding, 2002). In Zhang et al. (2001–2003) the parity space, $H_\infty$ optimal and $H_2$ optimal (a frequency domain approach) methods were adopted to design residual generators for sampled-data systems. All the methods were based on introducing appropriate operators that capture the intersample behavior which is a well-known technique in designing controllers for sampled-data systems (Chen & Francis, 1995).
All the above methods are successful extensions of the known design techniques to the sampled-data case. Unfortunately, introducing one individual operator for each problem and approach makes those methods complicated and not easy to follow. In this paper, we try to develop a general framework for sampled-data fault detection which offers a convenient tool for both design and analysis. By clearly defining norms of sampled systems and the so-called norm invariant transformation, this framework allows us to easily extend any known ($H^2$ or $H^\infty$) norm-based method of (discrete-time) fault detection to sampled-data systems. An alternative solution to the $H^2$ optimal residual generator for sampled-data systems in Zhang et al. (2003) is proposed by using this framework.

In Section 2, the norms of sampled systems and the norm invariant transformation are defined. In Section 3, the description of the system under consideration is given. A review on residual generation in continuous-time systems is also presented. In Section 4, main results including residual generator design for sampled-data systems based on discretization are developed. Conclusions are drawn in Section 5.

1.1. Notation

- $S$: the ideal sampling operator with sampling period $h$.
- $H$: the (zero-order) hold operator with sampling period $h$.
- $G$: a linear time-invariant strictly proper continuous-time system with impulse response function $g(t)$ and transfer function

$$\hat{g}(s) = C(sI - A)^{-1}B = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$  

- $G_D$: step invariant transformation of $G$ (Chen & Francis, 1995) with impulse response function $g_D(k)$ and transfer function

$$\hat{g}_D(z) = \begin{bmatrix} A_D & B_D \\ C & 0 \end{bmatrix}.$$  

where $A_D = e^{Ah}$ and $B_D = \int_0^h e^{At}B \, d\tau$.

- $G_J$: norm invariant transformation of $G$ (to be defined later) with impulse response function $g_J(k)$ and transfer function

$$\hat{g}_J(z) = \begin{bmatrix} A_D & B_J \\ C & 0 \end{bmatrix},$$  

where $B_J$ satisfies

$$B_J B_J^T = \int_0^h e^{A^T}BB^T e^{A^T} \, d\tau.$$  

2. Norms of sampled systems

In a variety of fault detection methods, a suitably chosen norm is used to design and analyze residual generators. In sampled-data systems, norms are also required to extend the known design techniques. To define appropriate norms for sampled systems we try to generalize the concepts of $H^2$-norm and $H^\infty$-norm. Let us assume that $G : \mathcal{L}_2(\mathbb{R}) \rightarrow \mathcal{L}_2(\mathbb{R})$ is a stable and strictly proper continuous-time system with $p$ inputs and $m$ outputs, and

$$\hat{g}(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$  

The sampled system $SG : \mathcal{L}_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{Z})$ maps continuous-time signals to discrete-time signals.

2.1. $H^\infty$-norm of $SG$

For system $G$, the $H^\infty$-norm is

$$\|\hat{g}(s)\|_\infty = \sup_{\|u\|_2 \leq 1} \|Gu\|_2.$$  

Similarly, the $H^\infty$-norm (also known as induced norm) of $SG$ is defined to be (Chen & Francis, 1995)

$$\|SG\|_\infty = \sup_{\|u\|_2 \leq 1} \|SGu\|_2.$$  

To compute $\|SG\|_\infty$ the following lemma is useful:

**Lemma 1** (Chen & Francis, 1995; Zhang et al., 2002). The $H^\infty$-norm of $SG$ is given by

$$\|SG\|_\infty = \|\hat{g}_J(z)\|_\infty.$$
2.2. \(H_2\)-norm of \(SG\)

For a SISO system, the \(H_2\)-norm is

\[
\|\hat{g}(s)\|_2^2 = \|g(t)\|_2^2 = \int_{-\infty}^{\infty} g(t)^2 \, dt,
\]

i.e., the \(H_2\)-norm of the transfer function \(\hat{g}(s)\) equals the \(L_2\)-norm of its impulse response. In the multivariable case, the \(H_2\)-norm is

\[
\|\hat{g}(s)\|_2^2 = \sum_{i=1}^{p} \|G\delta(t)e_i\|_2^2,
\]

where \(e_i, i = 1, \ldots, p\), denote the standard basis vectors in \(\mathbb{R}^p\) and \(\delta(t)\) denotes the unit impulse function. Thus, \(\delta(t)e_i\) is an impulse applied to the \(i\)th input channel.

To generalize the definition of \(H_2\)-norm to sampled systems, notice that \(SG\) is a time-varying but \(h\)-periodic system. Therefore, we define the \(H_2\)-norm of \(SG\) as the total energy of the outputs when impulses are applied in one period (sampling interval) to the input channels, i.e.,

\[
\|SG\|_2^2 = \sum_{i=1}^{p} \left( \int_0^h \|SG\delta(t) e_i\|_2^2 \, dt \right).
\]  

(1)

Similar to the \(H_\infty\)-norm, the \(H_2\)-norm of \(SG\) is related to the \(H_2\)-norm of discrete-time system \(G_J\) as shown in the following lemma:

Lemma 2. The \(H_2\)-norm of \(SG\) is given by

\[
\|SG\|_2 = \|\hat{g}_J(z)\|_2.
\]

Proof. We know that

\[
G\delta(t) = g(t) = Ce^{At}B1(t),
\]

where \(1(t)\) denotes the unit step function. We have

\[
G\delta(t - \tau) = Ce^{A(t-\tau)}B1(t - \tau),
\]

\[
SG\delta(t - \tau) = \{0, Ce^{A(h-\tau)}B, \ldots, Ce^{A(kh-\tau)}B, \ldots\},
\]

\[
\sum_{i=1}^{p} \|SG\delta(t - \tau)e_i\|_2^2 = \operatorname{tr} \left( \sum_{k=0}^{\infty} Ce^{A(kh-\tau)}B B^T e^{A^T(kh-\tau)} C^T \right),
\]

\[
\int_0^h \left( \sum_{i=1}^{p} \|SG\delta(t - \tau)e_i\|_2^2 \right) \, d\tau = \operatorname{tr} \left( \sum_{k=0}^{\infty} Ce^{Akh} \left( \int_0^h e^{-A\tau} B B^T e^{-A^T\tau} \, d\tau \right) e^{A^Tk\tau} C^T \right).
\]

On the other hand, we have (Chen & Francis, 1995)

\[
\|\hat{g}_J(z)\|_2 = \operatorname{tr} \left( \sum_{k=0}^{\infty} CA_k^T B_J B_J^T (A_k^T)^4 C^T \right).
\]

(2)

Comparing (2) and (3) and using definition (1) complete the proof. \(\square\)

2.3. Norm invariant transformation

Lemmas 1 and 2 show that the norm of the sampled system \(SG\) is equal to the norm of certain discrete-time system \(G_J\). Since this specific type of discretization preserves the \(H_2\)- and \(H_\infty\)-norms of the sampled system, we call it norm invariant transformation. Note that the inputs of the discrete-time system \(G_J\) are not related to the actual inputs of the original continuous-time system \(G\). It is also important that the number of independent inputs of \(G_J\) is greater than or equal to the number of independent inputs of \(G\) (Zhang et al., 2001, 2002). In other words, \(G_J\) introduces some fictitious inputs which are used for design purposes only and have no physical meaning.

3. Problem definition and review

3.1. System description

The continuous-time system under consideration has the following input–output description:

\[
y(t) = Gu(t) + GDd(t) + Gf f(t),
\]

(4)

where \(y \in \mathbb{R}^m\) is the vector of plant outputs, \(u \in \mathbb{R}^n\) the vector of control signals, \(d \in \mathbb{R}^d\) the vector of unknown inputs (disturbances) and \(f \in \mathbb{R}^f\) the vector of faults to be detected. \(G_u\), \(G_d\) and \(G_f\) are linear time-invariant strictly proper systems.

In a sampled-data control scheme, the output vector is sampled and discretized using an A/D converter modeled by

\[
\psi(k) = y(kh),
\]

(5)

and the control signal is generated by a computer and sent to the actuator using a zero-order hold D/A converter modeled by

\[
u(t) = v(k), \quad kh \leq t < (k + 1)h.
\]

Hence, \(\psi(k) = Sy(t)\) and \(u(t) = Hv(k)\).
3.2. Residual generator for continuous-time systems

A residual generator in a fault detection system uses the process input and output to generate a residual signal. The control literature is quite rich of different residual generation schemes and design approaches (Chen & Patton, 1999; Frank & Ding, 1997; Frank, Ding, & Marcu, 2000; Gertler, 1998). We use the well-known factorization approach here (Frank & Ding, 1994; Frank et al., 2000).

For the continuous-time system in (4), an LTI residual generator can be constructed as

\[ r_c(t) = R(P_u y(t) - Q_u u(t)) \]
\[ = RP_u G_d d(t) + RP_u G_f f(t), \]

where \( \hat{r}(s) \in \mathcal{R}_\infty \) is a designable post-filter and \( P_u, Q_u \) are a left coprime factorization of \( G_u \) satisfying

\[ \hat{g}_u(s) = \hat{p}_u(s)^{-1} \hat{q}_u(s). \]

If a post-filter \( R \) can be found such that

\[ RP_u G_d \equiv 0, \]
\[ RP_u G_f \neq 0, \] (7)

then perfect decoupling of the residual signal from the unknown inputs is possible. Otherwise in order to compromise between the sensitivity of the residual signal to the faults and its robustness to the unknown inputs, an optimization becomes necessary. As a widely accepted approach the following optimization problem is considered:

\[ \min_{\hat{r}(s) \in \mathcal{R}_\infty} \| \hat{r}(s) \hat{p}_u(s) \hat{g}_d(s) \|_\eta. \]

For \( \eta = 2 \) or \( \eta = \infty \) the analytical solutions are known in both continuous-time and discrete-time cases (Ding & Frank, 1989; Ding, Jeinsch, Frank, & Ding, 2000; Frank & Ding, 1994).

4. Fault detection in sampled-data systems

4.1. Residual generator

In the sampled-data scheme, the residual generator uses the discrete-time process input \( v(k) \) and output \( \psi(k) \) to generate the residual signal. Using the factorization approach the residual generator is

\[ \rho(k) = R(M_u \psi(k) - N_u v(k)). \] (8)

\( M_u, N_u \) are a left coprime factorization of \( G_{ud} \) satisfying

\[ \hat{g}_{ud}(z) = \hat{m}_u(z)^{-1} \hat{n}_u(z). \]

Substituting \( \psi(k) = S_y(t), u(t) = H v(k) \) and the system model (4) in (8), the dynamics of the residual generator with respect to the continuous-time signals \( d(t) \) and \( f(t) \) is

\[ \rho(k) = R M_u S_d d(t) + R M_u S_f f(t). \] (9)

Here \( R \) and \( M_u \) are discrete-time systems while \( G_d \) and \( G_f \) are continuous-time ones. \( R M_u S_d \) and \( R M_u S_f \) are two operators that map continuous-time signals to discrete-time signals. Eq. (9) shows how continuous-time signals \( d(t) \) and \( f(t) \) affect the discrete-time residual signal \( \rho(k) \). If a discrete-time post-filter \( R \) can be found such that

\[ R M_u S_d \equiv 0, \]
\[ R M_u S_f \neq 0, \]

then perfect decoupling is achievable. Otherwise the following optimization problem is considered

\[ \min_{\hat{r}(z) \in \mathcal{H}_\infty} J_\eta = \min_{\hat{r}(z) \in \mathcal{H}_\infty} \| R M_u S_d \|_\eta, \]

where \( \eta = 2 \) or \( \eta = \infty \).

4.2. Optimization using norm invariant transformation

The norm preserving property of the norm invariant transformation makes it appropriate to approach the optimization problem given in (10). Similar to Lemmas 1 and 2, the following theorem can be proved:

**Theorem 3.** For the sampled-data residual generator given in (9), the following equations hold:

\[ \| R M_u S_d \|_\eta = \| \hat{r}(z) \hat{m}_u(z) \hat{g}_d(z) \|_\eta, \]
\[ \| R M_u S_f \|_\eta = \| \hat{r}(z) \hat{m}_u(z) \hat{g}_f(z) \|_\eta, \]

for \( \eta = 2 \) and \( \eta = \infty \).

The theorem suggests that to design a residual generator for the sampled-data system in (4)–(6), as long as we are concerned about the norms of the operators relating the fault and disturbance signals to the residual signal, we can replace the sampled-data system with the following discrete-time system:

\[ \psi(k) = G_{ud} \psi(k) + G_d \hat{g}_d(k) + G_f \hat{g}_f(k). \] (11)

Any optimal (\( \mathcal{H}_2 \) or \( \mathcal{H}_\infty \)) norm-based residual generator designed for this discrete-time system will be optimal for the original sampled-data system as well. Therefore, to design the optimal discrete-time post-filter \( R \) in (10), we can solve the following discrete-time optimization problem:

\[ \min_{\hat{r}(z) \in \mathcal{H}_\infty} J_\eta = \min_{\hat{r}(z) \in \mathcal{H}_\infty} \| \hat{r}(z) \hat{m}_u(z) \hat{g}_d(z) \|_\eta. \] (12)

For \( \eta = \infty \) this is the same result as given in Zhang et al. (2002). For \( \eta = 2 \), this provides an alternative solution to the optimal design given in Zhang et al. (2003). It can be shown that the performance index in (12) is same as the one...
A necessary condition for (14) is

\[
\hat{g}_d(j\omega + jk\omega_s)\hat{g}_f^T(-j\omega - jk\omega_s).
\]

Here \(\omega_s = 2\pi/h\). Eq. (13) can be easily shown by using the results given in Hagiwara and Araki (1995) and the well-known Poisson sampling formula (Braslavsky, Meinsma, Middleton, & Freudenberg, 1997).

4.3. Perfect disturbance decoupling

The perfect disturbance decoupling in continuous-time is achievable if a post-filter \(R\) can be found such that (7) holds. The necessary and sufficient condition for perfect disturbance decoupling is (Frank & Ding, 1994)

\[
\text{rank } [\hat{g}_d(s) \hat{g}_f(s)] > \text{rank } [\hat{g}_d(s)].
\]

A necessary condition for (14) is

\[
\text{rank } [\hat{g}_d(z)] < m,
\]

which means that for perfect disturbance decoupling, the number of independent inputs should be less than the number of measurements (Frank & Ding, 1994; Frank, J., et al., 2000).

Using the norm invariant transformation, the necessary and sufficient condition for perfect disturbance decoupling in sampled-data system is obtained from the equivalent discrete-time model in (11)

\[
\text{rank } [\hat{g}_{dJ}(z) \hat{g}_{fJ}(z)] > \text{rank } [\hat{g}_{dJ}(z)].
\]

Since \(\hat{g}_{dJ}(z)\) has more inputs than \(\hat{g}_d(s)\), perfect disturbance decoupling is more difficult in the sampled-data case than in the continuous-time case (Zhang et al., 2001–2003).

5. Conclusion

In this contribution, the fault detection problem for sampled-data systems was studied. We showed that in order to design a norm-based residual generator, it is enough to replace the sampled-data system with a certain discrete-time system. Any optimal norm-based residual generator for this discrete-time system will be optimal for the original sampled-data system as well. This result was obtained based on introducing the norm invariant transformation, a method of discretization that preserves the norms of sampled systems. The proposed framework unifies the \(\mathcal{H}_2\) (via an alternative solution) and \(\mathcal{H}_\infty\) optimal design in sampled-data FDI.

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References


