Automatica 69 (2016) 275–288

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Event-based state estimation of linear dynamic systems with unknown exogenous inputs[☆]

Dawei Shi[a], Tongwen Chen[b], Mohamed Darouach[c]

[a] State Key Laboratory of Intelligent Control and Decision of Complex Systems, School of Automation, Beijing Institute of Technology, Beijing, 100081, PR China

[b] Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2V4

[c] CRAN-CNRS UMR 7039, Université de Lorraine, IUT de Longwy, 186 Rue de Lorraine, 54400 Cosnes-et-Romain, France

ARTICLE INFO

Article history:
Received 21 February 2015
Received in revised form
9 December 2015
Accepted 13 February 2016

Keywords:
Event-based state estimation
Unknown exogenous inputs
Optimal filtering

ABSTRACT

In this work, an event-based optimal state estimation problem for linear-time varying systems with unknown inputs is investigated. By treating the unknown input as a process with a non-informative prior, the event-based minimum mean square error (MMSE) estimator is obtained in a recursive form. It is shown that for the general time-varying case, the closed-loop matrix of the optimal event-based estimator is exponentially stable and the estimation error covariance matrix is asymptotically bounded for each sample path of the event-triggering process. The results are also extended to the multiple sensor scenario, where each sensor is allowed to have its own event-triggering condition. The efficiency of the proposed results is illustrated by a numerical example and comparative simulation with the MMSE estimators obtained based on time-triggered measurements. The results are potentially applicable to event-based secure state estimation of cyber-physical systems.

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1. Introduction

The increasing demand on safe, secure and high performance operation of civil and industrial engineering systems has given birth to cyber-physical systems (CPSs). Different from the traditional control systems, CPSs are normally composed of networks of interacting components (e.g., sensor/actuator networks). Despite the encouraging features and new opportunities brought on by this type of systems, CPSs have introduced several new challenges to control system design.

One of these challenges is the limitation of communication and power resources. In many CPS applications, a large number of sensors and actuators linked through communication networks are utilized to accomplish certain tasks (e.g., quality control, remote monitoring). When all the components transmit their updates to each other or the computing center, the number of available communication channels acts as a natural limitation; moreover, in many occasions, some of these components are battery powered (e.g., mobile sensor networks), making the amount of available power a restriction on system performance as well. The event-triggered data transmission policies (Åström & Bernhardsson, 2002; Meng & Chen, 2012; Yook, Tilbury, & Soparkar, 2002) provide an efficient remedy to handle these limitations, and event-based state estimation, which is the scope of this work, has received a lot of attention in the control community during the past few years.

Earlier results on this topic focus on optimal event-triggering policy design. The optimal event-based finite-horizon sensor transmission scheduling problems were studied in Imer and Başar (2005) and Rabi, Moustakides, and Baras (2006) for continuous-time and discrete-time scalar linear systems, respectively; the results were extended to vector linear systems in Li, Lemmon, and Wang (2010). In Marck and Sijs (2010), a sampling protocol was proposed based on the Kullback–Leibler divergence of the probability distributions obtained when incorporating or not incorporating a measurement. Adaptive sampling for state estimation was considered in Rabi, Moustakides, and Baras (2012) for continuous-time linear systems. For further results on this line of research, see also Shi, Johansson, and Qiu (2011), Li and Lemmon (2011), Molin and Hirche (2012), Wang and Fu (2014) and references therein.

☆ This work was supported in part by the National Natural Science Foundation of China under Grant 61503027, in part by Natural Sciences and Engineering Research Council of Canada, and in part by a Visiting Professorship from CRAN, Université de Lorraine, France. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Antonio Vicino under the direction of Editor Torsten Söderström.

E-mail addresses: dawei.shi@outlook.com (D. Shi), tchen@ualberta.ca (T. Chen), mohamed.darouach@univ-lorraine.fr (M. Darouach).

http://dx.doi.org/10.1016/j.automatica.2016.02.031
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Recent developments focus on the optimal estimator design for certain pre-specified event-triggering conditions. A general description of event-based sampling was proposed in Sijs and Lazar (2012), and a low-complexity event-based estimator with a hybrid update was proposed based on the approximation of the uniform distribution with the sum of a finite number of Gaussian distributions. Utilizing a Gaussian assumption on the distribution of the state conditioned on all past available measurement information, the event-based minimum mean square error (MMSE) estimator was derived in Wu, Jia, Johansson, and Shi (2013), and the tradeoff between communication rate and performance was explicitly analyzed; the extension to more general event-triggering conditions and the multiple sensor scenario was considered in Shi, Chen, and Shi (2014a). The Gaussian assumption was shown to be maintained in Han, Mo, Wu, Sinopoli, and Shi (2013), based on a stochastic event-triggering condition that introduced randomization in the triggering sets. In Sijs, Noack, and Hanebeck (2013), an event-based state estimator was obtained by minimizing the maximum possible mean squared error and treating the event-triggering conditions as non-stochastic uncertainty. In Shi, Chen, and Shi (2014b), a constrained optimization approach was utilized to solve an event-based estimation problem for a triggering scheme quantifying the magnitude of the innovation. A variance-triggered state estimation problem was considered in Trimpe and D'Andrea (2014), and the asymptotic periodicity of the triggering pattern was proved for an unstable scalar system. A nonlinear event-based state estimation problem was considered in Lee, Liu, and Hwang (2014), where a Markov chain approximation algorithm was proposed. The event-triggered estimation problem of systems with mixed time delays was considered by Zou, Wang, Gao, and Liu (2015) using sampled-data information for the continuous-time case. The problem of event-based state estimation for discrete-state hidden Markov models was investigated in Shi, Elliott, and Chen (2016).

On the other hand, the complex structure and extensive utilization of communication networks have made CPSs fragile and prone to unknown and unpredictable cyber attacks, which can cause disastrous consequences to infrastructure, national security, and even human life. In this context, a number of interesting attempts on secure detection and estimation have been recently reported, through graph-theoretic methods (Pasqualetti, Dorfler, & Bullo, 2013), by exploring the sparsity of the attack signals (Fawzi, Tabuada, & Diggavi, 2014), and using game-theoretic approaches (Miao, Pajic, & Pappas, 2013; Mo & Sinopoli, 2014). An alternative way to consider the secure estimation problems, however, is to treat the attack signals as unknown exogenous inputs and solve a problem of estimating the states in the existence of the unknown inputs. For the scenario of time-triggered measurements, this type of problems has been extensively investigated using the unbiased minimum variance (UMV) estimation approach in the literature, see, e.g., Kitanidis (1987), Darouach and Zasadzinski (1997), Darouach, Zasadzinski, and Boutayeb (2003), Cheng, Ye, Wang, and Zhou (2009), Fang, Shi, and Yi (2011) and references therein for the related developments; however, for the case of event-triggered measurement information, the estimation problem for systems with unknown inputs has not been investigated. The main difficulty is that when the measurements are assumed to be available at each time instant, the UMV estimators are normally obtained by directly minimizing the estimation error covariance matrices and the effect of the measurement information on the conditional distributions of the states (which is crucial in optimal event-based estimator design) is not explored; and thus the UMV estimation approach developed for the time-triggered measurement case cannot be generalized to consider the event-triggered scenario.

Meanwhile, it is interesting to note that by treating the unknown input as a process with non-informative prior, the Bayesian inference approach was successfully utilized to find the optimal MMSE estimate for systems with partially observed inputs (Li, 2013), and the results were shown to reduce to those obtained by the UMV approach for the unknown input case. For a system with an unknown exogenous input, it is normally not possible to find an appropriate state estimate or prediction, when no information about the current state is available from the sensor.[1] Although the Bayesian approach allows the exploitation of the implicit information contained in the event-triggering conditions (Han et al., 2013; Shi et al., 2014a; Wu et al., 2013), it is not yet clear whether this implicit information is "informative" enough to ensure the existence of an appropriate state estimate that is optimal in certain sense without exactly knowing the value of the current sensor measurement, which is investigated in this paper. To do this, an event-based optimal state estimation problem for linear time-varying systems with unknown exogenous inputs and stochastic event-triggering conditions is considered. The main contributions are summarized as follows:

(1) Under some mild conditions, it is shown that the conditional distribution of the state on the event-triggered measurement information is Gaussian, and the event-based MMSE estimate is developed in a recursive form. The obtained results generalize the time-triggered state estimation results obtained in Kitanidis (1987), Darouach and Zasadzinski (1997), Li (2013) to the case of event-triggered measurements.
(2) For each sample path of the event-triggering process, we show that the event-based MMSE estimator is exponentially stable with bounded estimation error covariance for the linear time-varying case. The results are equally applicable to the UMV estimator for linear time-varying systems as well, as it has the similar filter structure with the proposed event-based MMSE estimator.
(3) For the multiple sensor scenario with separate event-triggering conditions on each sensor, we show that the event-based MMSE estimator can also be developed under a rank condition on the lumped measurement matrices of all sensors. The differences of the estimator equations from the classic time-triggered Kalman filter as well as approximate event-based MMSE estimator developed for deterministic event-triggering conditions in the multiple sensor scenario are discussed.

For the multiple-sensor scenario, the problems of distributed event-based state estimation have been extensively investigated in Trimpe and D'Andrea (2011), Weimer, Araujo, and Johansson (2012) and Trimpe (2014). The main differences of the results developed for the multiple-sensor scenario in this work from Trimpe and D'Andrea (2011); Weimer et al. (2012) and Trimpe (2014) include: (1) the communication among sensors is not considered and the problem of event-based state estimation is considered in a centralized fashion; and (2) the effect of an unknown exogenous input term in the process equation is considered in event-based estimator design.

On the other hand, the problem of the unknown input observer design for continuous-time deterministic systems was presented in Darouach, Zasadzinski, and Xu (1994), where the necessary and sufficient conditions were provided; these conditions were related to the relative degree of the transfer function between the unknown input and the measured output. In Cristofaro and Johansen (2014), the authors utilized the unknown input observers for detection, isolation and control reconfiguration in overactuated systems. Recently, Johansen, Cristofaro, Sorensen, Hansen, and Fossen (2015) used the unknown input observer

[1] The intuition is that without the sensor measurement update, no clue of the current unknown input can be inferred.

for the estimation of wind velocity, angle-of-attack and sideslip angle for small UAVs; the approach proposed was based on Kalman Bucy filtering. In addition, Floquet, Edwards, and Spurgeon (2007) considered the unknown input observer design for linear continuous-time systems by combining the sliding mode observer with sliding mode exact differentiations to weaken the relative degree condition necessary for the unknown input observer design. All these results considered the problems for continuous-time systems; the results in this work, however, focus on discrete-time linear time-varying systems disturbed by stochastic noises. Moreover, in the current work, we consider the effect of event-triggers in estimator design; to investigate the event-triggered unknown input observer design problems for continuous-time systems, potentially different approaches are necessary to handle the issues caused by the event-triggers (see e.g., Heemels, Donkers, & Teel, 2013; Meng & Chen, 2014; Zhang & Han, 2015).

The rest of the paper is organized as follows: Section 2 provides the system description and problem formulation. The solution to the optimal event-based estimation problem is presented in Section 3. In Section 4, the asymptotic properties of the event-based MMSE estimator are investigated. The extension of the results to the multiple sensor scenario is provided in Section 5. Section 6 presents a numerical example to illustrate the efficiency of the proposed results, followed by some concluding remarks in Section 7.

Notations. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{N}$ denotes the set of nonnegative integers. Let $m, n \in \mathbb{N}$; $\mathbb{R}^{m\times n}$ denotes the set of m by n real-valued matrices. For brevity, denote $\mathbb{R}^m := \mathbb{R}^{m\times 1}$. If $m > n$, $y^{m:n}$ and $y_{m,n}$ denote the sets $\{y^m, \dots, y^n\}$ and $\{y_m, \dots, y_n\}$, respectively, and $\mathbb{N}_{m,n}$ denotes the integer subset $\{m, m+1, \dots, n\}$. The symbol I_n denotes an $n \times n$ identity matrix, and $\mathbf{0}$ denotes a zero matrix with a context-dependent size. Finally, $A \otimes B$ denotes the Kronecker product between matrices A and B.

2. Problem description

Consider the remote estimation scheme in Fig. 1. The process is linear time-varying and evolves in discrete time driven by white noise:

$$x_{k+1} = A_k x_k + G_k d_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^n$ is the noise input, which is zero-mean Gaussian with covariance $Q_k > 0$ and $d_k \in \mathbb{R}^p$ is the unknown input. The initial state x_0 is Gaussian with $\mathrm{E}(x_0) = \hat{x}_0^-$ and covariance $P_0^- > 0$. As we assume that d_k is unknown, no information of d_k is available. Following the conventions employed in the joint state and unknown input estimation literature (Darouach & Zasadzinski, 1997; Kitanidis, 1987), we assume rank $G_k = p$. This assumption implies $p < n$, which means that there are more states than disturbances, and is intuitive in that if the number of disturbance channels is equal to or larger than that of the states, it would be unlikely to obtain a good state estimate based on the measurement information provided.

The problem of estimating x_k under the unknown input $d_{1:k}$ and the time-driven measurement information has been extensively investigated in the literature. In this work, the estimation problem is explored under intermittent, event-triggered measurement information. In particular, we consider the scenario that the state information is measured by a smart sensor composed of a sensor measurement module and an event-based data scheduler (see Fig. 1). The measurement equation of the sensor measurement module is assumed to take the form

$$y_k = C_k x_k + v_k, \tag{2}$$

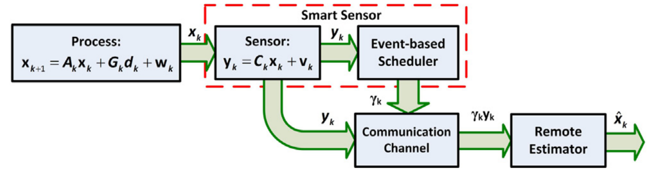

Fig. 1. Block diagram of the overall system.

where $v_k \in \mathbb{R}^m$ is zero-mean Gaussian with covariance $R_k > 0$. In addition, we assume x_0, w_k, v_k and d_k are independent of each other. The smart sensor measures the state information at each time instant through the sensor measurement module, and communicates with the remote estimator through a reliable communication channel. Considering the limitations on communication and energy resources, the measurement information y_k is not sent to the remote estimator at every time instant; alternatively, an event-based data scheduler with an event-triggering process γ_k is equipped in the smart sensor and decides whether to connect the sensor to the communication channel or not. The measurement information y_k is sent to the remote estimator only if $\gamma_k = 1$. Let

$$\mathcal{I}_k := \{(\gamma_0, \gamma_0 y_0), (\gamma_1, \gamma_1 y_1), \dots, (\gamma_k, \gamma_k y_k)\}$$

denote the available measurement information to the remote estimator up to time instant k.

In this work, we consider the case that the value of γ_k is determined through a stochastic event-triggering condition (Han et al., 2013). Unlike the deterministic event-triggering conditions (Shi et al., 2014b; Shi, Chen, & Shi, 2015), the stochastic event-triggering condition assigns each point y in the measurement space $\mathbb{R}^m$ a transmission probability $\phi_k(y)$ such that

$$\mathrm{P}(\gamma_k = 0 | y_k = y, \mathcal{I}_{k-1}) = \phi_k(y), \tag{3}$$

where $\phi_k(\cdot)$ is a known function given $\mathcal{I}_{k-1}$. In this way, γ_k is a random process that depends only on $\phi_k(y_k)$. Note that when $\gamma_k = 1$, the remote estimator knows the exact value of y_k; when $\gamma_k = 0$, although the exact value of y_k is not known, the remote estimator can still infer some information of y_k through the transmission probability function defined in (3); specifically, this information is given in the form of $\mathrm{P}(\gamma_k | y_k = y, \mathcal{I}_{k-1})$, which can be exploited to improve the performance of the estimator for the hidden state x_k.

In this work, we consider a transmission probability function of the exponential form

$$\phi_k(y) = \exp\left[-\frac{1}{2}(y - \xi_k)^\top Y_k (y - \xi_k)\right], \tag{4}$$

where Y_k is a positive definite weighting matrix, and the requirement on ξ_k is that either ξ_k is a deterministic parameter or the value of ξ_k can be inferred based on the available measurement information $\mathcal{I}_{k-1}$ to the remote estimator up to time instant $k-1$, so that the value of ξ_k is known to the estimator at each time instant without utilizing additional communication resources. Generally, Y_k is designed to compromise the tradeoff between sensor-to-estimator communication rates and estimation performance, and a smaller Y_k implies a relatively lower average communication rate. The transmission probability distribution in (3) can be implemented by realizing a random variable ζ_k uniformly distributed on $[0, 1]$ at each time instant; if $\zeta_k < \phi_k(y_k)$, then $\gamma_k = 0$ and otherwise $\gamma_k = 1$. Note that if $\xi_k = 0$, this condition reduces to the open-loop stochastic event-triggering condition introduced in Han et al. (2013); if $\xi_k = y_{\tau_k}$, with τ_k being defined by

$$\tau_k = \max_{t<k,\gamma_t=1} t,$$

and y_{τ_k} denoting the previously transmitted sensor measurement which is known by both the sensor and the estimator, this condition becomes a stochastic version of the well known "send-on-delta" condition (Miskowicz, 2006). Another choice of ζ_k is to choose $\zeta_k = C_k A_{k-1}\hat{x}_{k-1}$ which corresponds to a stochastic version of the triggers considered in Trimpe and D'Andrea (2011) and Sijs, Kester, and Noack (2014); here $\hat{x}_{k-1}$ denotes the estimate of x_{k-1} based on the available measurement information $\mathcal{I}_{k-1}$, and a copy of the event-based state estimator needs to be embedded on the sensor side.

The consideration of stochastic event-triggering conditions introduces randomization in the event-triggering process. Although the full control of the triggering decision procedure is lost in the sense that a favored measurement value may not be transmitted to the remote estimator by some nonzero probability, stochastic event-triggering schemes in the form of Eq. (4) help maintain the Gaussianity of the posterior distributions of the states on the event-triggered measurement information, even under the existence of the unknown input term d_k, as will be shown later. On the other hand, similar to the normal deterministic event-triggering conditions (e.g., the conditions considered in Trimpe and D'Andrea (2011); Wu et al. (2013)), the stochastic event-triggering conditions still reflect the designer's preference on the measurement data: the preferred data is transmitted by a larger probability, while the unimportant data is transmitted by a smaller one.

The objectives of this work include

(1) to find the optimal estimate $\hat{x}_k$ of x_k governed by an unknown input term d_k, given the event-triggered measurement information $\mathcal{I}_k$;
(2) to analyze the stability properties of the obtained event-based estimator.

3. Results on recursive state estimation

In this work, we investigate the event-based estimation problem utilizing a Bayesian inference approach. As no information is available for the unknown input d_k, we model it with a non-informative prior distribution, i.e.,

$$f(d_k|\mathcal{I}_k) \propto 1, \tag{5}$$

and as before, assume that x_0, w_k, v_k and d_k are independent of each other. The intuition of using this non-informative prior is that all possible values of the unknown input d_k are equally likely to occur, which is consistent with the assumption that no information of d_k is available. In the literature of statistical inference, this type of prior is termed as an improper prior (which describes the case that the sum or integral of the prior probability distribution is not finite), and has been extensively adopted in Bayesian inference and statistical signal processing (Burghaus & Dette, 2014; Dalton & Dougherty, 2011; Figueiredo & Nowak, 2001; Jeffreys, 1946; Li, 2013; Sivaganesan & Lingham, 2000; Svensson & Lundberg, 2005).

Remark 1. To further explain the rationale of using this type of improper prior, let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ be a sequence of mutually exclusive and exhaustive events and let $\mathcal{B}$ be an arbitrary event. From the Bayesian law, we have

$$\mathrm{P}(\mathcal{A}_i|\mathcal{B}) = \frac{\mathrm{P}(\mathcal{B}|\mathcal{A}_i)\mathrm{P}(\mathcal{A}_i)}{\sum_{j=1}^{N}\mathrm{P}(\mathcal{B}|\mathcal{A}_j)\mathrm{P}(\mathcal{A}_j)}.$$

The reason of using an improper prior is that if the term $\sum_{j=1}^{N}\mathrm{P}(\mathcal{B}|\mathcal{A}_j)\mathrm{P}(\mathcal{A}_j)$ converges, the posterior probabilities $\mathrm{P}(\mathcal{A}_i|\mathcal{B})$ still sum to 1 even if the prior probabilities $\mathrm{P}(\mathcal{A}_i)$ do not; in this sense, the prior probabilities only need to be given in the correct proportion.

Let $G_k^{\perp}$ denote a matrix such that $[G_k\ G_k^{\perp}] \in \mathbb{R}^{n\times n}$, rank $[G_k\ G_k^{\perp}] = n$ and $G_k^{\top}G_k^{\perp} = \mathbf{0}$. Since x_0 is Gaussian by assumption and d_k does not affect the measurement equation, it is easy to verify that $x_0|\mathcal{I}_0$ is Gaussian; for simplicity we denote its mean and covariance by $\hat{x}_0$ and P_0, respectively, which can be calculated in a way similar to Eqs. (11)–(13) in Han et al. (2013) on the basis of $\hat{x}_0^-$ and P_0^-. Let

$$T_k := [G_k\ G_k^{\perp}]^{-1} \tag{6}$$

and

$$L_k := [\mathbf{0}\ I_{n-p}]\,T_k. \tag{7}$$

The following result provides the optimal estimate $\hat{x}_k$ conditioned on the available measurement information $\mathcal{I}_k$ on the estimator side for $k \geq 1$.

Theorem 2. *For the remote state estimation scheme in Fig. 1 with exogenous unknown input and the stochastic event-triggering condition in (3)–(4) and $k \geq 1$, the conditional distribution of x_k on $\mathcal{I}_k$ is Gaussian with mean $\hat{x}_k$ and covariance P_k provided*

$$\text{rank}\begin{bmatrix} C_k^{\top} & L_{k-1}^{\top}\end{bmatrix}^{\top} = n. \tag{8}$$

If the condition in (8) is satisfied, $\hat{x}_k$ and P_k evolve according to the following recursive form:

$$P_{k|k-1} = A_{k-1}P_{k-1}A_{k-1}^{\top} + Q_{k-1}; \tag{9}$$

if $\gamma_k = 0$,

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + P_kC_k^{\top}(R_k + Y_k^{-1})^{-1}\xi_k - P_kC_k^{\top}(R_k + Y_k^{-1})^{-1}C_kA_{k-1}\hat{x}_{k-1}, \tag{10}$$

$$P_k = [C_k^{\top}(R_k + Y_k^{-1})^{-1}C_k + L_{k-1}^{\top}(L_{k-1}P_{k|k-1}L_{k-1}^{\top})^{-1}L_{k-1}]^{-1}; \tag{11}$$

if $\gamma_k = 1$,

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + P_kC_k^{\top}R_k^{-1}y_k - P_kC_k^{\top}R_k^{-1}C_kA_{k-1}\hat{x}_{k-1}, \tag{12}$$

$$P_k = [C_k^{\top}R_k^{-1}C_k + L_{k-1}^{\top}(L_{k-1}P_{k|k-1}L_{k-1}^{\top})^{-1}L_{k-1}]^{-1}. \tag{13}$$

In addition, the event-based MMSE estimate satisfies (9)–(13).

Proof. The proof of this result takes several steps. According to the standard results on optimal estimation (Anderson & Moore, 1979; Levy, 2008), the MMSE estimate is given by the mean of the conditional distribution of the state on the available measurement information. In this way, what we need to do in optimal estimator design is to find the recursion of the conditional distribution of the state on the event-triggered measurement information. First we define an augment variable z_k through a nonsingular linear transformation $z_k := T_{k-1}x_k$, and Eq. (1) becomes

$$z_k = F_{k-1}x_{k-1} + D_{k-1}d_{k-1} + \tilde{w}_{k-1} \tag{14}$$

where $F_{k-1} = T_{k-1}A_{k-1}$, $D_{k-1} = T_{k-1}G_{k-1} = [I_p\ \mathbf{0}]^{\top}$, and $\tilde{w}_{k-1} = T_{k-1}w_{k-1}$. Note that $\tilde{w}_k$ is a Gaussian white noise with zero mean and covariance $\tilde{Q}_k = T_kQ_kT_k^{\top} > 0$, and is independent of x_0, d_k and v_k.

Now we derive the distribution of $x_k|\mathcal{I}_k$ inductively. Since $x_0|\mathcal{I}_0$ is Gaussian, we assume at time instant $k-1$, the conditional distribution of $x_{k-1}|\mathcal{I}_{k-1}$ is Gaussian with mean $\hat{x}_{k-1}$ and covariance P_{k-1}. The conditional distributions of $x_k|\mathcal{I}_k$ are derived for $\gamma_k = 0$ and $\gamma_k = 1$, respectively, as has been done in Han et al. (2013) (but following a different line of argument due to the existence of d_k). For $\gamma_k = 0$, following the Bayesian law, we have

$$\begin{aligned} f(z_k|\mathcal{I}_k) &= f(z_k|\mathcal{I}_{k-1}, \gamma_k = 0) \\ &\propto \mathrm{P}(\gamma_k = 0|z_k, \mathcal{I}_{k-1}) f(z_k|\mathcal{I}_{k-1}) \\ &= f(z_k|\mathcal{I}_{k-1}) \\ &\cdot \int_{\mathbb{R}^m} \mathrm{P}(\gamma_k = 0|y_k, z_k, \mathcal{I}_{k-1}) f(y_k|z_k, \mathcal{I}_{k-1}) \mathrm{d}y_k \\ &= f(z_k|\mathcal{I}_{k-1}) \int_{\mathbb{R}^m} \mathrm{P}(\gamma_k = 0|y_k, \mathcal{I}_{k-1}) f(y_k|z_k, \mathcal{I}_{k-1}) \mathrm{d}y_k \\ &= f(z_k|\mathcal{I}_{k-1}) \int_{\mathbb{R}^m} \mathrm{P}(\gamma_k = 0|y_k, \mathcal{I}_{k-1}) f(y_k|x_k, \mathcal{I}_{k-1}) \mathrm{d}y_k, \end{aligned} \tag{15}$$

where the last equation follows from the fact that $z_k = T_{k-1}x_k$ and T_{k-1} is nonsingular, so that the conditional distribution of y_k on $\{z_k, \mathcal{I}_{k-1}\}$ is same as that of y_k on $\{x_k, \mathcal{I}_{k-1}\}$, and the second last equation is due to the fact that the event-trigger is fully determined by y_k and $\mathcal{I}_{k-1}$.

Since x_0, v_k and $\tilde{w}_k$ are independent, and from (3)–(4) and the second item of Lemma 15 in Appendix A, we have

$$\begin{aligned} &\int_{\mathbb{R}^m} \mathrm{P}(\gamma_k = 0|y_k, \mathcal{I}_{k-1}) f(y_k|x_k, \mathcal{I}_{k-1}) \mathrm{d}y_k \\ &\propto \int_{\mathbb{R}^m} \exp\left[-\frac{1}{2}(y_k - \xi_k)^\top Y_k (y_k - \xi_k)\right] \\ &\quad \cdot \exp\left[-\frac{1}{2}(y_k - C_k x_k)^\top R_k^{-1}(y_k - C_k x_k)\right] \mathrm{d}y_k \qquad (16) \\ &= \int_{\mathbb{R}^m} \exp\left\{-\frac{1}{2}\left[(y_k - C_k x_k)^\top R_k^{-1}(y_k - C_k x_k)\right.\right. \\ &\quad + [(y_k - C_k x_k) - (\xi_k - C_k x_k)]^\top \\ &\quad \left.\left. \cdot Y_k[(y_k - C_k x_k) - (\xi_k - C_k x_k)]\right]\right\} \mathrm{d}y_k \qquad (17) \\ &\propto \exp\left[-\frac{1}{2}(C_k x_k - \xi_k)^\top (R_k + Y_k^{-1})^{-1}(C_k x_k - \xi_k)\right], \end{aligned}$$

where Eq. (16) is due to the assumption that either ξ_k is a deterministic parameter or the value of ξ_k can be inferred based on $\mathcal{I}_{k-1}$. For $f(z_k|\mathcal{I}_{k-1})$, we have

$$f(z_k|\mathcal{I}_{k-1}) = \int_{\mathbb{R}^p} f(z_k|\mathcal{I}_{k-1}, d_{k-1}) f(d_{k-1}|\mathcal{I}_{k-1}) \mathrm{d}d_{k-1}.$$

From Eqs. (5) and (14) and the properties of the marginal distribution of a multivariate Gaussian random variable, the above equation further implies

$$\begin{aligned} f(z_k|\mathcal{I}_{k-1}) &\propto \int_{\mathbb{R}^p} \exp\Big(-\frac{1}{2}\left[z_k - \left(F_{k-1}\hat{x}_{k-1} + D_{k-1}d_{k-1}\right)\right]^\top \\ &\quad (P^z_{k|k-1})^{-1}\left[z_k - \left(F_{k-1}\hat{x}_{k-1} + D_{k-1}d_{k-1}\right)\right]\Big) \mathrm{d}d_{k-1} \\ &\propto \exp\Big[-\frac{1}{2}(z_k - F_{k-1}\hat{x}_{k-1})^\top \\ &\quad \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1}(z_k - F_{k-1}\hat{x}_{k-1})\Big], \end{aligned} \tag{18}$$

where $\bar{D}_{k-1} := [\mathbf{0}\ I_{n-p}]$ and

$$P^z_{k|k-1} = F_{k-1}P_{k-1}F_{k-1}^\top + \tilde{Q}_{k-1};$$

the detailed explanation of (18) is provided in Appendix C. Combining (15)–(18) and letting $H_k := C_k T_{k-1}^{-1}$, we have

$$\begin{aligned} f(z_k|\mathcal{I}_k) &\propto \exp\left[-\frac{1}{2}(H_k z_k - \xi_k)^\top (R_k + Y_k^{-1})^{-1}(H_k z_k - \xi_k)\right] \\ &\cdot \exp\Big[-\frac{1}{2}(z_k - F_{k-1}\hat{x}_{k-1})^\top \\ &\quad \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1}(z_k - F_{k-1}\hat{x}_{k-1})\Big]. \end{aligned} \tag{19}$$

Since T_{k-1} is nonsingular and

$$\begin{bmatrix} H_k \\ \bar{D}_{k-1} \end{bmatrix} T_{k-1} = \begin{bmatrix} C_k \\ L_{k-1} \end{bmatrix},$$

Eq. (8) implies rank $\begin{bmatrix} H_k^\top & \bar{D}_{k-1}^\top \end{bmatrix}^\top = n$, and therefore

$$\operatorname{rank}\left(\begin{bmatrix} H_k^\top & \bar{D}_{k-1}^\top \end{bmatrix} \begin{bmatrix} (R_k + Y_k^{-1})^{-1} & 0 \\ 0 & (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \end{bmatrix} \begin{bmatrix} H_k \\ \bar{D}_{k-1} \end{bmatrix}\right) = n.$$

From (19) and the first item of Lemma 15 in Appendix A, we conclude that for $\gamma_k = 0$, $z_k|\mathcal{I}_k$ is a Gaussian distribution with mean $\hat{z}_k$ and covariance P^z_k defined by

$$\begin{aligned} \hat{z}_k &= P^z_k[H_k^\top (R_k + Y_k^{-1})^{-1}\xi_k \\ &\quad + \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1} F_{k-1}\hat{x}_{k-1}] \\ &= F_{k-1}\hat{x}_{k-1} + P^z_k[H_k^\top (R_k + Y_k^{-1})^{-1}\xi_k \\ &\quad - H_k^\top (R_k + Y_k^{-1})^{-1} H_k F_{k-1}\hat{x}_{k-1}], \end{aligned} \tag{20}$$

$$\begin{aligned} P^z_k &= [H_k^\top (R_k + Y_k^{-1})^{-1} H_k \\ &\quad + \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1}]^{-1}. \end{aligned} \tag{21}$$

For $\gamma_k = 1$, by the Bayesian law, we similarly have

$$\begin{aligned} f(z_k|\mathcal{I}_k) &= f(z_k|\mathcal{I}_{k-1}, y_k) \\ &\propto f(y_k|\mathcal{I}_{k-1}, z_k) f(z_k|\mathcal{I}_{k-1}). \end{aligned} \tag{22}$$

Noticing that $f(z_k|\mathcal{I}_{k-1})$ satisfies the relationship in (18) and $f(y_k|\mathcal{I}_{k-1}, z_k)$ satisfies

$$f(y_k|\mathcal{I}_{k-1}, z_k) \propto \exp\left[-\frac{1}{2}(y_k - H_k z_k)^\top R_k^{-1}(y_k - H_k z_k)\right],$$

and from item 1 of Lemma 15 in Appendix A, we obtain that for $\gamma_k = 1$, $z_k|\mathcal{I}_k$ is a Gaussian distribution with mean $\hat{z}_k$ and covariance P^z_k defined by

$$\hat{z}_k = F_{k-1}\hat{x}_{k-1} + P^z_k[H_k^\top R_k^{-1} y_k - H_k^\top R_k^{-1} H_k F_{k-1}\hat{x}_{k-1}], \tag{23}$$

$$P^z_k = [H_k^\top R_k^{-1} H_k + \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1}]^{-1}, \tag{24}$$

which completes the derivation of the conditional distribution $z_k|\mathcal{I}_k$.

Finally, we obtain the results for $x_k|\mathcal{I}_k$ based on the results for $z_k|\mathcal{I}_k$. Since $x_k = T_{k-1}^{-1}z_k$, $F_{k-1} = T_{k-1}A_{k-1}$, $C_k = H_k T_{k-1}$ and $L_{k-1} = \bar{D}_{k-1}T_{k-1}$, $x_k|\mathcal{I}_k$ is Gaussian with mean $\hat{x}_k$ and covariance P_k defined by

$$\begin{aligned} P_k &= T_{k-1}^{-1}[H_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} H_k \\ &\quad + \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1}]^{-1} (T_{k-1}^{-1})^\top, \\ &= [C_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} C_k \\ &\quad + T_{k-1}^\top \bar{D}_{k-1}^\top (\bar{D}_{k-1} P^z_{k|k-1} \bar{D}_{k-1}^\top)^{-1} \bar{D}_{k-1} T_{k-1}]^{-1} \\ &= [C_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} C_k \\ &\quad + L_{k-1}^\top (L_{k-1} T_{k-1}^{-1} P^z_{k|k-1} (T_{k-1}^{-1})^\top L_{k-1}^\top)^{-1} L_{k-1}]^{-1} \\ &= [C_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} C_k \\ &\quad + L_{k-1}^\top (L_{k-1} P_{k|k-1} L_{k-1}^\top)^{-1} L_{k-1}]^{-1}, \end{aligned} \tag{25}$$

$$\begin{aligned} P_{k|k-1} &= T_{k-1}^{-1} P^z_{k|k-1} (T_{k-1}^{-1})^\top \\ &= A_{k-1} P_{k-1} A_{k-1}^\top + Q_{k-1}, \end{aligned} \tag{26}$$

$$
\begin{aligned}
\hat{x}_k &= T_{k-1}^{-1}F_{k-1}\hat{x}_{k-1} + T_{k-1}^{-1}P_k^z \\
&\quad \cdot [\gamma_k H_k^\top R_k^{-1} y_k + (1-\gamma_k)H_k^\top (R_k + Y_k^{-1})^{-1}\xi_k \\
&\quad - H_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} H_k F_{k-1}\hat{x}_{k-1}] \\
&= A_{k-1}\hat{x}_{k-1} + T_{k-1}^{-1}T_{k-1}P_k T_{k-1}^\top \\
&\quad \cdot [\gamma_k H_k^\top R_k^{-1} y_k + (1-\gamma_k)H_k^\top (R_k + Y_k^{-1})^{-1}\xi_k \\
&\quad - H_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} H_k F_{k-1}\hat{x}_{k-1}] \\
&= A_{k-1}\hat{x}_{k-1} + P_k \\
&\quad \cdot [\gamma_k C_k^\top R_k^{-1} y_k + (1-\gamma_k)C_k^\top (R_k + Y_k^{-1})^{-1}\xi_k \\
&\quad - C_k^\top (R_k + (1-\gamma_k)Y_k^{-1})^{-1} C_k A_{k-1}\hat{x}_{k-1}]. \qquad (27)
\end{aligned}
$$

Finally, since $\hat{x}_k$ is the mean of $x_k|\mathcal{I}_k$, it is the MMSE estimate (Anderson & Moore, 1979; Levy, 2008), which completes the proof. □

Remark 3. In Theorem 2, a sufficient rank condition is employed to guarantee the existence of a non-degenerate Gaussian distribution for $x_k|\mathcal{I}_k$. This is due to the existence of the unknown input term, rather than the event-triggering condition. Noticing that $[C_k^\top \; L_{k-1}^\top]^\top \in \mathbb{R}^{[m+(n-p)]\times n}$, the condition in (8) implies $m \geq p$, indicating that the number of independent sensor measurement channels should be at least as many as the number of linearly independent channels of unknown inputs. In addition, it is easy to verify that the condition is consistent with the conditions needed for the time-triggered measurement case with partially unknown inputs (Li, 2013) and the time-triggered unbiased minimum variance estimator (Darouach & Zasadzinski, 1997), which is rank $(C_kG_{k-1}) = p$; the current form indicates that when C_k has full column rank, the condition holds automatically, which can be easily satisfied for the multiple sensor scenario (see Section 5). Although matrices C_k and L_k vary with time for generic time-varying systems, it is possible to test the condition in (8), since the expressions of C_k and L_k are known and the rank condition can be checked based on these expressions; in particular, there are several classes of special cases that the verification of this rank condition is numerically favorable, e.g., the class of switched systems that switch among a finite number of LTI systems, in which case it suffices to test a finite number of rank conditions, and the class of periodic time-varying systems, for which the values of C_k and L_k repeat periodically and thus the number of rank conditions need to be checked is equal to the period of the system.

Remark 4. The above derivation also reveals the effect of the unknown input on state estimation. For the standard Kalman filter, it is well known that the filter update equations are composed of two steps—state prediction and measurement update; for the event-based estimator provided here, a state prediction might be chosen as

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1};$$

but Eq. (9) no longer represents the corresponding prediction error covariance. Due to the assumption on d_k, the prior covariance has directions that are infinite, which makes the prediction error covariance incomputable in its current form. One potential way to overcome this issue, however, is to use the information filtering approach (Mutambara, 1998) to design the event-based estimators, in which the inverse of the covariance matrix is recursively updated. However, when some information of the unknown input d_{k-1} is available, an estimate $\hat{x}_k$ can be generated; as is revealed by the obtained result, this information does not have to be the exact point-valued measurement information y_k—the implicit information provided by the event-triggering conditions can be sufficiently "informative" to guarantee the existence/uniqueness of a closed-form estimator as well.

4. Stability analysis

In this section, the stability properties of the proposed event-based estimator are investigated, the goal of which is to analyze the asymptotic behavior of the estimation error covariance matrix as $k \to \infty$. Due to the existence of the event-triggering process $\{\gamma_k\}$ that depends on both the past measurements and the stochastic event-triggering conditions, the estimation error covariance P_k becomes a random process as well, which is different from that of an estimator evolving according to an a priori determined sensor transmission schedule. In this way, the stability analysis needs to be performed from two aspects:

(1) How would P_k behave along an arbitrary sample path of $\{\gamma_k\}$? This problem is of primary concern because it is the particular sequence of $\{P_k\}$ corresponding to the actual sample path $\{\gamma_k\}$ that determines the existence and performance of the event-based estimator in a specific application. In particular, if P_k blows up for some sample path of $\{\gamma_k\}$, we will face the risk of not being able to compute the estimate $\hat{x}_k$ (which is calculated based on the value of P_k) when k becomes sufficiently large.
(2) What is the performance of P_k on the average? This problem is important as well, as P_k is a random process for an event-based estimator and in particular, the sample path of $\{\gamma_k\}$ can never be known a priori. For some applications, if an estimator results in a bounded expectation of the estimation error covariance matrix (namely, $\mathrm{E}(P_k) < \infty$) with some small probability that P_k goes to infinity as $k \to \infty$, it might be tolerable as well in the sense that the estimator behaves well on the average.

In the following, we answer the above two questions for the proposed estimator. First, we look into the stability of the proposed event-based estimator for an arbitrary sample path of $\{\gamma_k\}$. Specifically, we recall that the estimation error covariance equation has the form

$$P_k = [C_k^\top \tilde{R}_k^{-1} C_k + L_{k-1}^\top (L_{k-1}P_{k|k-1}L_{k-1}^\top)^{-1} L_{k-1}]^{-1}, \qquad (28)$$

with

$$P_{k|k-1} = A_{k-1}P_{k-1}A_{k-1}^\top + Q_{k-1}, \qquad (29)$$

$$\tilde{R}_k = R_k + (1-\gamma_k)Y_k^{-1}. \qquad (30)$$

The goal here is to find the conditions under which P_k is asymptotically bounded and analyze the relationship of the conditions found with the event-triggering process. From Eq. (28), one possible way to achieve this goal is to analyze the asymptotic behavior of P_k for a generic $\tilde{R}_k$ and investigate the effect of $\tilde{R}_k$ on the conditions that guarantee the boundedness of P_k.

To do this, consider a linear Gaussian system of the form (1)–(2):

$$x_{k+1} = A_k x_k + G_k d_k + w_k, \qquad (31)$$

$$\tilde{y}_k = C_k x_k + \tilde{v}_k, \qquad (32)$$

with the same assumptions on x_0, w_k and d_k as those of Section 2, but with a different measurement noise process $\tilde{v}_k$ with zero mean and covariance $\tilde{R}_k > 0$. Note that $\tilde{R}_k$ here is generic and does not have to satisfy (30).

In the literature of UMV state estimation for linear time-varying systems with unknown exogenous inputs based on time-triggered measurements (i.e., the measurements are sent to the estimator at each time instant through a reliable communication channel without packet dropout or transmission delays), it is known that the UMV estimator for the system in (31) and (32) satisfies (Darouach & Zasadzinski, 1997; Kitanidis, 1987; Li, 2013)

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + \Gamma_k(\tilde{y}_k - C_kA_{k-1}\hat{x}_{k-1}), \qquad (33)$$

$$
\begin{aligned}
\Gamma_k &= P_{k|k-1}C_k^\top \Pi_k^{-1} + (G_{k-1} - P_{k|k-1}C_k^\top \Pi_k^{-1} C_k G_{k-1}) \\
&\quad \cdot (G_{k-1}^\top C_k^\top \Pi_k^{-1} C_k G_{k-1})^{-1} G_{k-1}^\top C_k^\top \Pi_k^{-1} \qquad (34)
\end{aligned}
$$

$$\Pi_k = C_k P_{k|k-1} C_k^\top + \tilde{R}_k, \quad (35)$$

$$P_{k|k-1} = A_{k-1} P_{k-1} A_{k-1}^\top + Q_{k-1},$$

$$\begin{aligned} P_k = {} & P_{k|k-1} - P_{k|k-1} C_k^\top \Pi_k^{-1} C_k P_{k|k-1} \\ & + (G_{k-1} - P_{k|k-1} C_k^\top \Pi_k^{-1} C_k G_{k-1}) \\ & \cdot (G_{k-1}^\top C_k^\top \Pi_k^{-1} C_k G_{k-1})^{-1} \\ & \cdot (G_{k-1} - P_{k|k-1} C_k^\top \Pi_k^{-1} C_k G_{k-1})^\top. \end{aligned} \quad (36)$$

On the other hand, noticing $L_k = [\mathbf{0}\ I_{n-p}][G_k\ G_k^\perp]^{-1}$ and rank $[C_k^\top\ L_{k-1}^\top]^\top = n$, it is easy to verify that a direct application of Lemma 16 in Appendix B to Eq. (28) leads to Eq. (36). Therefore it suffices to analyze the asymptotic boundedness of Eq. (36). For the UMV estimator given in (33)–(36), the stability and convergence properties have been analyzed in Darouach and Zasadzinski (1997) and Fang and de Callafon (2012) for the LTI case; the idea of the latter work was also utilized in Su, Li, and Chen (2015) to analyze the time-triggered MMSE estimator with partially observed inputs for LTI systems. For the evolution equation of P_k in (36), however, since A_k, G_k, C_k, Q_k and in particular, $\tilde{R}_k$ are assumed to be time-varying, the results in Fang and de Callafon (2012) and Su et al. (2015) cannot be applied to obtain the asymptotic properties. To address this issue, the asymptotic properties of the UMV estimator for the time-varying case are investigated here, which serve as a nontrivial extension of the results in Fang and de Callafon (2012).

We now recall a few properties of the UMV estimator obtained in Kerwin and Prince (2000), Gillijns and De Moor (2007) and Fang and de Callafon (2012).[2] Let

$$\tilde{A}_k(M, K) := (I_n - KC_k)(I_n - G_{k-1} M C_k) A_{k-1},$$

$$\tilde{F}_k(M, K) := -(I_n - KC_k)(I_n - G_{k-1} M C_k),$$

$$\tilde{G}_k(M, K) := (I_n - KC_k) G_{k-1} M + K.$$

In Kerwin and Prince (2000) and Gillijns and De Moor (2007), it is proved that the UMV estimator in Eqs. (33)–(36) is the global optimal estimator of the form

$$\tilde{x}_k = A_k(M_k, K_k)\tilde{x}_{k-1} + \tilde{G}_k(M_k, K_k)\tilde{y}_k \quad (37)$$

with M_k satisfying $M_k C_k G_{k-1} = I_p$ in the sense of minimum estimation error covariance (see also Lemma 1 in Fang and de Callafon (2012)). Note that the form in (37) arises from the idea of designing separate linear gain matrices to estimate the unknown input d_{k-1} and state x_k:

$$\tilde{d}_{k-1} = M_k(y_k - C_k A_{k-1}\tilde{x}_{k-1}),$$

$$\begin{aligned} \tilde{x}_k = {} & A_{k-1}\tilde{x}_{k-1} + G_{k-1}\tilde{d}_{k-1} \\ & + K_k(y_k - C_k A_{k-1}\tilde{x}_{k-1} - C_k G_{k-1}\tilde{d}_{k-1}). \end{aligned}$$

Let M_k^* and K_k^* denote the gain matrices of the UMV estimator in Eqs. (33)–(36). From Eqs. (33)–(36), these matrices can be expressed as

$$K_k^* = P_{k|k-1} C_k^\top \Pi_k^{-1}, \quad (38)$$

$$M_k^* = (G_{k-1}^\top C_k^\top \Pi_k^{-1} C_k G_{k-1})^{-1} G_{k-1}^\top C_k^\top \Pi_k^{-1}. \quad (39)$$

For an estimator of the form in (37), the estimation error $\tilde{e}_k := x_k - \tilde{x}_k$ evolves according to

$$\tilde{e}_k = \tilde{A}_k(M_k, K_k)\tilde{e}_{k-1} - \begin{bmatrix}\tilde{F}_k(M_k, K_k) & \tilde{G}_k(M_k, K_k)\end{bmatrix}\begin{bmatrix} w_{k-1} \\ \tilde{v}_k \end{bmatrix}. \quad (40)$$

[2] Some of these properties were derived for the LTI case in Fang and de Callafon (2012); however, it is straightforward to verify that they equally apply to the time-varying case as well.

The reason that the unknown input term d_k does not appear in (40) is due to the requirement $M_k C_k G_{k-1} = I_p$ on M_k, which guarantees the unbiasedness of the estimate $\tilde{d}_k$ of the unknown input d_k (Gillijns & De Moor, 2007). Following the notations in Fang and de Callafon (2012), we represent the covariance matrix update equation as

$$\begin{aligned} \tilde{P}_k = {} & \phi_k(M_k, K_k, \tilde{P}_{k-1}) \\ = {} & \tilde{A}_k(M_k, K_k)\tilde{P}_{k-1}\tilde{A}_k^\top(M_k, K_k) + \begin{bmatrix}\tilde{F}_k(M_k, K_k) & \tilde{G}_k(M_k, K_k)\end{bmatrix} \\ & \cdot \begin{bmatrix} Q_k & 0 \\ 0 & \tilde{R}_k \end{bmatrix}\begin{bmatrix}\tilde{F}_k^\top(M_k, K_k) \\ \tilde{G}_k^\top(M_k, K_k)\end{bmatrix}. \end{aligned} \quad (41)$$

In this way, the estimation error covariance of the UMV estimator evolves according to

$$P_k = \phi_k(M_k^*, K_k^*, P_{k-1}). \quad (42)$$

Before continuing, we introduce the following notion of uniform detectability.

Definition 5. The triplet (A_k, G_k, C_k) is uniformly detectable if there exist bounded matrix sequences $\{(M_k, K_k)\}$ such that

(1) $M_k C_k G_{k-1} = I_p$;
(2) $(I_n - K_k C_k)(I_n - G_{k-1} M_k C_k) A_{k-1}$ is exponentially stable.

The above definition of uniform detectability is motivated from the ideas of defining uniform detectability for a pair (A_k, C_k) introduced in Anderson and Moore (1981), in which two equivalent definitions were proposed—the first one was based on the idea that the unstable modes of a system should be observable while the second one was to require the existence of an observer gain such that the closed-loop state observation system is exponentially stable. As will be shown later, the detectability notion introduced above does not guarantee that the estimation error $x_k - \hat{x}_k$ goes to 0 asymptotically (due to the effect of noises and disturbances); instead what can be guaranteed is that the estimation error covariance matrix in (36) remains bounded as $k \to \infty$. One possible way to test this uniform detectability condition is to first determine a sequence of M_k such that $M_k C_k G_{k-1} = I_p$, and then test the uniform detectability of the pair $(A_{k-1}, C_k(I_n - G_{k-1} M_k C_k) A_{k-1})$, for which spectral test exists (Peters & Iglesias, 1999). Note that a necessary condition for a triplet (A_k, G_k, C_k) to be uniformly detectable is that rank $C_k G_{k-1} = p$. Now we are in a position to provide the first result on the asymptotic properties.

Theorem 6. *If the triplet (A_k, G_k, C_k) is uniformly detectable, then the covariance matrix P_k satisfying (42) is asymptotically bounded and the closed-loop matrix $\tilde{A}_k(M_k^*, K_k^*)$ of the UMV estimate for the system in (31)–(32) is exponentially stable. Consequently, for the event-based MMSE estimator in (9)–(13), the estimation error covariance matrix P_k satisfying (9), (11) and (13) is asymptotically bounded and the closed-loop matrix of the estimator is exponentially stable for each sample path of $\{\gamma_k\}$.*

Proof. Since (A_k, G_k, C_k) is uniformly detectable, there exists a filter of the form in (37) parameterized by a bounded pair $\{M_k, K_k\}$ satisfying $M_k C_k G_{k-1} = I_p$ such that $\tilde{A}_k(M_k, K_k)$ is exponentially stable. Since M_k and K_k are bounded, $\tilde{F}_k(M_k, K_k)$ and $\tilde{G}_k(M_k, K_k)$ are bounded. From Lemma 4.2 in Anderson and Moore (1981), the solution $\tilde{P}_k$ to the Lyapunov equation

$$\tilde{P}_k = \phi_k(M_k, K_k, \tilde{P}_{k-1})$$

is bounded as $k \to \infty$. On the other hand, by the optimality of M_k^* and K_k^*, we have $P_k \leq \tilde{P}_k$, which indicates that P_k is bounded as $k \to \infty$. Since matrices P_k, Q_k and $\tilde{R}_k$ are bounded, the matrices $\tilde{A}_k(M_k^*, K_k^*)$ and $[\tilde{F}_k(M_k^*, K_k^*)\ \tilde{G}_k(M_k^*, K_k^*)]$ are bounded as well. Furthermore, for any matrix A with its eigenvalues lying in the unit circle, we have

$$\tilde{A}_k(M_k^*, K_k^*) + \left[\tilde{F}_k(M_k^*, K_k^*)Q_k^{1/2}\ \tilde{G}_k(M_k^*, K_k^*)R_k^{1/2}\right] \times \begin{bmatrix} Q_k^{-1/2}(-A + A_{k-1}) \\ R^{-1/2}C_k A \end{bmatrix} = A,$$

which implies that the pair

$$\left(\tilde{A}_k(M_k^*, K_k^*),\ \left[\tilde{F}_k(M_k^*, K_k^*)Q_k^{1/2}\ \tilde{G}_k(M_k^*, K_k^*)R_k^{1/2}\right]\right)$$

is uniformly stabilizable. The exponential stability of $\tilde{A}_k(M_k^*, K_k^*)$ follows from the asymptotical boundedness of P_k and Theorem 4.3 of Anderson and Moore (1981).

From the above derivations, we observe that the condition required to guarantee the asymptotic boundedness of P_k of the form in (28) depends only on the detectability notion defined on the triplet (A_k, G_k, C_k) and does not depend on $\tilde{R}_k$. Since the effect of the event-triggering process γ_k is only reflected on $\tilde{R}_k$ in the estimation error covariance update equations in (28)–(30) for the event-based MMSE estimator, we conclude that the estimation error covariance matrix P_k satisfying (9), (11) and (13) is asymptotically bounded and the closed-loop matrix of the estimator is exponentially stable along each sample path of $\{\gamma_k\}$, which completes the proof. □

Remark 7. For the standard Kalman filter, the key requirement for stability of the closed-loop matrix and the boundedness/convergence of the solution to the Riccati equation is the uniform detectability of (A_k, C_k) and the uniform stabilizability of (A_k, Q_k). As Q_k is assumed to be positive definite in this work, which is necessary to guarantee the existence of the event-based estimator, the requirement on stabilizability is automatically fulfilled. Due to the existence of the unknown input d_k, an alternative notion of uniform detectability is necessary to guarantee the asymptotic properties. Note that the introduction of the $\tilde{R}_k$ matrix in the form of (30) does not affect the asymptotic properties, which is consistent with the results for the Kalman filter and the time-varying Riccati equation theory (De Nicolao, 1992).

Apart from the boundedness and stability properties, one more question to ask is the effect of the initial condition P_0 on the asymptotic performance. Let $\psi_k(P)$ denote the solution to Eq. (42) at time k with initial condition $P_0 = P$. We have the following result.

Proposition 8. *Let P_1 and P_2 be two $n \times n$ positive semidefinite matrices. If the triplet (A_k, G_k, C_k) is uniformly detectable,*

$$\lim_{k\to\infty} [\psi_k(P_1) - \psi_k(P_2)] = 0.$$

Proof. This result can be proved following a similar argument of the convergence proof for Theorem 1 in Fang and de Callafon (2012) and thus is omitted. □

Remark 9. This result implies that for the time-varying case, the estimation error covariance of the UMV estimator (and thus the proposed event-based estimator for each sample path of $\{\gamma_k\}$) approaches the unique "moving equilibrium" as $k \to \infty$, which does not depend on the initial condition P_0.

Before continuing, we emphasize that since the detectability notion is defined based on A_k, G_k and C_k (rather than $\tilde{R}_k$), the event trigger does not affect the asymptotic boundedness of the estimation error covariance matrix along a single sample path of $\{\gamma_k\}$, which is another benefit of the event-triggering scheme considered together with the estimator proposed. The above results investigate the asymptotic properties of the event-based estimator for an arbitrary sample path of $\{\gamma_k\}$, and answer the first question listed at the beginning of this section. The next result summarizes the properties of $\mathrm{E}(P_k)$, which provides an answer to the second question.

Theorem 10. *For the event-based MMSE estimator together with its estimation error covariance matrix P_k defined in (9), (11) and (13), $\mathrm{E}(P_k)$ is bounded for all choices of event-triggering conditions that satisfy $Y_k > 0$. In particular,*

$$\mathrm{E}(P_k) \le \bar{P}_k$$

holds, where

$$\bar{P}_k = [C_k^\top (R_k + Y_k^{-1})^{-1} C_k + L_{k-1}^\top (L_{k-1}\bar{P}_{k|k-1}L_{k-1}^\top)^{-1} L_{k-1}]^{-1}, \tag{43}$$

$$\bar{P}_{k|k-1} = A_{k-1}\bar{P}_{k-1}A_{k-1}^\top + Q_{k-1} \tag{44}$$

and $\bar{P}_0 = P_0$.

Proof. As the estimation error covariance P_k is bounded for each sample path and $\{\gamma_k\}$ has a countable number of sample paths, it follows that the expectation of the estimation error covariance $\mathrm{E}(P_k)$ is bounded as well. In addition, since $Y_k > 0$ and $R_k + \gamma_k Y_k^{-1} \le R_k + Y_k^{-1}$, we have

$$P_k \le \bar{P}_k$$

for each sample path of $\{\gamma_k\}$, which completes the proof of the theorem. □

The above theorem presents a discussion on the performance of the event-based estimator in terms of average estimation error covariance. Due to the existence of the unknown term d_k with a non-informative prior, however, the behavior of the system in (1)–(2) at steady state, even for the LTI case, becomes unpredictable. As a result, the communication rate analysis is not feasible in general, making it difficult to obtain tight lower bounds for $\mathrm{E}(\gamma_k)$. In this regard, unlike the results obtained for a Gaussian system without the unknown input term d_k (e.g., Han et al., 2013), it is not likely to obtain tighter upper and lower bounds for $\mathrm{E}(P_k)$ based on communication rate analysis.

5. Extension to the multiple sensor scenario

In this section, we extend the results obtained in the previous sections to the multiple sensor scenario, in which case each sensor determines whether to send the measurement information to the remote estimator according to their own event-triggering conditions (see Fig. 2).

To do this, consider the process in (1) measured by M sensors[3] with the ith sensor described by

$$y_k^i = C_k^i x_k + v_k^i, \tag{45}$$

where $v_k^i \in \mathbb{R}^m$ is zero-mean Gaussian with covariance $R_k^i > 0$. In addition, x_0, w_k and v_k^i are independent from each other. Again we consider the case that the communication between a sensor and the estimator is reliable in the sense that the effect of time delays and packet dropouts is ignored. To reduce the communication

[3] For notational brevity, we assume all sensors have m channels, although the results obtained apply equally to the case of sensors with different or a time-varying number of channels as well.

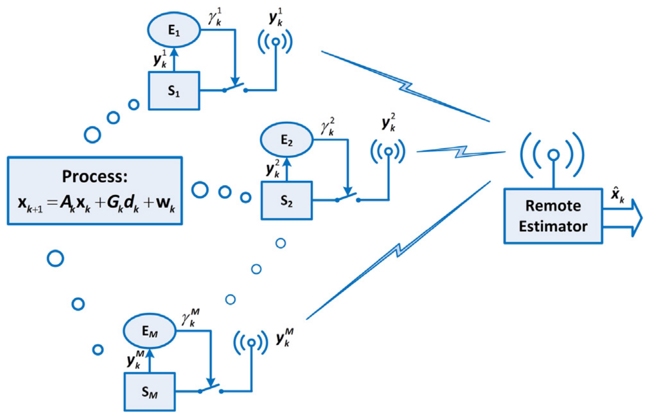

Fig. 2. Block diagram of the overall system for the multiple sensor scenario, where S_i and E_i denote the ith sensor and the ith event-triggered data scheduler, respectively.

burden, each sensor is equipped with its own event-triggering condition:

$$\Pr(\gamma_k^i | y_k^i = y) = \phi_k^i(y) \tag{46}$$

with γ_k^i depending only on y_k^i and

$$\phi_k^i(y) = \exp\left[-\frac{1}{2}(y - \xi_k^i)^\top Y_k^i (y - \xi_k^i)\right], \tag{47}$$

ξ_k^i being a known parameter at time instant k and Y_k^i being a positive definite weighting matrix. Let

$$C_k := [(C_k^1)^\top, (C_k^2)^\top, \ldots, (C_k^M)^\top]^\top, \tag{48}$$

$$R_k := \mathrm{diag}\{R_k^1, R_k^2, \ldots, R_k^M\}, \tag{49}$$

and

$$\bar{Y}_k := \mathrm{diag}\{(1-\gamma_k^1)\cdot(Y_k^1)^{-1}, \ldots, (1-\gamma_k^M)\cdot(Y_k^M)^{-1}\}.$$

For this scenario, we have the following result on the MMSE estimator.

Theorem 11. *For the multiple-sensor remote state estimation scheme in Fig. 2 with exogenous unknown input and the stochastic event-triggering condition in (47), the conditional distribution of x_k on $\mathcal{I}_k$ is Gaussian with mean $\hat{x}_k$ and covariance P_k provided*

$$\mathrm{rank}\begin{bmatrix} C_k^\top & L_{k-1}^\top \end{bmatrix}^\top = n. \tag{50}$$

If the condition in (50) is satisfied, $\hat{x}_k$ and P_k evolve according to the following recursive form:

$$\begin{aligned}\hat{x}_k = {} & A_{k-1}\hat{x}_{k-1} + \sum_{i=1}^M P_k (C_k^i)^\top [R_k^i + (1-\gamma_k^i)\cdot(Y_k^i)^{-1}]^{-1} \\ & \cdot \left[\gamma_k^i y_k^i + (1-\gamma_k^i)\xi_k^i - C_k^i A_{k-1}\hat{x}_{k-1}\right],\end{aligned} \tag{51}$$

$$\begin{aligned}P_k = {} & [C_k^\top (R_k + \bar{Y}_k)^{-1} C_k \\ & + L_{k-1}^\top (L_{k-1} P_{k|k-1} L_{k-1}^\top)^{-1} L_{k-1}]^{-1},\end{aligned} \tag{52}$$

$$P_{k|k-1} = A_{k-1} P_{k-1} A_{k-1}^\top + Q_{k-1}, \tag{53}$$

which is also the event-based MMSE estimator for the multiple sensor scenario.

Proof. The proof of this result follows a similar procedure as that for Theorem 2 and thus only the key differences are stated. First we note that T_k is not affected by the sensor measurement equations, and for sensor i, the measurement equation can be written as

$$y_k^i = H_k^i z_k + v_k^i, \tag{54}$$

with $H_k^i := C_k^i T_{k-1}^{-1}$. Define $\mathbb{I}_k$ as

$$\mathbb{I}_k := \{i \in \mathbb{N}_{1:M} | \gamma_k^i = 0\},$$

and let $\bar{\mathbb{I}}_k := \mathbb{N}_{1:M} \setminus \mathbb{I}_k$. By Bayesian law, we have

$$\begin{aligned}
f(z_k|\mathcal{I}_k) &= f\left(z_k | \mathcal{I}_{k-1}, \{\gamma_k^i y_k^i | i \in \mathbb{N}_{1:M}\}\right) \\
&\propto \mathrm{P}\left(\{\cap_{i\in\mathbb{I}_k} \gamma_k^i = 0\} \cap \{\cap_{i\in\bar{\mathbb{I}}_k} y_k^i = y^i\} | z_k, \mathcal{I}_{k-1}\right) \cdot f(z_k|\mathcal{I}_{k-1}) \\
&= f(z_k|\mathcal{I}_{k-1}) \cdot \int_{\mathbb{R}^m} \mathrm{P}\left(\{\cap_{i\in\mathbb{I}_k} \gamma_k^i = 0\} \cap \{\cap_{i\in\mathbb{N}_{1:M}\setminus\mathbb{I}_k} y_k^i = y^i\}\right. \\
&\quad \left. | y_k^{1:M}, z_k, \mathcal{I}_{k-1}\right) f(y_k^{1:M} | z_k, \mathcal{I}_{k-1}) \mathrm{d}y_k \\
&= f(z_k|\mathcal{I}_{k-1}) \\
&\quad \cdot \prod_{i=1}^M \int_{\mathbb{R}^m} \left[\mathrm{P}(\gamma_k^i = 0 | y_k^i, \mathcal{I}_{k-1})\right]^{1-\gamma_k^i} f(y_k^i | z_k, \mathcal{I}_{k-1}) \mathrm{d}y_k^i \\
&\propto f(z_k|\mathcal{I}_{k-1}) \\
&\quad \cdot \prod_{i=1}^M \int_{\mathbb{R}^m} \left[\exp\left[-\frac{1}{2}(y_k^i - \xi_k^i)^\top Y_k^i (y_k^i - \xi_k^i)\right]\right]^{1-\gamma_k^i} \\
&\quad \cdot \exp\left[-\frac{1}{2}(y_k^i - H_k^i z_k)^\top (R_k^i)^{-1} (y_k^i - H_k^i z_k)\right] \mathrm{d}y_k^i \\
&\propto \exp\left\{-\frac{1}{2}\left[\sum_{i\in\mathbb{I}_k} (H_k^i z_k - \xi_k^i)^\top [R_k^i + (Y_k^i)^{-1}]^{-1}\right.\right. \\
&\quad \left.\left. \cdot (H_k^i z_k - \xi_k^i) + \sum_{j\in\bar{\mathbb{I}}_k} (y_k^j - H_k^j z_k)^\top (R_k^j)^{-1} (y_k^j - H_k^j z_k)\right]\right\}
\end{aligned} \tag{55}$$

where (55) follows from the fact that γ_k^i depends on y_k^i and that y_k^i are mutually independent given z_k. The rest of the proof can be completed following a similar argument as that of Theorem 2. □

Remark 12. The above result indicates that to guarantee the existence of the event-based estimator for the multiple sensor scenario, the rank condition does not have to be satisfied for each sensor; instead, a weaker requirement is necessary, which imposed the rank condition on the lumped C_k matrix, see Eq. (50). Due to the rank condition, the covariance matrix can only be updated in a lumped fashion in (52), unlike the case of the standard Kalman filter, which can be calculated by sequentially updating each (C_k^i, R_k^i) pair as well. In addition, from (52), it is also observed that the change of sensor fusion sequences does not affect the fusion results, and that introducing additional sensor measurements always improves the estimation performance in terms of estimation error covariance.

Remark 13. Write

$$\underline{\gamma}_k := \mathrm{diag}\{\gamma_k^1, \gamma_k^2, \ldots, \gamma_k^M\} \otimes I_m,$$

$$y_k := [(y_k^1)^\top, (y_k^2)^\top, \ldots, (y_k^M)^\top]^\top,$$

$$\xi_k := [(\xi_k^1)^\top, (\xi_k^2)^\top, \ldots, (\xi_k^M)^\top]^\top.$$

It is easy to show that Eq. (51) can be rewritten as

$$\begin{aligned}\hat{x}_k = {} & A_{k-1}\hat{x}_{k-1} + P_k C_k^\top [R_k + \bar{Y}_k]^{-1} \\ & \cdot \left[\underline{\gamma}_k y_k + (I_{Mm} - \underline{\gamma}_k)\xi_k - C_k A_{k-1}\hat{x}_{k-1}\right],\end{aligned} \tag{56}$$

which indicates that despite the introduction of the event-triggering conditions, the estimator can be updated in a lumped fashion as well, treating ξ_k^i as a virtual sensor measurement when y_k^i is not available; this is different from the multiple sensor scenario with deterministic event-triggers, for which the estimates have to be updated by sequentially fusing the information from different sensors (Shi et al., 2014a). The estimator in the form of (51), however, indicates that based on the value of P_k, it is possible to implement the estimator in a distributed fashion on the side of the remote estimator.

As the estimation error covariance equation has a similar form as (11), the asymptotic properties of the multiple sensor case can be proved following a similar argument as that in Theorem 11, which is summarized in the next result but the proof is omitted.

Theorem 14. *If the triplet (A_k, G_k, C_k) is uniformly detectable, then for each sample path of $\{\gamma_k^{1:M}\}$,*

(1) *the estimation error covariance matrix P_k satisfying (52)–(53) is asymptotically bounded and approaches the unique moving equilibrium as $k \to \infty$; and*
(2) *the closed-loop matrix of the event-based estimator in (51) for the multiple sensor scenario is exponentially stable.*

Finally, following the same argument as that in the proof of Theorem 10, $\mathrm{E}(P_k) \le \bar{P}_k$ holds for the multiple sensor scenario as well, with $\bar{P}_k$ defined by

$$\bar{P}_k = [C_k^\top (R_k + Y_k^{-1})^{-1} C_k + L_{k-1}^\top (L_{k-1}\bar{P}_{k|k-1}L_{k-1}^\top)^{-1} L_{k-1}]^{-1},$$
$$\bar{P}_{k|k-1} = A_{k-1}\bar{P}_{k-1}A_{k-1}^\top + Q_{k-1},$$

where C_k and R_k are defined in (48)–(49), and $Y_k := \mathrm{diag}\{Y_k^1, \ldots, Y_k^M\}$.

6. Numerical example

In this section, we illustrate the proposed event-based estimator for systems with exogenous unknown inputs by a numerical example. Consider a stable linear time-varying system of the form in (1) with the following matrix parameters

$$A_k = \begin{bmatrix} a_{11,k} & a_{12,k} & a_{13,k} \\ a_{21,k} & a_{22,k} & a_{23,k} \\ a_{31,k} & a_{32,k} & a_{33,k} \end{bmatrix},$$

with

$$a_{11,k} = \exp[-h + \sin(kh) - \sin(kh - h)],$$
$$a_{12,k} = 0,\ a_{13,k} = 0,$$
$$a_{21,k} = 2\sinh(h/2)\exp[-3h/2 + \sin(kh) - \sin(kh - h)],$$
$$a_{22,k} = \exp[-2h + \sin(kh) - \sin(kh - h)],\ a_{23,k} = 0,$$
$$a_{31,k} = 0,\ a_{32,k} = 0,$$
$$a_{33,k} = \exp[-2h + \sin(kh) - \sin(kh - h)],$$

$h = 0.2$ and

$$Q_k = \begin{bmatrix} 0.6050 & 0.6000 & 0.1700 \\ 0.6000 & 1.0000 & 0.5200 \\ 0.1700 & 0.5200 & 0.9240 \end{bmatrix}.$$

For the purpose of illustrating the proposed results, a stable system obtained by construction is utilized here to ensure the trajectory of the states are bounded, so that the system will not blow up in finite time. In accordance with the proposed results, two different scenarios of sensor measurement and unknown input settings are considered.

(1) Multiple sensor scenario. In this case, the system is measured by three sensors with matrix parameters

$$C_k^1 = \begin{bmatrix} 1 & \cos(kh) & \sin(kh) \end{bmatrix},$$
$$C_k^2 = \begin{bmatrix} \sin(kh) & 2 & \cos(kh) \end{bmatrix},$$
$$C_k^3 = \begin{bmatrix} \cos(kh) & \sin(kh) & 1.5 \\ 1 & \sin(2kh) & \cos(2kh) \end{bmatrix},$$
$$R_k^1 = 0.2,\ R_k^2 = 0.3,$$
$$R_k^3 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.25 \end{bmatrix}.$$

For this case, the unknown signal d_k utilized is taken to have the form shown in Fig. 3 with

$$G_k = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.2 & 0.3 & 0 \end{bmatrix}^\top.$$

For this case, the condition in (50) is satisfied for all k, as it can be verified that C_k has full column rank. Each sensor determines whether or not to send their current measurement to the estimator according to their own event-triggering conditions, which is taken to have the stochastic "send-on-delta" form by taking

$$\xi_k^i = y_{\tau_k^i}^i,$$

with $y_{\tau_k^i}^i$ being the previously transmitted measurement of sensor i. To investigate the estimation performance under different sensor-to-estimator communication rates, two sets of Y_k^i's are considered:

(1) Parameter setting I: $Y_k^1 = 1.8$, $Y_k^2 = 0.9$, and

$$Y_k^3 = \begin{bmatrix} 1.4 & 0.4 \\ 0.4 & 1.6 \end{bmatrix};$$

(2) Parameter setting II: $Y_k^1 = 90$, $Y_k^2 = 45$, and

$$Y_k^3 = \begin{bmatrix} 35 & 10 \\ 10 & 40 \end{bmatrix}.$$

The event-based MMSE estimator in (51) is implemented to estimate the state trajectories, and the performance is shown in Figs. 4 and 5, respectively, where the average communication rates $\sum_{i=1}^{3} \gamma_k^i/3$ at each time instant are included to provide the access rates of different sensors. The average communication rates $\sum_{k=1}^{160} \gamma_k^i$ for the three sensors obtained using parameter setting I are 0.57, 0.70 and 0.86, respectively, while the average communication rates for the three sensors obtained using parameter setting II are 0.16, 0.26 and 0.36, respectively. To illustrate the effect of the unknown inputs, the undisturbed states obtained according to

$$x_{k+1} = A_k x_k + w_k$$

and using the same realization of w_k are also included in Figs. 4 and 5. For comparison, the time-triggered MMSE estimate obtained by using all past measurement information is also plotted. Compared with the event-based MMSE estimate, more information is exploited to generate this estimate, and therefore theoretically it has improved performance in terms of estimation covariance; indeed, the actual average estimation errors for the event-based MMSE estimate and the time-triggered MMSE estimate are 0.5811 and 0.5073 for parameter setting I and 3.4924 and 0.5190 for parameter setting II, respectively. The observation here is that despite the obviously decreased communication cost between the sensors and the remote estimator, the event-based MMSE estimates still track the states of the system under the existence of the unknown input term. One potential antetype of this example is the secure estimation of cyber-physical systems (CPS) with a network of sensors, in which case the communication and energy resources can be limited and external hazardous attack inputs do sometimes exist to deviate the process from its normal operating points. The results shown here, however, indicate that despite the fact that no information is available for the attack inputs and the limited access to the communication channels, the states of the processes under attack can still be safely estimated and monitored with satisfactory performance.

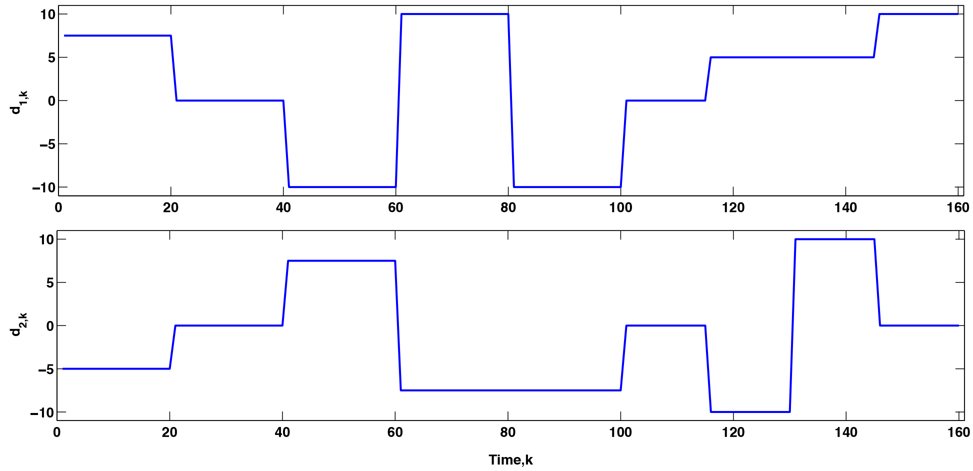

Fig. 3. The unknown input signals.

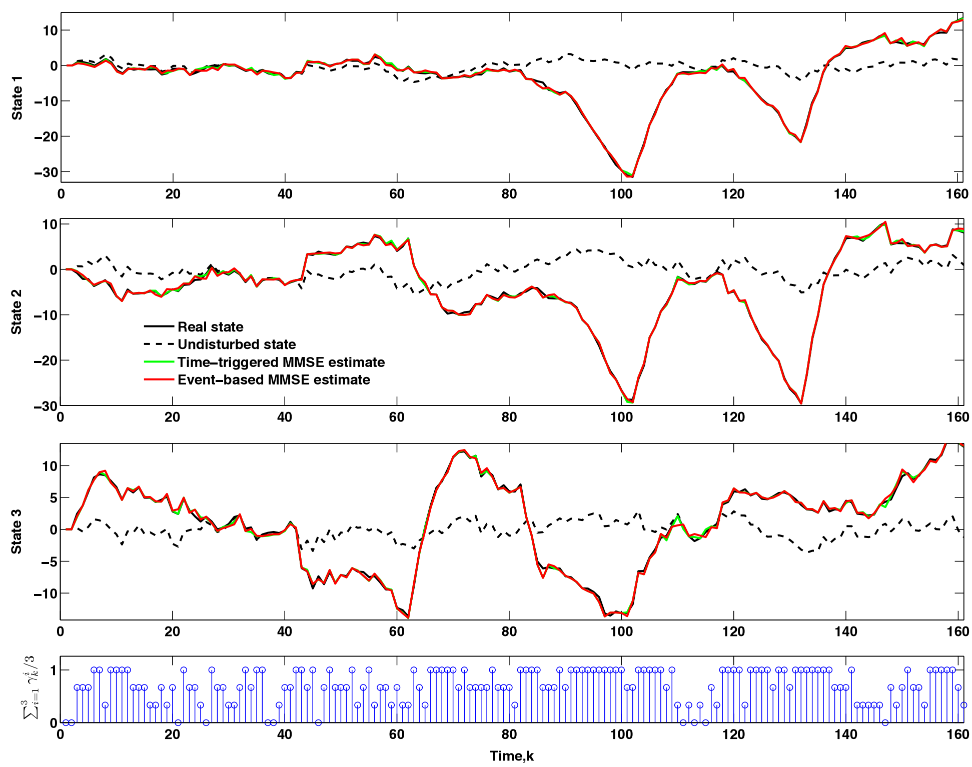

Fig. 4. Performance of the event-based MMSE estimator for parameter setting I.

(2) Single sensor scenario. This case is employed to investigate the tradeoff between the estimation performance (in terms of average estimation error) and average communication rate for the event-based MMSE estimator, which can be achieved by varying the Y_k parameter in the event-triggering condition in (4). To do this, consider a sensor with parameter matrices

$$C_k = \begin{bmatrix} 1 & \cos(kh) & \sin(kh) \end{bmatrix}$$

and $R_k = 0.2$. For this case, a scalar-valued unknown input signal d_k is randomly generated according to the uniform distribution between 0 and 10, and G_k is given by

$$G_k = \begin{bmatrix} 0.1 & 0.3 & 0.2 \end{bmatrix}^\top .$$

Since C_k and G_k are both rank 1 matrices, the condition in (8) holds according to Remark 3. The stochastic "send-on-delta" condition is still utilized with a constant Y_k. To evaluate the tradeoff between the communication rate and estimation performance, Monte-Carlo simulation experiments are performed for different values of Y_k between 0.01 and 10^3. The obtained results are provided in Fig. 6, where the relative estimation errors of the proposed event-based MMSE estimate obtained by deducting the average estimation error of the time-triggered MMSE estimator from those of the event-based MMSE estimator under different average communication rates are provided. From this figure, we observe that the performance of the event-based MMSE estimator stays close to that of the time-triggered MMSE estimator even when the communication rate is cut down by more than 50%. Therefore, with the help of the event-triggered data scheduler, it is possible to decrease the average communication rate while maintaining the estimation performance at a satisfactory level.

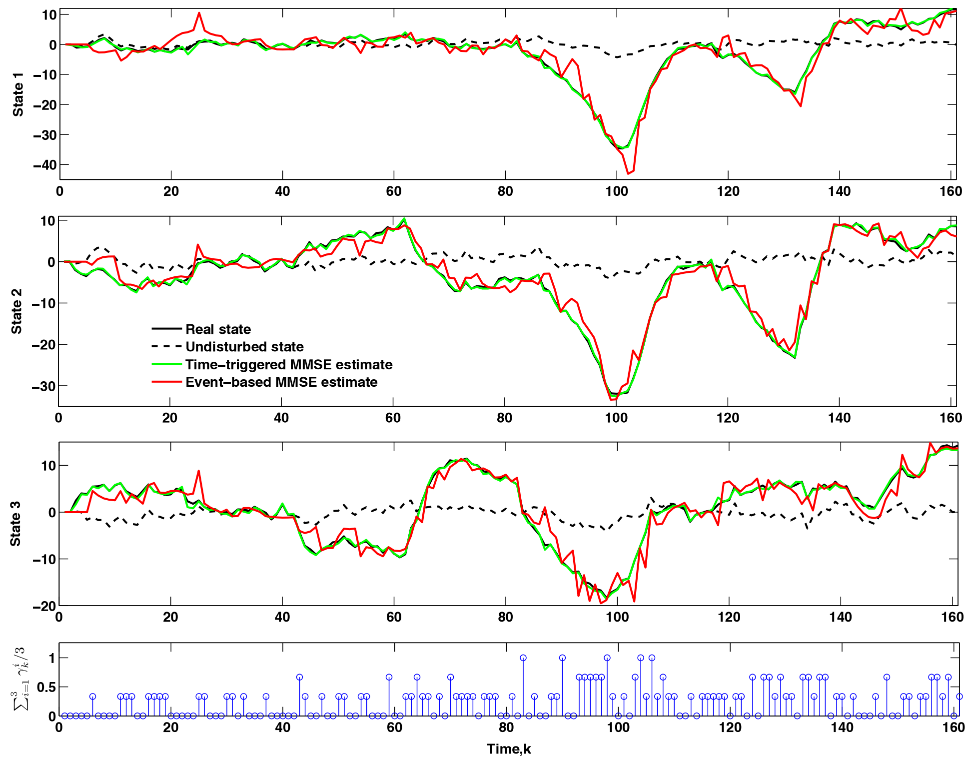

Fig. 5. Performance of the event-based MMSE estimator for parameter setting II.

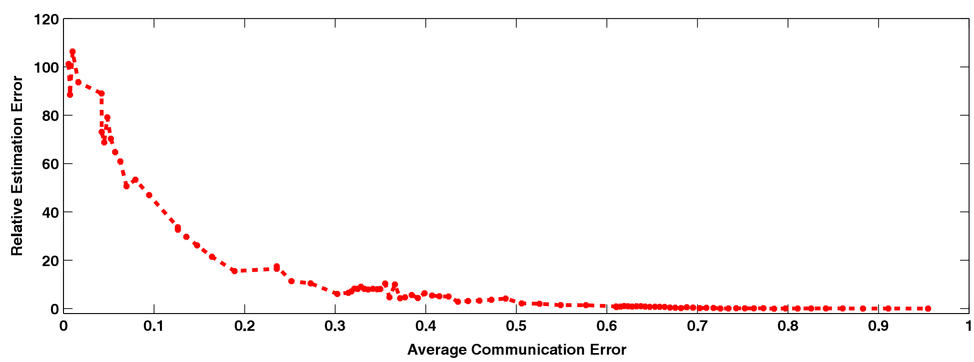

Fig. 6. Tradeoff between the estimation performance and the average communication rate.

7. Conclusion

In this work, the event-based state estimation problem for linear-time varying systems with unknown inputs has been investigated for both the single sensor case and the multiple sensor case. By treating the unknown input as a process with a non-informative prior, the event-based MMSE estimates are obtained in a recursive form. It is proved that the obtained optimal event-based estimators are exponentially stable and the estimation error covariance matrices are asymptotically bounded for each sample path of the event-triggering process, which implies that the expectation of the estimation error covariance is asymptotically bounded as well. In the present event-based estimation framework, the unknown input signal affects only the state equation of the system; an interesting extension is to consider systems with unknown inputs in both state equations and sensor measurement equations, which will be investigated in our next step. On the other hand, the communication channel between a sensor and the estimator is assumed to be reliable in the current work; for the case that the unknown input term does not exist, it is known that the consideration of the effect of packet dropouts and time delays induced by the communication network is in general difficult in optimal event-based estimator design. One possible way of handling this issue, however, is to consider alternative modeling frameworks (e.g., using finite-state hidden Markov models (Elliott, Aggoun, & Moore, 1995)), which forms another interesting direction for our future work.

Acknowledgments

The authors would like to thank the Associate Editor and the anonymous reviewers for their suggestions which have improved the quality of the work.

Appendix A. Lemma 15

Lemma 15. *Let $A \in \mathbb{R}^{p\times n}$, $X \in \mathbb{R}^{p\times p}$, $C \in \mathbb{R}^{q\times n}$, $Y \in \mathbb{R}^{q\times q}$. Let x, b, d be vectors with appropriate dimensions.*

(1) *If* $\text{rank}(A^\top XA + C^\top YC) = n$, *then*

$$(Ax+b)^\top X(Ax+b) + (Cx+d)^\top Y(Cx+d)$$
$$= \left[x + (A^\top XA + C^\top YC)^{-1}(A^\top Xb + C^\top Yd)\right]^\top$$

$$(A^\top XA + C^\top YC)\left[x + (A^\top XA + C^\top YC)^{-1}(A^\top Xb + C^\top Yd)\right] + \star,$$

where $\star$ denotes a term unrelated with x.

(2) *If X and Y are nonsingular,*

$$x^\top Xx + (x-b)^\top Y(x-b) = \left[x - (X+Y)^{-1}Yb\right]^\top (X+Y)\left[x - (X+Y)^{-1}Yb\right] + b^\top (X^{-1}+Y^{-1})^{-1}b.$$

Proof. The first expression can be verified by completing the squares and noticing that $A^\top XA + C^\top YC$ is nonsingular. The second expression can be obtained by completing the squares and the matrix inversion lemma. □

Appendix B. Lemma 16

Lemma 16 (*Lemma 2 in Li (2013)*). *Let C and D be matrices such that $D^\top$ and $[C^\top\ D^\top]^\top$ have full column ranks. Let F be the orthogonal complement of $D^\top$ such that $DF = \mathbf{0}$. For positive definite matrices P and R with appropriate dimensions, the following equations hold:*

$$\begin{aligned}&[D^\top(DPD^\top)^{-1}D + C^\top R^{-1}C]^{-1}\\ &\quad = P - PC^\top H^{-1}CP + (F - PC^\top H^{-1}CF)\\ &\qquad\cdot (F^\top C^\top H^{-1}CF)^{-1}(F - PC^\top H^{-1}CF)^\top,\end{aligned} \tag{B.1}$$

with $H = CPC^\top + R$.

Appendix C. Explanation of (18)

First we observe that

$$D_{k-1}d_{k-1} = \begin{bmatrix} d_{k-1}\\ \mathbf{0}\end{bmatrix}$$

with $d_{k-1} \in \mathbb{R}^p$. Noticing that $D_{k-1} = [I_p\ \mathbf{0}]^\top$ and $\bar{D}_{k-1} = [0\ I_{n-p}]$, we have

$$\begin{bmatrix} D_{k-1}^\top\\ \bar{D}_{k-1}\end{bmatrix} = I_n.$$

Let $\hat{z}_k := z_k - F_{k-1}\hat{x}_{k-1}$ and write

$$\hat{z}_k := \begin{bmatrix}\hat{z}_{k,1}\\ \hat{z}_{k,2}\end{bmatrix} = \begin{bmatrix} D_{k-1}^\top \hat{z}_k\\ \bar{D}_{k-1}\hat{z}_k\end{bmatrix}. \tag{C.1}$$

For $P^z_{k|k-1}$, we have

$$P^z_{k|k-1} = \begin{bmatrix} D_{k-1}^\top\\ \bar{D}_{k-1}\end{bmatrix} P^z_{k|k-1}\left[D_{k-1}\ \bar{D}_{k-1}^\top\right] \tag{C.2}$$

$$= \begin{bmatrix} D_{k-1}^\top P^z_{k|k-1}D_{k-1} & D_{k-1}^\top P^z_{k|k-1}\bar{D}_{k-1}^\top\\ \bar{D}_{k-1}P^z_{k|k-1}D_{k-1} & \bar{D}_{k-1}P^z_{k|k-1}\bar{D}_{k-1}^\top\end{bmatrix}. \tag{C.3}$$

To obtain (18), we observe that

$$\begin{aligned} f(z_k|\mathcal{I}_{k-1}) &\propto \int_{\mathbb{R}^p}\exp\Big(-\frac{1}{2}\left[z_k - (F_{k-1}\hat{x}_{k-1} + D_{k-1}d_{k-1})\right]^\top (P^z_{k|k-1})^{-1}\left[z_k - (F_{k-1}\hat{x}_{k-1} + D_{k-1}d_{k-1})\right]\Big)\mathrm{d}d_{k-1}\\ &= \int_{\mathbb{R}^p}\exp\left(-\frac{1}{2}\begin{bmatrix} d_{k-1} - \hat{z}_{k,1}\\ -\hat{z}_{k,2}\end{bmatrix}^\top \begin{bmatrix} D_{k-1}^\top P^z_{k|k-1}D_{k-1} & D_{k-1}^\top P^z_{k|k-1}\bar{D}_{k-1}^\top\\ \bar{D}_{k-1}P^z_{k|k-1}D_{k-1} & \bar{D}_{k-1}P^z_{k|k-1}\bar{D}_{k-1}^\top\end{bmatrix}^{-1} \begin{bmatrix} d_{k-1} - \hat{z}_{k,1}\\ -\hat{z}_{k,2}\end{bmatrix}\right)\mathrm{d}d_{k-1}\end{aligned} \tag{C.4}$$

$$\propto \exp\left[-\frac{1}{2}(z_k - F_{k-1}\hat{x}_{k-1})^\top \bar{D}_{k-1}^\top(\bar{D}_{k-1}P^z_{k|k-1}\bar{D}_{k-1}^\top)^{-1}\bar{D}_{k-1}(z_k - F_{k-1}\hat{x}_{k-1})\right], \tag{C.5}$$

where (C.5) is due to the properties of the marginal distribution of a multivariate Gaussian random variable.

References

Anderson, B. D. O., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs, NJ: Prentice-Hall.

Anderson, B. D. O., & Moore, J. B. (1981). Detectability and stabilizability of time-varying discrete-time linear systems. *SIAM Journal on Control and Optimization*, *19*(1), 20–32.

Burghaus, I., & Dette, H. (2014). Optimal designs for nonlinear regression models with respect to non-informative priors. *Journal of Statistical Planning and Inference*, *154*, 12–25.

Cheng, Y., Ye, H., Wang, Y., & Zhou, D. (2009). Unbiased minimum-variance state estimation for linear systems with unknown input. *Automatica*, *45*(2), 485–491.

Cristofaro, A., & Johansen, T. (2014). Fault tolerant control allocation using unknown input observers. *Automatica*, *50*(7), 1891–1897.

Dalton, L., & Dougherty, E. (2011). Bayesian minimum mean-square error estimation for classification error – part i: Definition and the Bayesian mmse error estimator for discrete classification. *IEEE Transactions on Signal Processing*, *59*(1), 115–129.

Darouach, M., & Zasadzinski, M. (1997). Unbiased minimum variance estimation for systems with unknown exogenous inputs. *Automatica*, *33*(4), 717–719.

Darouach, M., Zasadzinski, M., & Boutayeb, M. (2003). Extension of minimum variance estimation for systems with unknown inputs. *Automatica*, *39*(5), 867–876.

Darouach, M., Zasadzinski, M., & Xu, S. (1994). Full-order observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, *39*(3), 606–609.

De Nicolao, G. (1992). On the time-varying riccati difference equation of optimal filtering. *SIAM Journal on Control and Optimization*, *30*(6), 1251–1269.

Elliott, R. J., Aggoun, L., & Moore, J. B. (1995). *Hidden Markov models: Estimation and control*. Springer.

Fang, H., & de Callafon, R. A. (2012). On the asymptotic stability of minimum-variance unbiased input and state estimation. *Automatica*, *48*(12), 3183–3186.

Fang, H., Shi, Y., & Yi, J. (2011). On stable simultaneous input and state estimation for discrete-time linear systems. *International Journal of Adaptive Control and Signal Processing*, *25*(8), 671–686.

Fawzi, H., Tabuada, P., & Diggavi, S. (2014). Secure estimation and control for cyber-physical systems under adversarial attacks. *IEEE Transactions on Automatic Control*, *59*(6), 1454–1467.

Figueiredo, M., & Nowak, R. (2001). Wavelet-based image estimation: an empirical bayes approach using Jeffrey's noninformative prior. *IEEE Transactions on Image Processing*, *10*(9), 1322–1331.

Floquet, T., Edwards, C., & Spurgeon, S. K. (2007). On sliding mode observers for systems with unknown inputs. *International Journal of Adaptive Control and Signal Processing*, *21*(8–9), 638–656.

Gillijns, S., & De Moor, B. (2007). Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, *43*(1), 111–116.

Han, D., Mo, Y., Wu, J., Sinopoli, B., & Shi, L. (2013). Stochastic event-triggered sensor scheduling for remote state estimation. In *52nd IEEE conference on decision and control* (pp. 6079–6084).

Heemels, W., Donkers, M., & Teel, A. (2013). Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, *58*(4), 847–861.

Imer, O., & Başar, T. (2005). Optimal estimation with limited measurements. In *Proceedings of the 44th IEEE conference on decision and control, and the European control conference* (pp. 1029–1034). Dec.

Jeffreys, H. (1946). An invariant form for the prior probability in estimation problems. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, *186*(1007), 453–461.

Johansen, T., Cristofaro, A., Sorensen, K., Hansen, J., & Fossen, T. (2015). On estimation of wind velocity, angle-of-attack and sideslip angle of small UAVs using standard sensors. In *2015 International conference on unmanned aircraft systems, ICUAS* (pp. 510–519).

Kerwin, W., & Prince, J. (2000). On the optimality of recursive unbiased state estimation with unknown inputs. *Automatica*, *36*(9), 1381–1383.

Kitanidis, P. K. (1987). Unbiased minimum-variance linear state estimation. *Automatica*, *23*(6), 775–778.

Lee, S., Liu, W., & Hwang, I. (2014). Markov chain approximation algorithm for event-based state estimation. *IEEE Transactions on Control Systems Technology PP*, (99), 1–1.

Levy, B. C. (2008). *Principles of signal detection and parameter estimation*. Springer.

Li, B. (2013). State estimation with partially observed inputs: A unified Kalman filtering approach. *Automatica*, *49*(3), 816–820.

Li, L., & Lemmon, M. (2011). Performance and average sampling period of sub-optimal triggering event in event triggered state estimation. In *Proceedings of the 50th IEEE conference on decision and control and European control conference* (pp. 1656–1661). Dec.

Li, L., Lemmon, M., & Wang, X. (2010). Event-triggered state estimation in vector linear processes. In *Proceedings of the 2010 American control conference* (pp. 2138–2143). July.

Marck, J., & Sijs, J. (2010). Relevant sampling applied to event-based state estimation. In *Proceedings of the 4th international conference on sensor technologies and applications* (pp. 618–624). Venice, Italy.

Meng, X., & Chen, T. (2012). Optimal sampling and performance comparison of periodic and event based impulse control. *IEEE Transactions on Automatic Control*, *57*(12), 3252–3259.

Meng, X., & Chen, T. (2014). Event detection and control co-design of sampled-data systems. *International Journal of Control*, *87*(4), 777–786.

Miao, F., Pajic, M., & Pappas, G. (2013). Stochastic game approach for replay attack detection. In *IEEE 52nd annual conference on decision and control* (pp. 1854–1859).

Miskowicz, M. (2006). Send-on-delta concept: An event-based data reporting strategy. *Sensors*, *6*(1), 49–63.

Mo, Y., & Sinopoli, B. (2014). Secure estimation in the presence of integrity attacks. *IEEE Transactions on Automatic Control*.

Molin, A., & Hirche, S. (2012). An iterative algorithm for optimal event-triggered estimation. In *4th IFAC conference on analysis and design of hybrid systems* (pp. 64–69). Eindhoven, Netherlands.

Mutambara, A. (1998). *Decentralized estimation and control for multisensor systems*. Boca Raton, Florida, USA: CRC Press, Inc..

Pasqualetti, F., Dorfler, F., & Bullo, F. (2013). Attack detection and identification in cyber-physical systems. *IEEE Transactions on Automatic Control*, *58*(11), 2715–2729.

Peters, M., & Iglesias, P. (1999). A spectral test for observability and reachability of time-varying systems. *SIAM Journal on Control and Optimization*, *37*(5), 1330–1345.

Rabi, M., Moustakides, G., & Baras, J. (2006). Multiple sampling for estimation on a finite horizon. In *Proceedings of the 45th IEEE conference on decision and control* (pp. 1351–1357).

Rabi, M., Moustakides, G., & Baras, J. (2012). Adaptive sampling for linear state estimation. *SIAM Journal on Control and Optimization*, *50*(2), 672–702.

Shi, D., Chen, T., & Shi, L. (2014a). An event-triggered approach to state estimation with multiple point-and set-valued measurements. *Automatica*, *50*(6), 1641–1648.

Shi, D., Chen, T., & Shi, L. (2014b). Event-triggered maximum likelihood state estimation. *Automatica*, *50*(1), 247–254.

Shi, D., Chen, T., & Shi, L. (2015). On set-valued Kalman filtering and its application to event-based state estimation. *IEEE Transactions on Automatic Control*, *60*(5), 1275–1290.

Shi, D., Elliott, R. J., & Chen, T. (2016). Event-based state estimation of discrete-state hidden Markov models. *Automatica*, *65*, 12–26.

Shi, L., Johansson, K., & Qiu, L. (2011). Time and event-based sensor scheduling for networks with limited communication resources. In *Proceedings of the 18th IFAC world congress* (pp. 13263–13268).

Sijs, J., Kester, L., & Noack, B. (2014). A study on event triggering criteria for estimation. In *17th International conference on information fusion* (pp. 1–8).

Sijs, J., & Lazar, M. (2012). Event based state estimation with time synchronous updates. *IEEE Transactions on Automatic Control*, *57*(10), 2650–2655.

Sijs, J., Noack, B., & Hanebeck, U. (2013). Event-based state estimation with negative information. In *Proceedings of the 16th international conference on information fusion* (pp. 2192–2199).

Sivaganesan, S., & Lingham, R. T. (2000). Bayes factors for a test about the drift of the brownian motion under noninformative priors. *Statistics & Probability Letters*, *48*(2), 163–171.

Åström, K., & Bernhardsson, B. (2002). Comparison of Riemann and Lebesgue sampling for first order stochastic systems. In *Proceedings of the 41st IEEE conference on decision and control. Vol. 2* (pp. 2011–2016). vol.2.

Su, J., Li, B., & Chen, W.-H. (2015). On existence, optimality and asymptotic stability of the kalman filter with partially observed inputs. *Automatica*, *53*, 149–154.

Svensson, L., & Lundberg, M. (2005). On posterior distributions for signals in gaussian noise with unknown covariance matrix. *IEEE Transactions on Signal Processing*, *53*(9), 3554–3571.

Trimpe, S. (2014). Stability analysis of distributed event-based state estimation. In *Proceedings of the 53rd IEEE conference on decision and control* (pp. 2013–2019).

Trimpe, S., & D'Andrea, R. (2011). An experimental demonstration of a distributed and event-based state estimation algorithm. In *Proceedings of the 18th IFAC world congress* (pp. 8811–8818). Milano, Italy.

Trimpe, S., & D'Andrea, R. (2014). Event-based state estimation with variance-based triggering. *IEEE Transactions on Automatic Control*, *59*(12), 3266–3281.

Wang, B., & Fu, M. (2014). Comparison of periodic and event-based sampling for linear state estimation. In *Proceedings of IFAC world congress*.

Weimer, J., Araujo, J., & Johansson, K. H. (2012). Distributed event-triggered estimation in networked systems. In *Proceedings of the 4th IFAC conference on analysis and design of hybrid systems* (pp. 178–185). Eindhoven, Netherlands.

Wu, J., Jia, Q., Johansson, K., & Shi, L. (2013). Event-based sensor data scheduling: Trade-off between communication rate and estimation quality. *IEEE Transactions on Automatic Control*, *58*(4), 1041–1045.

Yook, J., Tilbury, D., & Soparkar, N. (2002). Trading computation for bandwidth: reducing communication in distributed control systems using state estimators. *IEEE Transactions on Control Systems Technology*, *10*(4), 503–518.

Zhang, X.-M., & Han, Q.-L. (2015). Event-based filtering for sampled-data systems. *Automatica*, *51*(0), 55–69.

Zou, L., Wang, Z., Gao, H., & Liu, X. (2015). Event-triggered state estimation for complex networks with mixed time delays via sampled data information: The continuous-time case. *IEEE Transactions on Cybernetics*, *45*(12), 2804–2815.

Dawei Shi received his B.Eng. degree in Electrical Engineering and Automation from the Beijing Institute of Technology in 2008. He received his Ph.D. degree in Control Systems from the University of Alberta in 2014. Since December 2014, he has been appointed as an Associate Professor at the School of Automation, Beijing Institute of Technology, China.

His research interests include event-based control and estimation, robust model predictive control and tuning, and wireless sensor networks. He is a reviewer for a number of international journals, including IEEE Transactions on Automatic Control, Automatica, and Systems & Control Letters. In 2009, he received the Best Student Paper Award in IEEE International Conference on Automation and Logistics.

Tongwen Chen is presently a Professor of Electrical and Computer Engineering at the University of Alberta, Edmonton, Canada. He received the B.Eng. degree in Automation and Instrumentation from Tsinghua University (Beijing) in 1984, and the M.A.Sc. and Ph.D. degrees in Electrical Engineering from the University of Toronto in 1988 and 1991, respectively.

His research interests include computer and network based control systems, process safety and alarm systems, and their applications to the process and power industries. He has served as an Associate Editor for several international journals, including IEEE Transactions on Automatic Control, Automatica, and Systems and Control Letters. He is a Fellow of IFAC, IEEE, as well as the Canadian Academy of Engineering.

Mohamed Darouach graduated from "Ecole Mohammadia d'Ingénieurs", Rabat, Morocco, in 1978, and received the Doctor Engineer and Doctor of Sciences degrees from Nancy University, France, in 1983 and 1986, respectively. From 1978 to 1986 he was Associate Professor and Professor of automatic control at "Ecole Hassania des Travaux Publics", Casablanca, Morocco. Since 1987 he is a Professor at Université de Lorraine. He has been a Vice Director of the Research Center in Automatic Control of Nancy (CRAN UMR 7039, Nancy-University, CNRS) from 2005 to 2013. He obtained a degree Honoris Causa from the Technical University of IASI and since 2010 he is a member of the Scientific council of Luxembourg University. Since July 2013 he is a Vice Director of the University Institute of Technology of Longwy (Université de Lorraine). He held invited positions at University of Alberta, Edmonton. His research interests span theoretical control, observers design, and control of large-scale uncertain systems with applications.