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This paper investigates the event-driven sampled-data consensus in switching networks of multiple integrators and studies both the bidirectional interaction and leader-following passive reaction topologies in a unified framework. In these topologies, each information link is modelled by an edge of the information graph and assigned a sequence of edge events, which activate the mutual data sampling and controller updates of the two linked agents. Two kinds of edge-event-detecting rules are proposed for the general asynchronous data-sampling case and the synchronous periodic event-detecting case. They are implemented in a distributed fashion, and their effectiveness in reducing communication costs and solving consensus problems under a jointly connected topology condition is shown by both theoretical analysis and simulation examples.

Keywords: sampled-data consensus; switching topologies; event-driven data sampling; asynchronous data sampling

1. Introduction

Sampled-data consensus originated as an interesting research topic in the distributed control of large-scale networks of multiple agents, in which each individual agent usually has limited data-collecting and communicating capacities. Its research focuses involve the design of consensus protocols (distributed control laws) to drive the concerned states of agents to an agreement based on local sampled information of their neighbours. Indeed, in networks with digital information channels and restricted bandwidth, sampled-data techniques offer many benefits, such as lower communication costs, satisfactory control accuracy/performance, and robustness against time delays and noise effect (Chen & Francis, 1995).

In most of the previous results, the implementation of sampled-data consensus protocols is based on periodic sampling and zero-order hold devices. One such example was given in Xie, Liu, Wang, and Jia (2009a, 2009b), where a synchronous protocol was given and sufficient conditions in terms of the length of sampling periods were presented for the state consensus of single-integrator networks with fixed and switching topologies. By tools of non-negative matrix theory and linear matrix inequalities (LMIs), the same type of data-sampling mechanism was investigated in Cao and Ren (2010), Gao and Wang (2010a), Qin, Zheng, and Gao (2010), and Qin, Gao, and Zheng (2012), in consensus control of double-integrator networks. The synchronous periodic sampling assumption facilitates theoretical analysis, but it is not realistic in a distributed network without any central agent. Furthermore, synchronous data sampling would burden information networks with package loss and time delays. Therefore, asynchronous data sampling is of particular interest in distributed consensus and has been attracting the attention of many researchers. In Cao, Morse, and Anderson (2008), they investigated an asynchronous sampled-data version of the Vicsek model, where each agent samples the heading of its neighbours at some discrete times and changes its heading from one-way point to the other in a monotonic and piecewise-continuous manner. These sampling events are time-driven and evenly spaced. Another version of asynchronous sampled-data consensus protocol was presented in Xiao and Wang (2008), and it requires that the data exchange and parameter adjustment occur in a time-driven manner with bounded sampling periods. There are also several publications on asynchronous sampled-data consensus analysis in double-integrator networks; interested readers may refer to Gao and Wang (2010b, 2011).

On the other hand, designing event-based consensus protocols is another effective solution to relieve the network burden in communication costs. Furthermore, compared with the time-driven data sampling, a better convergence performance with lower average data-sampling frequency is more likely to be obtained. Examples preferring event-based techniques can be found in Åström and Bernhardsson (2002). In Dimarogonas, Frazzoli, and Johansson (2012), they designed several event-driven protocols for the first-order consensus problem with a reduced number of actuator updates. The actuator updates depend on the ratio of a certain measurement error with respect to the norm of a state

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function. There are also event-based results, derived by a self-triggered approach with the aim to remove continuous monitoring of measurement errors and to reduce communications (Dimarogonas et al., 2012), and event-based results on general linear subsystems (Zhang, Hao, Zhang, & Wang, 2014; Zhu, Jiang, & Feng, 2014).

In this paper, we develop an edge-event-based theory for distributed sampled-data consensus. In the information graph, each edge models an information link connecting a pair of neighbouring agents and is assigned a sequence of edge events. If these events occur, the data exchange over the associated information link is executed and the controllers of the two linked agents are updated. To decrease unnecessary power consumption by inter-agent communication, edge events are defined as that the state displacement of any of the two linked agents goes beyond an interval decided by the prior sampled state difference. The similar event-based agent-to-agent data-sampling policies were adopted in Zhong and Cassandras (2010) to solve an asynchronous distributed optimisation problem on a complete graph, and in Meng and Chen (2014) to solve the state consensus of multiple agents with the same linear dynamics on a time-invariant graph. The occurrence of this kind of events would activate the communication devices of the corresponding agents with all their neighbours. In the edge-event-triggered scheme of our work, agent-to-agent communications are treated independently for different pairs of neighbouring agents and they are only triggered when necessary. Thus, the communication cost is expected to be reduced tremendously. This advantage was indicated in our preliminary result in Xiao, Meng, and Chen (2012) and will be also shown in our theoretical analysis. Furthermore, this paper unifies the non-reciprocal detailed balance networks and leader-following networks in one framework and presents convergence results under relaxed conditions of switching topologies. These results significantly expand our previous result on the average event-based sampled-data consensus in undirected networks (Xiao et al., 2012) and are also applicable in the traditional periodic systems of sampled-data consensus. It should be noted that, in Meng and Chen (2014), only the undirected time-invariant topology was considered by LMIs and agent-to-agent communications were always required at every event-detecting step.

We formulate the problem in Section 3, following the presentation of some preliminary notions in Section 2. Then, in Section 4, we propose a set of event-detecting rules for the general asynchronous data sampling, which introduce n switching variables to eliminate the Zeno behaviour of edge events. In Section 5, we restrict the event-detecting actions on some periodic instants and present a novel periodic-like event-triggered principle, which provides a method of decreasing the number of data sampling in the periodic sampled-data control. Simulation examples are given in Section 6. Finally, the paper is concluded in Section 7. Some preliminary lemmas are attached in the appendix.

2. Preliminaries

This section gives some basic notions in graph theory. Given an *undirected* simple graph $\mathcal{G}$ with vertex $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and edge set $\mathcal{E}$, a path in $\mathcal{G}$ from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ of finite vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}$ for $j = 1, 2, \dots, k-1$. Graph $\mathcal{G}$ is called *connected* if for any two vertices, there always exists a path connecting them. Let m denote the number of edges in graph $\mathcal{G}$. Label the m edges with 1 through m and assign each edge an arbitrary orientation. Then, for the assigned orientations, the n-by-m incidence matrix $D = [d_{ij}]$ is defined by

$$d_{ij} = \begin{cases} -1, & \text{if } v_i \text{ is the tail of the } j\text{th oriented edge,} \\ 1, & \text{if } v_i \text{ is the head of the } j\text{th oriented edge,} \\ 0, & \text{otherwise.} \end{cases}$$

If we associate $\mathcal{G}$ with a symmetric non-negative matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $(v_i, v_j) \in \mathcal{E}$, if and only if $a_{ij} \neq 0$, then we get a weighted undirected graph $\mathcal{G}(A)$, and a_{ij} is called the *weight* of edge (v_i, v_j). In what follows, we will simply use 'graph' instead of 'undirected graph' or 'weighted graph' if it is clear from the context. The *graph Laplacian* of $\mathcal{G}(A)$, denoted by $L(A)$, is defined by $L(A) = DWD^T$, where $W = [w_{ij}] \in \mathbb{R}^{m\times m}$ is a diagonal matrix with w_{ii} equal to the weight of the ith edge, $i = 1, 2, \dots, m$ (Mesbahi & Egerstedt, 2010). In particular, the *graph Laplacian* of $\mathcal{G}$ is defined by $L = DD^T$, which can be seen as a special case of $L(A)$ if we treat 1 as the common weight of all edges (Godsil & Royal, 2001).

For a group of graphs $\mathcal{G}_i$, $i \in \mathcal{I}$, with a common vertex set, the union graph of them is the graph on their shared vertex set with edge set given by the union of their edge sets, where $\mathcal{I}$ is the index set of the group.

3. Problem formulation

The system studied in this paper consists of n single integrators. Label these agents with 1 through n and let $x_i(t) \in \mathbb{R}$ denote the state of agent i, $i = 1, 2, \dots, n$. The dynamics of each agent is given by the following equation:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, n, \tag{1}$$

where $u_i(t)$ is a state feedback, called *protocol*, to be designed based on the state information received by agent i from its neighbours.

Suppose the interaction among agents is bidirectional and we use a time-varying weighted undirected graph $\mathcal{G}(A(t))$ with vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and edge set $\mathcal{E}(t)$ to model the topology of information links. $A(t) = [a_{ij}(t)]$ is an n-by-n piecewise constant non-negative symmetric matrix which belongs to a finite set and has all diagonal entries equal to zero. Vertices $v_1, v_2, \ldots, v_n$ model the n integrators. An edge $(v_i, v_j) \in \mathcal{E}(t)$ implies the existence of an available information link between agents i and j at time t, and $a_{ij}(t)$ is the *weighting factor* of the link (Ren & Beard, 2005). At any time t, define all the agents that are connected with agent i by edges as the *neighbours* of agent i, indexed by $\mathcal{N}_i(t)$. Mathematically, $\mathcal{N}_i(t) = \{j : (v_i, v_j) \in \mathcal{E}(t)\}$.

For each possible information link (v_i, v_j), we will give several event-detecting rules to generate a sequence of discrete times $t_0^{ij}, t_1^{ij}, t_2^{ij}, \ldots$, with the property that $t_k^{ij} = t_k^{ji}$ and $t_k^{ij} < t_{k+1}^{ij}$, $k = 0, 1, 2, \ldots$. At these times, agents i and j detect the availability of information link (v_i, v_j), and if it is available, agents i and j exchange the relative states between them and update their controllers simultaneously. We call the above events generating $t_0^{ij}, t_1^{ij}, t_2^{ij}, \ldots$ the *edge events* of (v_i, v_j). By a little abuse of notation, denote $k^{ij}(t) = \max\{k : t_k^{ij} \le t\}$, which indexes the most recent event time of (v_i, v_j) up to t. Clearly, $k^{ij}(t) = k^{ji}(t)$ for all t. Then, the protocol is given as follows[1]:

$$u_i(t) = \omega_i \sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})} s_{ij}\big(t^{ij}_{k^{ij}(t)}\big)a_{ij}\big(t^{ij}_{k^{ij}(t)}\big) \times \big(x_j\big(t^{ij}_{k^{ij}(t)}\big) - x_i\big(t^{ij}_{k^{ij}(t)}\big)\big),\ i = 1, 2, \ldots, n. \quad (2)$$

In the above protocol, $\omega_i \ge 0$ is a factor, weighting the tendency of agent i to state the change or inverse of some kind of reluctance or inertia of agent i. These weighting factors may be different for different agents and imply the *non-reciprocal* interaction among agents. Obviously, if $\omega_i = 0$, then agent i will never change its state, and thus it will play the role of a leader. This paper assumes that there exists *at most one agent* with ω_i equal to 0. $s_{ij}(\cdot) \in \{0, 1\}$ is a switching variable with $s_{ij}(t) = s_{ji}(t)$ for any t, and it is used to eliminate the Zeno behaviour of edge events.

Remarks:

(1) If we view $\omega_i a_{ij}(t)$ as the link weighting factor from v_j to v_i, the topology of information links becomes a directed graph, and if $\omega_i \ne 0$ for all i, then the directed graph satisfies *the detailed balance condition* (Haken, 1978). Such topologies were studied in the context of *non-reciprocal* swarming in Chu, Wang, Chen, and Mu (2006).

(2) If $t_k^{ij} = t_k^{i'j'}$ and $t_{k+1}^{ij} = t_k^{ij} + h$, $k = 0, 1, 2, \ldots$, hold for some $h > 0$ and all possible edges $(v_i, v_j), (v_{i'}, v_{j'})$, the system under protocol (2) becomes the typical periodic sampled-data consensus model studied in Xie et al. (2009a, 2009b).

Define variable[2]

$$\kappa(t) = \begin{cases} \frac{\sum_{i=1}^n \frac{x_i(t)}{\omega_i}}{\sum_{i=1}^n \frac{1}{\omega_i}}, & \text{if } \omega_i \ne 0 \text{ for all } i,\\ x_i(t), & \text{if } \omega_i = 0 \text{ for some } i, \end{cases}$$

where the first equation is a weighted average of agents' states. Under protocol (2), if there exists one agent i with $\omega_i = 0$, then $\frac{d\kappa(t)}{dt} = u_i(t) = 0$; otherwise, by the same arguments as in proving Equation (3) in Xiao et al. (2012), we have that

$$\begin{aligned}\frac{d\kappa(t)}{dt} =& \frac{1}{\sum_{i=1}^n \frac{1}{\omega_i}} \sum_{i=1}^n \sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})} s_{ij}\big(t^{ij}_{k^{ij}(t)}\big)a_{ij}\big(t^{ij}_{k^{ij}(t)}\big)\\ &\times \big(x_j\big(t^{ij}_{k^{ij}(t)}\big) - x_i\big(t^{ij}_{k^{ij}(t)}\big)\big)\\ =& \frac{1}{2\sum_{i=1}^n \frac{1}{\omega_i}} \sum_{i=1}^n \sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})} \Big(s_{ij}\big(t^{ij}_{k^{ij}(t)}\big)a_{ij}\big(t^{ij}_{k^{ij}(t)}\big)\\ &\times \big(x_j\big(t^{ij}_{k^{ij}(t)}\big) - x_i\big(t^{ij}_{k^{ij}(t)}\big)\big)\\ &+ s_{ji}\big(t^{ji}_{k^{ji}(t)}\big)a_{ji}\big(t^{ji}_{k^{ji}(t)}\big)\big(x_i\big(t^{ji}_{k^{ji}(t)}\big) - x_j\big(t^{ji}_{k^{ji}(t)}\big)\big)\Big)\\ =& 0. \end{aligned} \quad (3)$$

In what follows, notation κ instead of $\kappa(t)$ will be used. Therefore, if the states of agents converge to a common value as time goes on, then the common value should be κ defined above. Denote

$$x_i(t) = \kappa + \delta_i(t), \quad i = 1, 2, \ldots, n.$$

In consensus control, variables $\delta_i(t)$, $i = 1, 2, \ldots, n$, measure the differences between agent states and their common final state, and these variables were referred to as the *group disagreements* in Olfati-Saber and Murray (2004). It follows from Equation (3) that

$$\dot{\delta}_i(t) = u_i(t).$$

Specially, in the leader-following case, if $\omega_i = 0$, then $u_i(t) \equiv 0$ and $\delta_i(t) = \dot{\delta}_i(t) \equiv 0$.

Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^n \frac{1}{\omega_i}\delta_i(t)^2,$$

where we set $\frac{1}{0}0 = 0$ to unify the bidirectional non-reciprocal case and leader-following case in one equation. Obviously, $\lim_{t\to\infty} V(t) = 0$, if and only if the states of agents converge to κ as time increases; in other words, $\lim_{t\to\infty} V(t) = 0$ implies that system (1) solves *a consensus problem*.

4. Asynchronous sampled-data consensus driven by edge events

Differentiating $V(t)$ with respect to t gives

$$\begin{aligned}\frac{dV(t)}{dt} &= \sum_{i=1}^{n}\frac{1}{\omega_i}\delta_i(t)u_i(t)\\ &= \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})}\delta_i(t)s_{ij}\left(t^{ij}_{k^{ij}(t)}\right)a_{ij}\left(t^{ij}_{k^{ij}(t)}\right)\\ &\quad\times\left(x_j\left(t^{ij}_{k^{ij}(t)}\right)-x_i\left(t^{ij}_{k^{ij}(t)}\right)\right)\\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})}\Big(\delta_i(t)s_{ij}\left(t^{ij}_{k^{ij}(t)}\right)a_{ij}\left(t^{ij}_{k^{ij}(t)}\right)\\ &\quad\times\left(x_j\left(t^{ij}_{k^{ij}(t)}\right)-x_i\left(t^{ij}_{k^{ij}(t)}\right)\right)\\ &\quad+\delta_j(t)s_{ji}\left(t^{ji}_{k^{ji}(t)}\right)a_{ji}\left(t^{ji}_{k^{ji}(t)}\right)\left(x_i\left(t^{ji}_{k^{ji}(t)}\right)-x_j\left(t^{ji}_{k^{ji}(t)}\right)\right)\Big)\\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})}s_{ij}\left(t^{ij}_{k^{ij}(t)}\right)a_{ij}\left(t^{ij}_{k^{ij}(t)}\right)\\ &\quad\times\left(\delta_i(t)-\delta_j(t)\right)\left(x_i\left(t^{ij}_{k^{ij}(t)}\right)-x_j\left(t^{ij}_{k^{ij}(t)}\right)\right)\\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(t)})}s_{ij}\left(t^{ij}_{k^{ij}(t)}\right)a_{ij}\left(t^{ij}_{k^{ij}(t)}\right)\\ &\quad\times\left(x_i(t)-x_j(t)\right)\left(x_i\left(t^{ij}_{k^{ij}(t)}\right)-x_j\left(t^{ij}_{k^{ij}(t)}\right)\right).\end{aligned}\tag{4}$$

It can be observed from Equation (4) that if we can ensure that for any two neighbouring agents i and j, $x_i(t)-x_j(t)$ and $x_i(t^{ij}_{k^{ij}(t)})-x_j(t^{ij}_{k^{ij}(t)})$ share the same sign, then $V(t)$ will be non-increasing, which will be useful to guarantee consensus. Motivated by this observation, we propose the following event-detecting rules for agent i to determine the edge events of (v_i, v_j) with its neighbouring agent j:

(A1) *Agents i and j initialise an edge event of* (v_i, v_j) *when information link* (v_i, v_j) *is first available, set* t^{ij}_0 *to be that time, and set* $s_{ij}(t^{ij}_0)=1$*;* $s_{ij}(t)$ *will remain constant until a new value is given*;

for agent i and event time t^{ij}_k, $k=0,1,2,\ldots$ (agents i and j follow the same rule and collaborate in determining the edge events of (v_i, v_j)),

(A2) *If* $(v_i, v_j)\notin\mathcal{E}(t^{ij}_k)$ *(which means that* $a_{ij}(t^{ij}_k)=0$*; i.e., the data sampling is temporarily unavailable), then the controller update of agent i (as well as agent j) is still required; the next edge-event time* t^{ij}_{k+1} *is scheduled to be the earliest time when information link* (v_i, v_j) *is available in the time interval* $[\max\{t^{ij}_k, t^{ij}_{k-1}+T_s\},\infty)$ *in the case with* $s_{ij}(t^{ij}_k)=1$ *or in the time interval* $[t^{ij}_k+T_s,\infty)$ *in the case with* $s_{ij}(t^{ij}_k)=0$*; and set* $s_{ij}(t^{ij}_{k+1})=1$ *in the latter case, where* T_s *is a given positive real number; otherwise, execute rules (A3) and (A4):*

(A3) *If* $s_{ij}(t^{ij}_k)=0$ *or* $x_i(t^{ij}_k)-x_j(t^{ij}_k)=0$*, then the next edge event is scheduled to be at* $t^{ij}_{k+1}=t^{ij}_k+T_s$ *and set* $s_{ij}(t^{ij}_{k+1})=1$;

(A4) *If* $s_{ij}(t^{ij}_k)=1$ *and* $x_i(t^{ij}_k)-x_j(t^{ij}_k)\neq 0$*, then the next edge event will be determined by agent i at time t,* $t>t^{ij}_k$ *and set* $t^{ij}_{k+1}=t$*, if no edge event of* (v_i, v_j) *happens over* (t^{ij}_k, t) *and any one of the following inequalities in (A4-1,A4-2) under their associated conditions is violated:*

(A4-1) *If* $x_i(t^{ij}_k)-x_j(t^{ij}_k)>0$,

$$-\frac{1-\alpha}{2}\left(x_i(t^{ij}_k)-x_j(t^{ij}_k)\right)<x_i(t)-x_i(t^{ij}_k)<\frac{\sigma-1}{2}\left(x_i(t^{ij}_k)-x_j(t^{ij}_k)\right);$$

(A4-2) *If* $x_i(t^{ij}_k)-x_j(t^{ij}_k)<0$,

$$\frac{\sigma-1}{2}\left(x_i(t^{ij}_k)-x_j(t^{ij}_k)\right)<x_i(t)-x_i(t^{ij}_k)<-\frac{1-\alpha}{2}\left(x_i(t^{ij}_k)-x_j(t^{ij}_k)\right),$$

where $0\le\alpha<1$, $\sigma>2-\alpha$;

(A5) *In (A4), if the event at* t^{ij}_{k+1} *is determined by* $x_i(t^{ij}_{k+1})-x_i(t^{ij}_k)=-\frac{1-\alpha}{2}(x_i(t^{ij}_k)-x_j(t^{ij}_k))$ *and* $t^{ij}_{k+1}-t^{ij}_k<T_s$*, then set* $s_{ij}(t^{ij}_{k+1})=0$.

Remarks:

(1) The edge event of (v_i, v_j) at t^{ij}_{k+1} can be determined by either one of the agents i and j, which follow rules (A1–A5). All parameters $s_{ij}(t^{ij}_{k+1})=s_{ji}(t^{ij}_{k+1})$ are shared by both of them.
(2) If agent j is a leader, then the edge events of (v_i, v_j) can be only taken care of by agent i and no data sampling is needed for agent j.
(3) For any successful data updating at t^{ij}_k and any $t\in[t^{ij}_k, t^{ij}_{k+1})$, if $s_{ij}(t^{ij}_k)=1$, $a_{ij}(t^{ij}_k)\neq 0$ and $x_i(t^{ij}_k)-x_j(t^{ij}_k)\neq 0$, then rules (A4) can ensure that equations of (A4-1) and (A4-2) and the revised equations of (A4-1) and (A4-2) by interchanging indexes i and j hold; and these equations further ensure that $x_i(t)-x_j(t)$ is located between $\alpha(x_i(t^{ij}_k)-x_j(t^{ij}_k))$ and $\sigma(x_i(t^{ij}_k)-x_j(t^{ij}_k))$. Therefore, $\frac{dV(t)}{dt}\le 0$, and $x_i(t)$ and $u_i(t)$, $i=1,2,\ldots,n$, are bounded for all t.
(4) Obviously, the smaller the σ is, or the larger the α is, the more frequently the edge events occur. In simulations, one example will show that frequent

events may not necessarily lead to fast state convergence.

(5) T_s can be any positive number and is given to separate some consecutive edge events in rules (A3) and (A5). Without it, if $\alpha > 0$ and the states of agents differ a lot from each other, it is easy to construct examples to show the existence of Zeno behaviour. T_s can be chosen according to the processing capacities of agents' hardware in applications. But T_s is not exactly the minimum inter-event time.

(6) Since the edge events are asynchronous with respect to information links, any minimum interval between events of each agent cannot be guaranteed. One solution to such a problem is to choose only one effective information link for each agent at each time. This scheme raises another issue of how to ensure the connectivity condition of interaction graph, which is not easy to be solved cooperatively by agents in a distributed way.

Theorem 1: *Under protocol* (2) *with edge events defined by* ($A1$–$A5$)*, there exists no finite accumulation point* (*Zeno behaviour*) *in the event time sequence, and if there exists a positive real number T such that the union of interaction graph* $\mathcal{G}(A(\cdot))$ *over* $[t, t+T]$ *is connected for any t, then the states of agents converge to* κ *asymptotically as time goes to* ∞.

Proof: By Lemma A.2, we have that the union of graph $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ over $[t, t+T+T_s]$ is connected for any t. In what follows, we use T instead of $T+T_s$ and use the condition that the union of graph $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ over $[t, t+T]$ is connected for any t.

If there exists a finite accumulation point in the event times, then we have an infinite sequence of consecutive edge-event times t^{ij}_k of some edge (v_i, v_j), converging to this point, and the time distance between any two consecutive events in the sequence converges to 0. By noticing that rules (A2, A3, A5) will separate edge-event times by T_s, we have that for large k, the edge event at t^{ij}_{k+1} is only determined by $x_i(t) - x_i(t^{ij}_k) = \frac{\sigma-1}{2}(x_i(t^{ij}_k) - x_j(t^{ij}_k))$ or $x_j(t) - x_j(t^{ji}_k) = \frac{\sigma-1}{2}(x_j(t^{ji}_k) - x_i(t^{ji}_k))$, which implies that $|x_i(t^{ij}_{k+1}) - x_j(t^{ij}_{k+1})| \geq \frac{\sigma+\alpha}{2}|x_i(t^{ij}_k) - x_j(t^{ij}_k)|$. By the assumption that $\sigma > 2-\alpha$, $|x_i(t^{ij}_k) - x_j(t^{ij}_k)|$ becomes infinitely large as k goes to ∞. This leads to a contradiction due to the boundedness of agent states.

To prove the effectiveness of the proposed protocol in solving the consensus problem, we first show that $\lim_{t\to\infty} u_i(t) = 0$ for all i by contradiction. Assume that for some i, it is not true that $\lim_{t\to\infty} u_i(t) = 0$. Then, there exist a positive number $\varepsilon > 0$ and an infinite sequence of time instants $\tau_1, \tau_2, \tau_3, \ldots$, such that $\lim_{k\to\infty}\tau_k = \infty$ and for any k, $|u_i(\tau_k)| \geq \varepsilon$; equivalently, $\omega_i \neq 0$ and

$$\left|\sum_{j\in\mathcal{N}_i(t^{ij}_{k^{ij}(\tau_k)})} s_{ij}\left(t^{ij}_{k^{ij}(\tau_k)}\right)a_{ij}\left(t^{ij}_{k^{ij}(\tau_k)}\right)\left(x_j\left(t^{ij}_{k^{ij}(\tau_k)}\right) - x_i\left(t^{ij}_{k^{ij}(\tau_k)}\right)\right)\right| \geq \frac{\varepsilon}{\omega_i}.$$

Therefore, there exists some $j \in \mathcal{N}_i(t^{ij}_{k^{ij}(\tau_k)})$, such that

$$s_{ij}\left(t^{ij}_{k^{ij}(\tau_k)}\right)a_{ij}\left(t^{ij}_{k^{ij}(\tau_k)}\right)\left|x_i\left(t^{ij}_{k^{ij}(\tau_k)}\right) - x_j\left(t^{ij}_{k^{ij}(\tau_k)}\right)\right| \geq \frac{\varepsilon}{(n-1)\omega_i}. \tag{5}$$

Then, the next event at $t^{ij}_{k^{ij}(\tau_k)+1}$ is only determined by (A4), and thus,

$$t^{ij}_{k^{ij}(\tau_k)+1} - t^{ij}_{k^{ij}(\tau_k)} \geq \min\left\{\frac{1-\alpha}{2}, \frac{\sigma-1}{2}\right\}\frac{\varepsilon}{a_{\max}u_{\max}(n-1)\omega_i}, \tag{6}$$

where $a_{\max} = \max\{a_{i'j'}(t) \neq 0 : i', j', t\}$ and $u_{\max}$ is the upper bound of $|u_{i'}(t)|$, $i' = 1, 2, \ldots, n$, $t \geq 0$. Here, we allow the occurrence of $t^{ij}_{k^{ij}(\tau_k)+1} = \infty$. Furthermore, by Remark 3 right after rule (A), for any $t \in [t^{ij}_{k^{ij}(\tau_k)}, t^{ij}_{k^{ij}(\tau_k)+1})$,

$$\left(x_i(t) - x_j(t)\right)\left(x_i\left(t^{ij}_{k^{ij}(t)}\right) - x_j\left(t^{ij}_{k^{ij}(t)}\right)\right) \geq \alpha\left(x_i\left(t^{ij}_{k^{ij}(t)}\right) - x_j\left(t^{ij}_{k^{ij}(t)}\right)\right)^2$$

and by inequality (5),

$$s_{ij}\left(t^{ij}_{k^{ij}(t)}\right)a_{ij}\left(t^{ij}_{k^{ij}(t)}\right)\left(x_i(t) - x_j(t)\right)\left(x_i\left(t^{ij}_{k^{ij}(t)}\right) - x_j\left(t^{ij}_{k^{ij}(t)}\right)\right) \geq \frac{\alpha\varepsilon^2}{a_{\max}(n-1)^2{\omega_i}^2}.$$

Thus, by Equation (4),

$$\frac{dV(t)}{dt} \leq -\frac{\alpha\varepsilon^2}{a_{\max}(n-1)^2{\omega_i}^2},$$

which leads to

$$V\left(t^{ij}_{k^{ij}(\tau_k)+1}\right) \leq V\left(t^{ij}_{k^{ij}(\tau_k)}\right) - \left(t^{ij}_{k^{ij}(\tau_k)+1} - t^{ij}_{k^{ij}(\tau_k)}\right) \times \frac{\alpha\varepsilon^2}{a_{\max}(n-1)^2{\omega_i}^2}.$$

Since the above inequality holds for any k and there is a non-zero upper bound of all non-zero ω_i, by Equation (6), we get $\lim_{t\to\infty} V(t) = -\infty$, a contradiction. Therefore, we have the conclusion that $\lim_{t\to\infty} u_i(t) = 0$ for all i.

Next, we will give a proof of $\lim_{t\to\infty} V(t) = 0$ by contradiction. For simplicity, let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $\delta(t) = [\delta_1(t), \delta_2(t), \ldots, \delta_n(t)]^T$, and let $L(t)$ be the Laplacian matrix of unweighted union graph of $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ over $[t, t+T]$. Note that by definition of vector $\delta(t)$, the following properties hold for any t:

(P1) If $\omega_i \neq 0$ for all i, then vector $\delta(t)$ is perpendicular to vector $[\frac{1}{\omega_1}, \frac{1}{\omega_2}, \dots, \frac{1}{\omega_n}]^T$.
(P2) If $\omega_i = 0$ for some i, then the ith entry of vector $\delta(t)$ is 0.

If $V(t) \neq 0$, then

$$\frac{x(t)^T L(t) x(t)}{V(t)} \geq \omega_{\min} \frac{2\delta(t)^T L(t)\delta(t)}{\delta(t)^T \delta(t)},$$

where $\omega_{\min} = \min_{\omega_i \neq 0} \omega_i$. By Lemma A.1 and the finite possibilities of $L(t)$,

$$\hat{\lambda} = \min\big\{\xi^T L(t)\xi : t \geq 0, \xi^T\xi = 1, \xi \text{ has the same property as } \delta(t) \text{ described in (P1) or some entries of } \xi \text{ are zero}\big\}$$

exists and $\hat{\lambda} > 0$. Therefore, for all t,

$$x(t)^T L(t) x(t) \geq 2\hat{\lambda}\omega_{\min} V(t).$$

Assume that $\lim_{t\to\infty} V(t) = \epsilon > 0$. Then, by Lemma A.1 (4), the above equation yields that for any given time t, there exist $t', t' \in [t, t+T]$, and two neighbouring agents i, j at time $t^{ij}_{k^{ij}(t')}$, such that

$$\begin{cases} s_{ij}(t^{ij}_{k^{ij}(t')})a_{ij}(t^{ij}_{k^{ij}(t')}) \neq 0, \\ x_i(t) - x_j(t) \geq 2\sqrt{\dfrac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}. \end{cases} \tag{7}$$

Since $u_i(t) \to 0$ for all i, we assume that the considered time t is large enough so that for any $t'' \geq t - T$,

$$|u_i(t'')| \leq \frac{\min\{1-\alpha, \sigma-1\}}{3T(2+\min\{1-\alpha,\sigma-1\})}\sqrt{\frac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}.$$

And, without loss of generality, we assume that $T > T_s$. In fact, if this assumption does not hold, we can redefine T by any real number larger than $\max\{T, T_s\}$. Then, by Equation (7), for any $t'' \in [t-T, t+2T]$,

$$|x_i(t'') - x_j(t'')| > \frac{4}{2+\min\{1-\alpha,\sigma-1\}}\sqrt{\frac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}, \tag{8}$$

and if $t^{ij}_{k^{ij}(t')} \in [t-T, t+T]$,

$$\begin{cases} \left|x_i\left(t^{ij}_{k^{ij}(t')}\right) - x_j\left(t^{ij}_{k^{ij}(t')}\right)\right| > \dfrac{4}{2+\min\{1-\alpha,\sigma-1\}}\sqrt{\dfrac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}} \\ \left|x_i(t'') - x_i\left(t^{ij}_{k^{ij}(t')}\right)\right| \leq \dfrac{2\min\{1-\alpha,\sigma-1\}}{2+\min\{1-\alpha,\sigma-1\}}\sqrt{\dfrac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}} \\ \left|x_j(t'') - x_j\left(t^{ij}_{k^{ij}(t')}\right)\right| \leq \dfrac{2\min\{1-\alpha,\sigma-1\}}{2+\min\{1-\alpha,\sigma-1\}}\sqrt{\dfrac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}, \end{cases}$$

which, by rule (A4), implies that for any $t'' \in [t+T, t+2T]$,

$$k^{ij}(t'') = k^{ij}(t').$$

On the other hand, if $t^{ij}_{k^{ij}(t')} < t - T$, then for any $t'' \in [t-T, t] \subset [t-T, t']$,

$$k^{ij}(t'') = k^{ij}(t').$$

In this case, assumption $T > T_s$ guarantees that $x_i(t^{ij}_{k^{ij}(t')}) - x_j(t^{ij}_{k^{ij}(t')}) \neq 0$ by (A3).

In both of the above two cases, by Equations (4) and (8),

$$V(t+2T) - V(t-T) < -a_{\min}T \times \left(\frac{4}{2+\min\{1-\alpha,\sigma-1\}}\right)^2 \frac{\hat{\lambda}\omega_{\min}\epsilon}{\sigma n(n-1)},$$

where $a_{\min} = \min\{a_{ij}(t'') \neq 0 : i, j, t''\}$. Since the above inequality holds for any sufficiently large t, we have $V(t)$ converges to $-\infty$ as time goes on, which is a contradiction. Therefore,

$$\lim_{t\to\infty} V(t) = 0.$$

5. Period-like sampled-data consensus driven by edge events

In this section, we restrict the event-detecting/event instants to the discrete-time set $\{t_k\colon t_{k+1} = t_k + h, k = 0, 1, 2, \dots\}$, where h is the event-detecting period. In such a case, $t^{ij}_k \in \{t_k : k = 0, 1, 2, \dots\}$ and h is also the minimum length of data-sampling periods. Redefine

$$k^{ij}(t) = \max\big\{k : t_k \in \{t^{ij}_{k'} : t^{ij}_{k'} \leq t\}\big\}.$$

Then, a simplified version of protocol (2) for this case is given as follows:

$$u_i(t) = \omega_i \sum_{j \in \mathcal{N}_i(t_{k^{ij}(t)})} a_{ij}(t_{k^{ij}(t)})\big(x_j(t_{k^{ij}(t)}) - x_i(t_{k^{ij}(t)})\big), \quad i = 1, 2, \dots, n. \tag{9}$$

To get a compact form of system (1) under protocol (9), we introduce an incident matrix D of the complete simple graph over the n agents with edge (v_i, v_j) arbitrarily oriented, and let $W(t)$ be the diagonal matrix such that for any i, j, $i \neq j$, if (v_i, v_j) is indexed by q in D, then the qth diagonal entry of $W(t)$ equals $a_{ij}(t_{k^{ij}(t)})$. Denote $y(t) = [y_1(t), y_2(t), \ldots, y_{\frac{n(n-1)}{2}}(t)]^T = D^T x(t)$ and define $\hat{y}(t) = [\hat{y}_1(t), \hat{y}_2(t), \ldots, \hat{y}_{\frac{n(n-1)}{2}}(t)]^T$ in the way that for any q, $1 \leq q \leq \frac{n(n-1)}{2}$, $\hat{y}_q(t) = x_i(t_{k^{ij}(t)}) - x_j(t_{k^{ij}(t)})$, if (v_i, v_j) is indexed by q in D and $y_q(t) = x_i(t) - x_j(t)$. Then, under protocol (9), the system can be represented by

$$\dot{x}(t) = -\mathrm{diag}(\omega) D W(t) \hat{y}(t),$$

where $\mathrm{diag}(\omega)$ is the diagonal matrix with ω_i as the ith diagonal entry. Thus, for any $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\frac{dV(t)}{dt} &= -\,\delta(t)^T D W(t_k)\hat{y}(t_k) = -x(t)^T D W(t_k)\hat{y}(t_k)\\ &= -\,\hat{y}(t_k)^T W(t_k) y(t)\\ &= -\,\hat{y}(t_k)^T W(t_k)\\ &\quad \times (y(t_k) - (t - t_k) D^T \mathrm{diag}(\omega) D W(t_k)\hat{y}(t_k))\\ &= -\,\hat{y}(t_k)^T W(t_k) y(t_k)\\ &\quad + (t - t_k)\hat{y}(t_k)^T W(t_k) D^T \mathrm{diag}(\omega) D W(t_k)\hat{y}(t_k).\end{aligned}$$

Therefore,

$$\begin{aligned}V(t_{k+1}) - V(t_k) &= -h\hat{y}(t_k)^T W(t_k) y(t_k)\\ &\quad + \frac{h^2}{2}\hat{y}(t_k)^T W(t_k) D^T \mathrm{diag}(\omega) D W(t_k)\hat{y}(t_k). \quad (10)\end{aligned}$$

Let $W(t_k)^{\frac{1}{2}}$ be a diagonal square root of $W(t_k)$ where each entry along the diagonal is a square root of the corresponding entry of $W(t_k)$, and let λ_n be the maximum eigenvalue of matrix $W(t_k)^{\frac{1}{2}} D^T \mathrm{diag}(\omega) D W(t_k)^{\frac{1}{2}}$, $k = 0, 1, 2, \ldots$. Then, we have the following lemma.

Lemma 1: *If there exists some non-negative integer T such that, for any k, the union of interaction topologies $\mathcal{G}([a_{ij}(t_{k^{ij}(t)})])$ at $t_k, t_{k+1}, \ldots, t_{k+T}$ is connected and there exist positive numbers α, β with $0 < \alpha, \beta \leq 1$, such that*

$$\begin{cases}\alpha\hat{y}(t_k)^T W(t_k)\hat{y}(t_k) \leq \hat{y}(t_k)^T W(t_k) y(t_k)\\ \beta^2 y(t_k)^T W(t_k) y(t_k) \leq \hat{y}(t_k)^T W(t_k)\hat{y}(t_k),\end{cases} \quad (11)$$

then there exists a maximum event-detecting period $h_{\max} = \frac{2\alpha}{\lambda_n}$, such that all states of agents under protocol (9) *with $0 < h < h_{\max}$ converge to κ asymptotically.*

Proof: Combining Equations (10) and (11), we have for any k,

$$\begin{aligned}V(t_{k+1}) - V(t_k) &\leq -\alpha h\hat{y}(t_k)^T W(t_k)\hat{y}(t_k)\\ &\quad + \frac{h^2\lambda_n}{2}\hat{y}(t_k)^T W(t_k)\hat{y}(t_k)\\ &\leq -\beta^2 h\left(\alpha - \frac{h\lambda_n}{2}\right) y(t_k)^T W(t_k) y(t_k)\\ &\leq 0. \quad (12)\end{aligned}$$

Next, we prove that $\lim_{k\to\infty} u_i(t_k) = 0$ for all i. Assume that for some i, it is not true that $u_i(t_k) \to 0$. Then, there exist a positive number ε and an infinite sequence of integers $\tau_0, \tau_1, \tau_2, \ldots$, such that $\lim_{k\to\infty}\tau_k = \infty$ and for any k, $|u_i(t_{\tau_k})| \geq \varepsilon$, which implies that there exists some $j \in \mathcal{N}_i(t_{k^{ij}(t_{\tau_k})})$, such that

$$\left|x_j(t_{k^{ij}(t_{\tau_k})}) - x_i(t_{k^{ij}(t_{\tau_k})})\right| \geq \frac{\varepsilon}{a_{\max}(n-1)\omega_i},$$

where $\omega_i \neq 0$. By the second inequality in Equation (12),

$$V(t_{\tau_k+1}) - V(t_{\tau_k}) \leq -\beta^2 h\left(\alpha - \frac{h\lambda_n}{2}\right)\frac{a_{\min}\varepsilon^2}{{a_{\max}}^2(n-1)^2{\omega_i}^2},$$

which means $\lim_{k\to\infty} V(t_k) = -\infty$, a contradiction. Therefore, $\lim_{k\to\infty} u_i(t_k) = 0$ for all i.

Now we can employ the same argument as in proving Equation (7) and get that if $\lim_{k\to\infty} V(t_k) = \epsilon > 0$, then for any k, there exits k', $k \leq k' \leq k + T$, such that

$$\begin{cases} a_{ij}(t_{k^{ij}(t_{k'})}) > 0,\\ |x_i(t_k) - x_j(t_k)| \geq 2\sqrt{\dfrac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}.\end{cases}$$

Since $\lim_{k\to\infty} u_i(t_k) = 0$ for all i, we assume that k is large enough so that for any i' and any k'', $k'' \geq k$,

$$|u_{i'}(t_{k''})| \leq \frac{1}{2T}\sqrt{\frac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}},$$

which yields that

$$|x_i(t_{k'}) - x_j(t_{k'})| \geq \sqrt{\frac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)}}.$$

By the second inequality in Equation (12), for any k,

$$V(t_{k+T+1}) - V(t_k) \leq -\beta^2 h\left(\alpha - \frac{h\lambda_n}{2}\right) a_{\min}\frac{\hat{\lambda}\omega_{\min}\epsilon}{n(n-1)},$$

which leads to that $\lim_{k\to\infty} V(t_k) = -\infty$, a contradiction. Therefore, $\lim_{k\to\infty} V(t_k) = 0$, i.e, the system solves a consensus problem. □

Note that $\frac{2}{\lambda_n}$ was given as a necessary and sufficient condition in the context of periodic sampling in the fixed topology case in Xie et al. (2009a). Thus, the above convergence result provides a non-conservative maximum event-detecting period $h_{\max} = \frac{2\alpha}{\lambda_n}$.

In the next part, we aim to construct distributed event-detecting rules that ensure inequality (11) holds. The first set of distributed event-detecting rules for edge events of (v_i, v_j) are given as follows:

(B1) *Agents i and j collaborate in determining their edge events of* (v_i, v_j) *and initialise* t_0^{ij} *at some time* t_k *when* (v_i, v_j) *is first available*;

for time t_k, $k = k^{ij}(t_0^{ij}) + 1, k^{ij}(t_0^{ij}) + 2, \ldots,$

(B2) *If information link* (v_i, v_j) *is lost (unavailable) or recovers from loss at time* t_k*, or if any one of the following inequalities in (B2-1) and (B2-2) under their associated conditions is violated, an edge event of* (v_i, v_j) *occurs at time* t_k*:*
(B2-1) If $x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})}) \geq 0$,

$$\alpha(x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})})) \leq x_i(t_k) - x_j(t_k) \leq \sigma(x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})}));$$

(B2-2) If $x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})}) < 0$,

$$\sigma(x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})})) \leq x_i(t_k) - x_j(t_k) \leq \alpha(x_i(t_{k^{ij}(t_{k-1})}) - x_j(t_{k^{ij}(t_{k-1})})),$$

where parameters α *and* σ *are known to all agents with the property that* $0 < \alpha \leq 1$ *and* $\sigma \geq 1$.

Clearly, if (v_i, v_j) is indexed by q and oriented with v_i as the head in the definition of D, then the above rules can guarantee that $y_q(t_k)$ is always located between values $\alpha\hat{y}_q(t_k)$ and $\sigma\hat{y}_q(t_k)$, and thus inequality (11) holds with $\beta = \frac{1}{\sigma}$.

Note that the enforcement of rule (B) requires data sampling between neighbouring agents at event-detecting time t_k. This requirement can be relaxed if each agent can remember its control input after its recent edge events with respect to its incident edges. Rule (C) is an example of such event-detecting rules, which is proposed for agents *i* and *j* to determine the edge events of (v_i, v_j), and does not require the mutual data sampling of agents *i* and *j* between their event times:

(C1) *Agent i initialises* t_0^{ij} *with agent j at some time* t_k *when* (v_i, v_j) *is first available*;

for agent *i* and event time t_k^{ij}, $k = 0, 1, 2, \ldots$ (agents *i* and *j* follow the same rule and collaborate in determining the edge events of (v_i, v_j)),

(C2) *If* $(v_i, v_j) \notin \mathcal{E}(t_k^{ij})$*, agent i still updates its controller at* t_k^{ij} *(agent j also updates its controller), and the next edge event will occur at time* $t_{k'}$, $t_{k'} > t_k^{ij}$*, when information link* (v_i, v_j) *turns available*; otherwise

(C3) *Agent i will determine the next edge event of* (v_i, v_j) *at the earliest time* $t_{k'}$, $t_{k'} > t_k^{ij}$*, such that no edge event of* (v_i, v_j) *happens over* $(t_k^{ij}, t_{k'})$ *and any one of the following inequalities in (C3-1) and (C3-2) under their associated conditions is violated:*
(C3-1) If $x_i(t_k^{ij}) - x_j(t_k^{ij}) \geq 0$,

$$-\frac{1-\alpha}{2}\left(x_i(t_k^{ij}) - x_j(t_k^{ij})\right) \leq h\sum_{l=k^{ij}(t_k^{ij})}^{k'-1} u_i(t_l) \leq \frac{\sigma-1}{2}\left(x_i(t_k^{ij}) - x_j(t_k^{ij})\right);$$

(C3-2) If $x_i(t_k^{ij}) - x_j(t_k^{ij}) < 0$,

$$\frac{\sigma-1}{2}\left(x_i(t_k^{ij}) - x_j(t_k^{ij})\right) \leq h\sum_{l=k^{ij}(t_k^{ij})}^{k'-1} u_i(t_l) \leq -\frac{1-\alpha}{2}\left(x_i(t_k^{ij}) - x_j(t_k^{ij})\right),$$

where $0 < \alpha \leq 1$ and $\sigma \geq 1$.

As a consequence of Lemma 5.1, we have the following theorem.

Theorem 2: *If event-detecting period h satisfies* $0 < h < \frac{2\alpha}{\lambda_n}$ *and there exists some non-negative integer T such that, for any k, the union of graph* $\mathcal{G}(A(\cdot))$ *at* $t_k, t_{k+1}, \ldots, t_{k+T}$ *is connected, then under protocol* (9) *with event-detecting rule (B) or (C), the states of agents converge to* κ *asymptotically.*

Proof: Denote the equations of (C3-1) and (C3-2) with the interchange of indexes *i* and *j* by (C3-1)' and (C3-2)', respectively. The edge events of (v_i, v_j) are both determined by agents *i* and *j*, and thus they will be triggered by the violation of any one of (C3-1), (C3-2), (C3-1)', and (C3-2)'. Note that these four equations or equations in (B2-1) and (B2-2) ensure that Equation (5.3) holds. Moreover, that the union of graph $\mathcal{G}(A(\cdot))$ at $t_k, t_{k+1}, \ldots, t_{k+T}$ is connected implies that the union of $\mathcal{G}([a_{ij}(t_{k^{ij}(t)})])$ at $t_k, t_{k+1}, \ldots, t_{k+T}$ is connected. Therefore, we can get Theorem 5.2 by Lemma 5.1. □

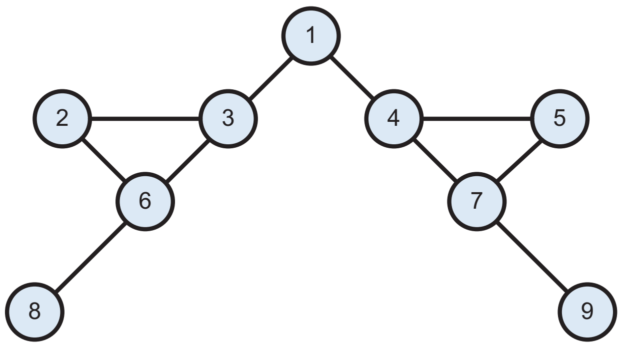

Figure 1. Communication topology.

6. Illustrative example

Consider the multi-agent system consisting of $N = 9$ single-integrator agents, random initial conditions generated from the uniform distribution on the interval [0, 10], and the variable topology. The edge set of the network topology is randomly chosen from the edge set of Figure 1 with 25% probability for each edge. After the dwell time 0.25, the network switches to another graph which is generated by the same manner, but the generation has to guarantee that the switching topology is uniformly jointly connected with time horizon $T = 1$. Such randomly switching process continues until the end of simulation.

The parameters ω_i in the control input in Equation (2) are randomly selected from the uniform distribution on the interval [0, 1]. The associated matrix A for the network in Figure 1 is given by

$$A = \begin{bmatrix} 0 & 0 & 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0.8 & 0.5 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.5 & 0.8 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \end{bmatrix}.$$

The parameters for rule (A) are set to $\alpha = 0.5$, $\sigma = 2$, and $T_s = 0.05$; the parameters for rules (B) and (C) are set to $\alpha = 0.5$, $\sigma = 2$, and $h = \frac{\alpha}{\lambda_n}$. The evolution of $x(t)$ by using rules (A), (B), and (C), and the time-triggered rule with period h are shown in Figures 2–5, respectively. The evolution of $V(t)$ for different rules is shown in Figure 6, where letters A, B, and C indicate rules (A), (B), and (C), respectively, and letter P indicates the periodic sampling case. It can be seen from the figures that the convergence rates of the Lyapunov function by using rules (B) and (C) are almost the same. Notice that the agent group eventually achieved rendezvous under the proposed control laws. Figures 7–9 show the number of events generated at every 10 second interval by using different rules. The number of edge events by event-

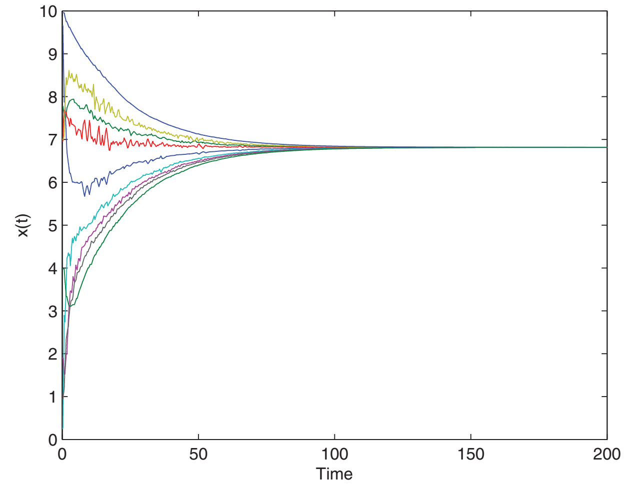

Figure 2. Evolution of each agent using event-detecting rule (A).

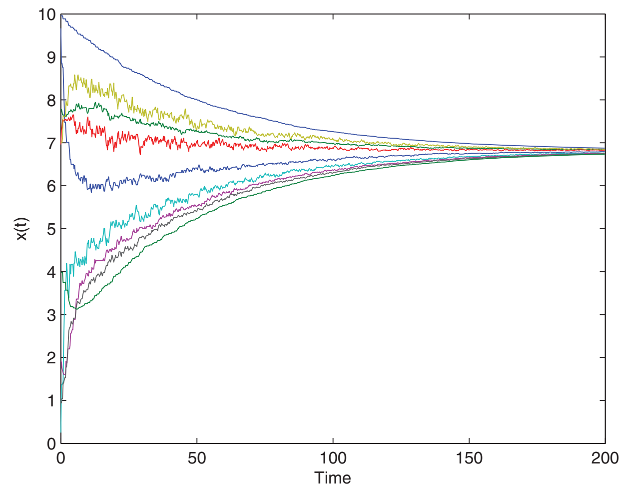

Figure 3. Evolution of each agent using event-detecting rule (B).

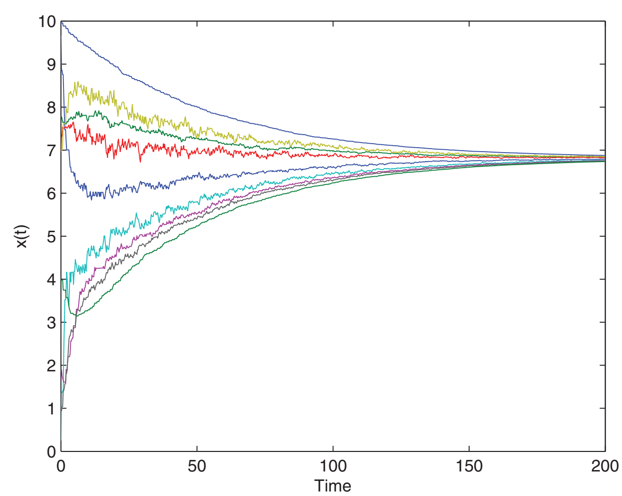

Figure 4. Evolution of each agent using event-detecting rule (C).

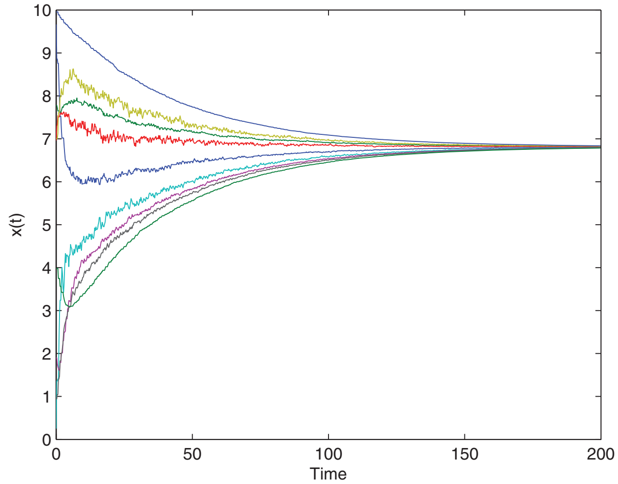

Figure 5. Evolution of each agent using periodic sampling.

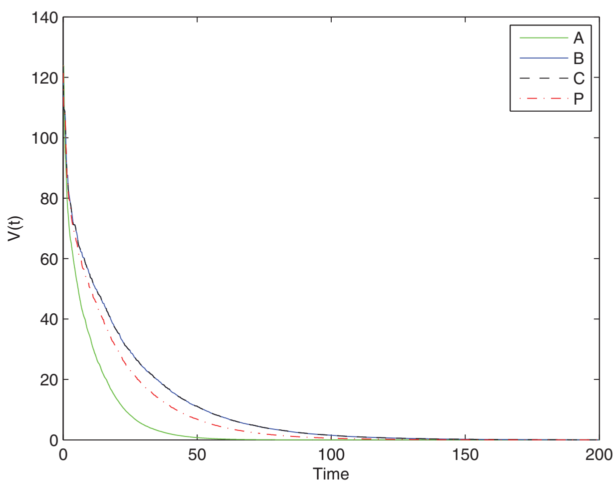

Figure 6. Evolution of the Lyapunov function.

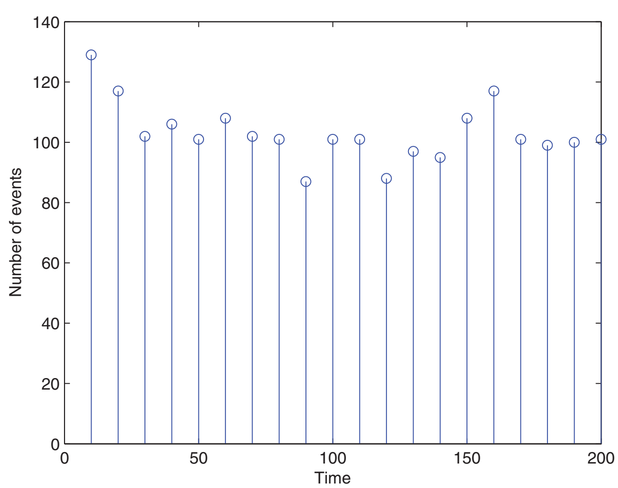

Figure 7. Number of events using event-detecting rule (A).

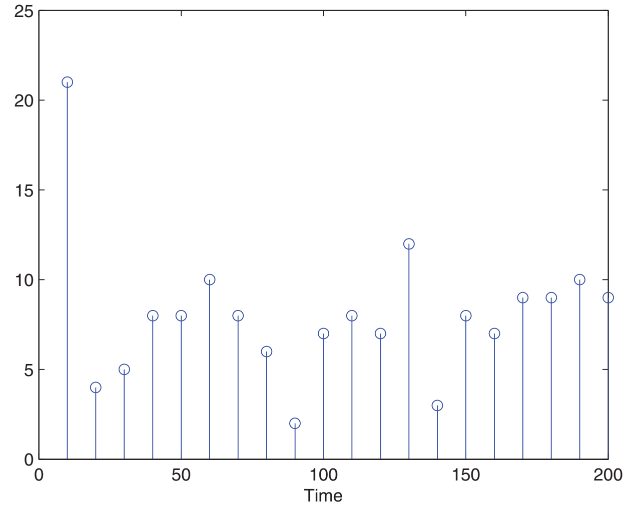

Figure 8. Number of events using event-detecting rule (B).

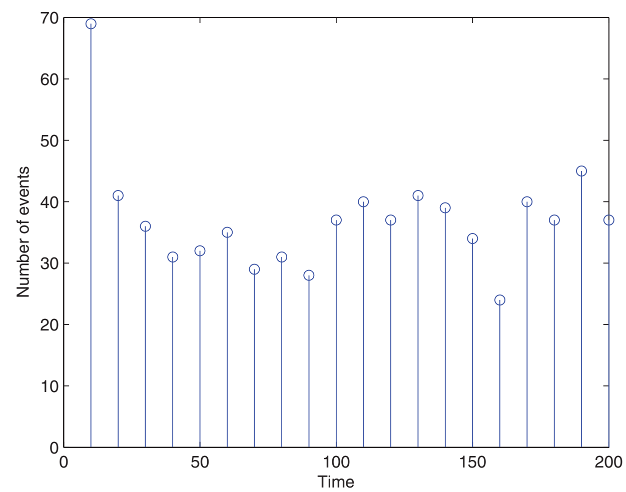

Figure 9. Number of events using event-detecting rule (C).

detecting rule B is fewer than that by event-detecting rule C, which is consistent with that event-detecting rule (C) is a sufficient condition for event-detecting rule (B).

Finally, let us investigate the relationship between the parameters α, β and the number of events by using the event-detecting rule (C). In this case, consider the fixed topology shown in Figure 1 and the sampling period $h = 0.0001$. Figure 10 shows the relationship between the parameter α and the number of events with $\beta = 0.5$, where the left y-axis denotes the time when the Lyapunov function $V(t)$ decreases to $0.95V(0)$ and the right y-axis denotes the total number of events until that time. Figure 11 has the same interpretation with Figure 10 when the parameter $\alpha = 0.5$. It is interesting to note that the convergence rate decreases as the number of events increases, which shows that frequent updating may not necessarily lead to a fast convergence rate. Comparing Figures 10 and 11, it can be

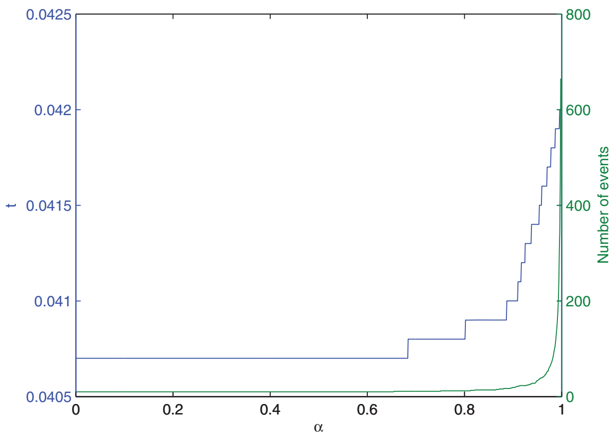

Figure 10. The relationship between the parameter α and the number of events.

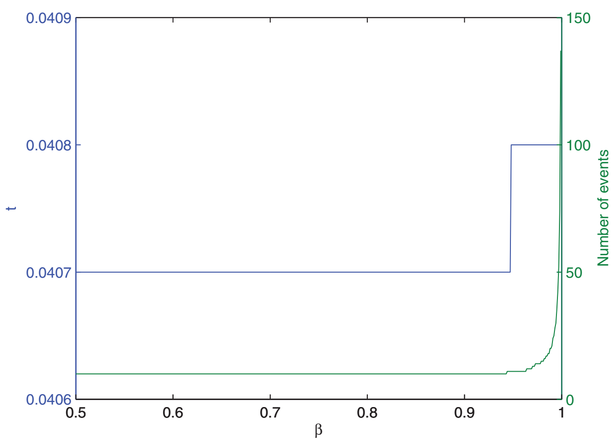

Figure 11. The relationship between the parameter β and the number of events.

concluded that the number of events is more sensitive to the parameter α than β.

7. Conclusion

In this paper, we presented two kinds of event-based consensus protocols for networks of multiple integrators with asynchronous and periodic-like data sampling. We defined edge events with respect to information links, and the occurrence of edge events will trigger the data sampling and controller updates of the associated agent pairs. We showed that the presented event-driven protocols can be implemented in a distributed manner with reduced communication costs.

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Notes

1. If $\mathcal{N}_i(t^{ij}_{k^{ij}(t)}) = \emptyset$, then $u_i(t) = 0$.
2. Note that this paper assumes that there exists *at most one agent* with ω_i equal to 0.

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Appendix 1. Preliminary lemmas

Lemma A1 (Godsil & Royal, 2001; Olfati-Saber & Murray, 2004)**:**

(1) 0 *is always an eigenvalue of* $L(A)$ *with* $[1, 1, \ldots, 1]^T \in \mathbb{R}^n$ *as the associated eigenvector;*

(2) *If graph* $\mathcal{G}(A)$ *is connected, then the second smallest eigenvalue, denoted by* λ_2*, is larger than* 0*;*

(3) $\lambda_2 = \min\{\frac{\xi^T L(A)\xi}{\xi^T\xi} : \xi \neq 0, \sum_{i=1}^n \xi_i = 0\}$*, where* $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n$*;*

(4) $\xi^T L(A)\xi = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(\xi_i - \xi_j)^2$.

Lemma A2: *Under protocol* (2) *with edge events defined by rules* (*A*1–*A*5)*, if the union of graph* $\mathcal{G}(A(\cdot))$ *over* $[t, t+T]$ *is connected for any t, then the union of interaction graph* $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ *over* $[t, t+T+T_s]$ *is connected for any t.*

Proof: Suppose that (v_i, v_j) is an edge of $\mathcal{G}(A(t'))$ at some time t' with $t' \in [t+T_s, t+T+T_s]$. If $s_{ij}(t^{ij}_{k^{ij}(t)})a_{ij}(t^{ij}_{k^{ij}(t)}) > 0$, then (v_i, v_j) is an edge of $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(t)})a_{ij}(t^{ij}_{k^{ij}(t)})])$; otherwise, we have that $s_{ij}(t^{ij}_{k^{ij}(t)}) = 0$ or $a_{ij}(t^{ij}_{k^{ij}(t)}) = 0$.

(1) In the case with $a_{ij}(t^{ij}_{k^{ij}(t)}) = 0$, by rule (A2), an edge event of (v_i, v_j) must occur at some time t'' in $(t, t']$ with the property that $t'' = t^{ij}_{k^{ij}(t'')}$ and $s_{ij}(t'')a_{ij}(t'') > 0$.

(2) In the case with $s_{ij}(t^{ij}_{k^{ij}(t)}) = 0$ and $a_{ij}(t^{ij}_{k^{ij}(t)}) \neq 0$, then by rule (A3), an edge event of (v_i, v_j) must be scheduled to be at time $t'' = t^{ij}_{k^{ij}(t)} + T_s$ in $(t, t+T_s]$, $t'' = t^{ij}_{k^{ij}(t'')}$, and $s_{ij}(t'') = 1$. If $a_{ij}(t'') > 0$, then (v_i, v_j) belongs to the edge set of $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(t'')})a_{ij}(t^{ij}_{k^{ij}(t'')})])$; otherwise, by rule (A2), an edge event of (v_i, v_j) must occur at some time t''' in $(t'', t']$ with the property that $t''' = t^{ij}_{k^{ij}(t''')}$ and $s_{ij}(t''')a_{ij}(t''') > 0$.

To conclude, (v_i, v_j) is an edge of the union of interaction graph $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ over $[t, t+T+T_s]$. Therefore, the union of interaction graph $\mathcal{G}([s_{ij}(t^{ij}_{k^{ij}(\cdot)})a_{ij}(t^{ij}_{k^{ij}(\cdot)})])$ over $[t, t+T+T_s]$ is connected for any t. □