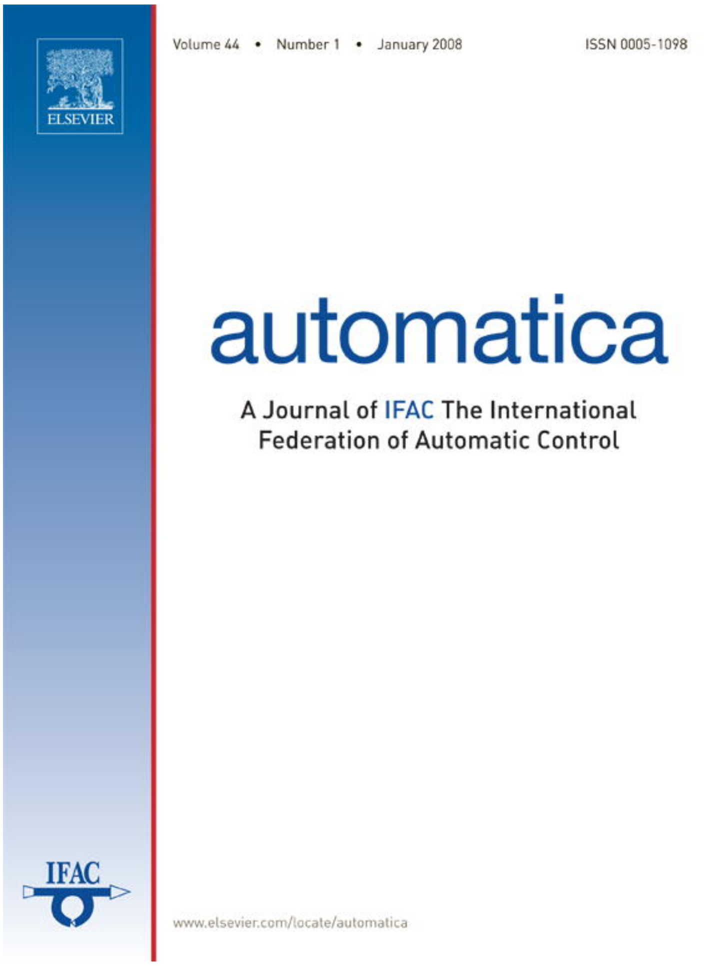
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A new delay system approach to network-based control ☆

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Abstract

This paper presents a new delay system approach to network-based control. This approach is based on a new time-delay model proposed recently, which contains multiple successive delay components in the state. Firstly, new results on stability and $\mathscr{H}_\infty$ performance are proposed for systems with two successive delay components, by exploiting a new Lyapunov–Krasovskii functional and by making use of novel techniques for time-delay systems. An illustrative example is provided to show the advantage of these results. The second part of this paper utilizes the new model to investigate the problem of network-based control, which has emerged as a topic of significant interest in the control community. A sampled-data networked control system with simultaneous consideration of network induced delays, data packet dropouts and measurement quantization is modeled as a nonlinear time-delay system with two successive delay components in the state and, the problem of network-based $\mathscr{H}_\infty$ control is solved accordingly. Illustrative examples are provided to show the advantage and applicability of the developed results for network-based controller design.

Keywords: Linear matrix inequality (LMI); Networked control systems (NCSs); Sampled-data systems; Stability; Time-delay systems

1. Introduction

Time-delay systems, also called systems with after effect or dead time, hereditary systems, equations with deviating argument or differential–difference equations, have been an active research area for the last few decades. The main reason is that many processes include after-effect phenomena in their inner dynamics, and engineers require models to behave more like real processes due to the ever-increasing expectations of dynamic performance. There have been a great number of research results concerning time-delay systems scattered in the literature. To mention a few, stability analysis is carried out in He, Wang, Lin, and Wu (2007), He, Wang, Xie, and Lin (2007b), He, Wu, She, and Liu (2004), Lin, Wang, and Lee (2006), Xia and Jia (2002); stabilizing and $\mathscr{H}_\infty$ controllers are designed in Hua, Guan, and Shi (2005), and Zhang, Wu, She, and He (2005); robust filtering is addressed in Gao and Wang (2004), Liu, Sun, He, and Sun (2004), Wang and Burnham (2001), Wang, Huang, and Unbehauen (1999); and model reduction/simplification is investigated in Gao, Lam, Wang, and Xu (2004) and Xu, Lam, Huang, and Yang (2001). The importance of the study on time-delay systems is further highlighted by the recent survey paper (Richard, 2003) and monographs (Gu, Kharitonov, & Chen, 2003; Niculescu, 2001).

Closely related to time-delay systems, network-based control has emerged as a topic of significant interest in the control community. It is well known that in many practical systems, the physical plant, controller, sensor and actuator are difficult to be located at the same place, and thus signals are required to be transmitted from one place to another. In modern industrial systems, these components are often connected

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over network media (typically digital band-limited serial communication channels), giving rise to the so-called networked control systems (NCSs). Compared with traditional feedback control systems, where these components are usually connected via point-to-point cables, the introduction of communication network media brings great advantages, such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability (Ishii & Francis, 2002). Therefore, NCSs receive more and more attention and become more and more popular in many practical applications in recent years. Modeling, analysis and synthesis of network-based feedback systems have been receiving increasing attention, which is highlighted by the recent special issue edited by Antsaklis and Baillieul (2004). Among the reported results on NCSs, to mention a few, stability issue is investigated in Montestruque and Antsaklis (2004), Walsh, Ye, and Bushnell (2002), and Zhang, Branicky, and Phillips (2001), stabilizing controllers are designed in Yang, Wang, Hung, and Gani (2006), Yu, Wang, and Chu (2005a), Yu, Wang, Chu, and Hao (2005b), Zhang, Shi, Chen, and Huang (2005), Zhivoglyadov and Middleton (2003), performance preserved control is studied in Lian, Moyne, and Tilbury (2003), Seiler and Sengupta (2005), and Yue, Han, and Lam (2005), and moving horizon control is proposed in Goodwin, Haimovich, Quevedo, and Welsh (2004). In an NCS, the most significant feature is the network induced delays, which are usually caused by limited bits rate of the communication channels, by a node waiting to send out a packet via a busy channel, or by signal processing and propagation. The existence of signal transmission delays generally brings negative effects on NCS stability and performance. This observation further enhances the importance of the study on time-delay systems.

The most commonly and frequently used state-space model to represent time-delay systems is

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)), \tag{1}$$

where $d(t)$ is a time delay in the state $x(t)$, which is often assumed to be either constant, or time-varying satisfying certain conditions, e.g.,

$$0 \leqslant d(t) \leqslant \bar{d} < \infty, \quad \dot{d}(t) \leqslant \tau < \infty. \tag{2}$$

Almost all the reported results on time-delay systems are based on this basic mathematical model.

In a recent paper (Lam, Gao, & Wang, 2007), the following new model for time-delay systems is proposed:

$$\dot{x}(t) = Ax(t) + A_d x\left(t - \sum_{i=1}^{s} d_i(t)\right), \tag{3}$$

$$0 \leqslant d_i(t) \leqslant \bar{d}_i < \infty, \quad \dot{d}_i(t) \leqslant \tau_i < \infty. \tag{4}$$

This model contains multiple delay components in the state, and a stability analysis result is reported in Lam et al. (2007) for systems with two successive delay components. The introduction of this new model is motivated by the observation that sometimes in practical situations, signals transmitted from one point to another may experience a few network segments, which can possibly induce successive delays with different properties due to variable network transmission conditions, and has been clearly justified by a state-feedback remote control problem. A numerical example has shown the advantage of the stability result.

The intention of Lam et al. (2007) is to expose the new model and to give a preliminary result on its stability analysis. It is worth noting that this stability condition leaves much room for improvement. A significant source of conservativeness that could be further reduced lies in the calculation of the time-derivative of the Lyapunov–Krasovskii functional. In addition, only stability is analyzed, while the $\mathscr{H}_\infty$ performance has not been investigated, and application of this new model to the emerging network-based control would likely yield better performance.

Following the work of Lam et al. (2007), it is our intention in this paper to present new stability and $\mathscr{H}_\infty$ performance conditions for systems with multiple successive delay components, and apply this new model to network-based control. To make our idea more lucid, we still consider the case where only two successive delay components appear in the state, and the idea behind this paper can be easily extended to systems with multiple successive delay components. New results on stability and $\mathscr{H}_\infty$ performance are proposed by exploiting a new Lyapunov–Krasovskii functional and by making use of novel techniques for time-delay systems. An illustrative example is provided to show the significant advantage of the developed results. These constitute the contents of Section 2.

In Section 3, we apply the new time-delay model to the problem of network-based control. As can be seen later, a sampled-data NCS with simultaneous consideration of network induced delays, data packet dropouts and measurement quantization can be modeled as a nonlinear time-delay system with two successive delay components in the state, which forms a solid background for the new model mentioned above. Then, the $\mathscr{H}_\infty$ performance condition developed in Section 2 is exploited to investigate the problem of network-based $\mathscr{H}_\infty$ control. Illustrative examples are provided to show the advantage and applicability of the developed results for network-based controller design.

Notation: The notation used throughout the paper is fairly standard. The superscripts "T" and "−1" stand for matrix transposition and matrix inverse, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and the notation $P > 0$ ($\geqslant 0$) means that P is real symmetric and positive definite (semi-definite). In symmetric block matrices, we use an asterisk ($*$) to represent a term that is induced by symmetry and diag{...} stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The space of square-integrable vector functions over $[0, \infty)$ is denoted by $L_2[0, \infty)$, and for $w = \{w(t)\} \in L_2[0, \infty)$, its norm is denoted by $\|w\|_2$.

2. Main results on the new delay model

2.1. Stability analysis

Consider the following system with two successive delay components in the state:

$$\Sigma:\quad \begin{aligned}&\dot{x}(t)=Ax(t)+A_dx(t-d_1(t)-d_2(t)),\\ &x(t)=\phi(t),\quad t\in[-\bar{d},0].\end{aligned}\tag{5}$$

Here $x(t)\in\mathbb{R}^n$ is the state vector; $d_1(t)$ and $d_2(t)$ represent the two delay components in the state and we denote $d(t)=d_1(t)+d_2(t)$; A, A_d are system matrices with appropriate dimensions. It is assumed that

$$\begin{aligned}&0\leqslant d_1(t)\leqslant\bar{d}_1<\infty,\quad \dot{d}_1(t)\leqslant\tau_1<\infty,\\ &0\leqslant d_2(t)\leqslant\bar{d}_2<\infty,\quad \dot{d}_2(t)\leqslant\tau_2<\infty,\end{aligned}\tag{6}$$

and $\bar{d}=\bar{d}_1+\bar{d}_2$, $\tau=\tau_1+\tau_2$. $\phi(t)$ is the initial condition on the segment $[-\bar{d},0]$.

The purpose of this subsection is to derive a new stability condition under which system Σ in (5) is asymptotically stable for all delays $d_1(t)$ and $d_2(t)$ satisfying (6). One possible approach to check the stability of this system is to simply combine $d_1(t)$ and $d_2(t)$ into one delay $h(t)$ with

$$0\leqslant h(t)\leqslant\bar{d}_1+\bar{d}_2<\infty,\quad \dot{h}(t)\leqslant\tau_1+\tau_2<\infty.\tag{7}$$

Then, system Σ in (5) becomes

$$\Sigma_1:\quad \begin{aligned}&\dot{x}(t)=Ax(t)+A_dx(t-h(t)),\\ &x(t)=\phi(t),\qquad t\in[-\bar{d},0].\end{aligned}\tag{8}$$

The stability of system Σ_1 in (8) can be readily checked by using some existing stability conditions. As discussed in Lam et al. (2007), however, since this approach does not make full use of the information on $d_1(t)$ and $d_2(t)$, it would be inevitably conservative for some situations. In the following, we present a new stability criterion.

Theorem 1. *System Σ in* (5) *with delays $d_1(t)$ and $d_2(t)$ satisfying* (6) *is asymptotically stable if there exist matrices $P>0$, $Q_1\geqslant Q_2\geqslant 0$, $R\geqslant 0$, $Z_1\geqslant Z_2>0$, $M>0$, and S, T, U, V satisfying*

$$\begin{bmatrix}\Xi_1+\Xi_2+\Xi_2^{\mathrm{T}}+\Xi_3 & \Xi_4\\ * & \Xi_5\end{bmatrix}<0,\tag{9}$$

where

$$\Xi_2=[S+V\ \ T-S\ \ U-T\ \ -U-V],$$

$$\Xi_1=\begin{bmatrix}\Xi_{11} & 0 & PA_d & 0\\ * & \Xi_{12} & 0 & 0\\ * & * & -(1-\tau)Q_2 & 0\\ * & * & * & -R\end{bmatrix},$$

$$\Xi_{11}=PA+A^{\mathrm{T}}P+Q_1+R,\ \Xi_{12}=-(1-\tau_1)(Q_1-Q_2),$$

$$\Xi_3=\Xi_{31}^{\mathrm{T}}[\bar{d}_1Z_1+\bar{d}_2Z_2+\bar{d}M]\Xi_{31},$$

$$\Xi_{31}=[A\ \ 0\ \ A_d\ \ 0],\quad \Xi_4=[S\ \ T\ \ U\ \ V],$$

$$\Xi_5=\mathrm{diag}\{-\bar{d}_1^{-1}Z_1,-\bar{d}_2^{-1}Z_2,-\bar{d}^{-1}Z_2,-\bar{d}^{-1}M\}.\tag{10}$$

Proof. Define a Lyapunov–Krasovskii functional as

$$V(t)=V_1(t)+V_2(t)+V_3(t)+V_4(t),$$

$$V_1(t)=x^{\mathrm{T}}(t)Px(t),$$

$$V_2(t)=\int_{t-d_1(t)}^{t}x^{\mathrm{T}}(s)Q_1x(s)\,\mathrm{d}s+\int_{t-d(t)}^{t-d_1(t)}x^{\mathrm{T}}(s)Q_2x(s)\,\mathrm{d}s,$$

$$V_3(t)=\int_{t-\bar{d}}^{t}x^{\mathrm{T}}(s)Rx(s)\,\mathrm{d}s,$$

$$\begin{aligned}V_4(t)=&\int_{-\bar{d}_1}^{0}\int_{\beta}^{0}\dot{x}^{\mathrm{T}}(t+\alpha)Z_1\dot{x}(t+\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &+\int_{-\bar{d}}^{-\bar{d}_1}\int_{\beta}^{0}\dot{x}^{\mathrm{T}}(t+\alpha)Z_2\dot{x}(t+\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &+\int_{-\bar{d}}^{0}\int_{\beta}^{0}\dot{x}^{\mathrm{T}}(t+\alpha)M\dot{x}(t+\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta,\end{aligned}\tag{11}$$

where $P>0$, $Q_1\geqslant Q_2\geqslant 0$, $R\geqslant 0$, $Z_1\geqslant Z_2>0$ and $M>0$ are matrices to be determined. Then, along the solution of system Σ in (5), the time derivative of $V(t)$ is given by

$$\dot{V}_1(t)=2x^{\mathrm{T}}(t)P[Ax(t)+A_dx(t-d(t))],\tag{12}$$

$$\begin{aligned}\dot{V}_2(t)\leqslant{}&x^{\mathrm{T}}(t)Q_1x(t)-(1-\tau)x^{\mathrm{T}}(t-d(t))Q_2x(t-d(t))\\ &-(1-\tau_1)x^{\mathrm{T}}(t-d_1(t))(Q_1-Q_2)x(t-d_1(t)),\end{aligned}\tag{13}$$

$$\dot{V}_3(t)=x^{\mathrm{T}}(t)Rx(t)-x^{\mathrm{T}}(t-\bar{d})Rx(t-\bar{d}),\tag{14}$$

$$\begin{aligned}\dot{V}_4(t)={}&\dot{x}^{\mathrm{T}}(t)[\bar{d}_1Z_1+\bar{d}_2Z_2+\bar{d}M]\dot{x}(t)-\int_{t-\bar{d}_1}^{t}\dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\,\mathrm{d}\alpha\\ &-\int_{t-\bar{d}}^{t-\bar{d}_1}\dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha-\int_{t-\bar{d}}^{t}\dot{x}^{\mathrm{T}}(\alpha)M\dot{x}(\alpha)\,\mathrm{d}\alpha\\ \leqslant{}&\dot{x}^{\mathrm{T}}(t)[\bar{d}_1Z_1+\bar{d}_2Z_2+\bar{d}M]\dot{x}(t)-\int_{t-\bar{d}}^{t}\dot{x}^{\mathrm{T}}(\alpha)M\dot{x}(\alpha)\,\mathrm{d}\alpha\\ &-\int_{t-d_1(t)}^{t}\dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\,\mathrm{d}\alpha-\int_{t-d(t)}^{t-d_1(t)}\dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha\\ &-\int_{t-\bar{d}}^{t-d(t)}\dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha.\end{aligned}\tag{15}$$

Note that in the above derivation, we have used the relationships $Q_1\geqslant Q_2\geqslant 0$ and $Z_1\geqslant Z_2>0$. By the Newton–Leibniz formula, for any appropriately dimensioned matrices S, T, U, V, we have $\Upsilon_i=0$ $(i=1,\ldots,4)$ with

$$\Upsilon_1\triangleq\zeta^{\mathrm{T}}(t)S\left(x(t)-x(t-d_1(t))-\int_{t-d_1(t)}^{t}\dot{x}(\alpha)\,\mathrm{d}\alpha\right),$$

$$\Upsilon_2\triangleq\zeta^{\mathrm{T}}(t)T\left(x(t-d_1(t))-x(t-d(t))-\int_{t-d(t)}^{t-d_1(t)}\dot{x}(\alpha)\,\mathrm{d}\alpha\right),$$

$$\Upsilon_3 \triangleq \zeta^{\mathrm{T}}(t)U\left(x(t-d(t)) - x(t-\bar{d}) - \int_{t-\bar{d}}^{t-d(t)} \dot{x}(\alpha)\,\mathrm{d}\alpha\right),$$

$$\Upsilon_4 \triangleq \zeta^{\mathrm{T}}(t)V\left(x(t) - x(t-\bar{d}) - \int_{t-\bar{d}}^{t} \dot{x}(\alpha)\,\mathrm{d}\alpha\right), \tag{16}$$

where $\zeta(t) = [x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-d_1(t))\ x^{\mathrm{T}}(t-d(t))\ x^{\mathrm{T}}(t-\bar{d})]^{\mathrm{T}}$. Then, from (11)–(16) we have

$$\begin{aligned}
\dot{V}(t) \leqslant\ & 2x^{\mathrm{T}}(t)P[Ax(t) + A_d x(t-d(t))] + x^{\mathrm{T}}(t)Q_1 x(t) \\
& - (1-\tau_1)x^{\mathrm{T}}(t-d_1(t))(Q_1 - Q_2)x(t-d_1(t)) \\
& - (1-\tau)x^{\mathrm{T}}(t-d(t))Q_2 x(t-d(t)) \\
& + x^{\mathrm{T}}(t)Rx(t) - x^{\mathrm{T}}(t-\bar{d})Rx(t-\bar{d}) \\
& + \dot{x}^{\mathrm{T}}(t)[\bar{d}_1 Z_1 + \bar{d}_2 Z_2 + \bar{d}M]\dot{x}(t) \\
& - \int_{t-d_1(t)}^{t} \dot{x}^{\mathrm{T}}(\alpha)Z_1\dot{x}(\alpha)\,\mathrm{d}\alpha - \int_{t-d(t)}^{t-d_1(t)} \dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha \\
& - \int_{t-\bar{d}}^{t-d(t)} \dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha - \int_{t-\bar{d}}^{t} \dot{x}^{\mathrm{T}}(\alpha)M\dot{x}(\alpha)\,\mathrm{d}\alpha \\
& + \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 \\
\leqslant\ & \zeta^{\mathrm{T}}(t)[\Xi_1 + \Xi_2 + \Xi_2^{\mathrm{T}} + \Xi_3 + \Xi_6]\zeta(t) + \sum_{i=7}^{10} \Xi_i,
\end{aligned} \tag{17}$$

where

$$\Xi_6 = \bar{d}_1 S Z_1^{-1} S^{\mathrm{T}} + \bar{d}_2 T Z_2^{-1} T^{\mathrm{T}} + \bar{d} U Z_2^{-1} U^{\mathrm{T}} + \bar{d} V M^{-1} V^{\mathrm{T}},$$

$$\Xi_7 = -\int_{t-d_1(t)}^{t} \Xi_{71}^{\mathrm{T}} Z_1^{-1} \Xi_{71}\,\mathrm{d}\alpha, \quad \Xi_{71} = S^{\mathrm{T}}\zeta(t) + Z_1\dot{x}(\alpha),$$

$$\Xi_8 = -\int_{t-d(t)}^{t-d_1(t)} \Xi_{81} Z_2^{-1} \Xi_{81}\,\mathrm{d}\alpha, \quad \Xi_{81} = T^{\mathrm{T}}\zeta(t) + Z_2\dot{x}(\alpha),$$

$$\Xi_9 = -\int_{t-\bar{d}}^{t-d(t)} \Xi_{91} Z_2^{-1} \Xi_{91}\,\mathrm{d}\alpha, \quad \Xi_{91} = U^{\mathrm{T}}\zeta(t) + Z_2\dot{x}(\alpha),$$

$$\Xi_{10} = -\int_{t-\bar{d}}^{t} \Xi_{101}^{\mathrm{T}} M^{-1} \Xi_{101}\,\mathrm{d}\alpha, \quad \Xi_{101} = V^{\mathrm{T}}\zeta(t) + M\dot{x}(\alpha). \tag{18}$$

Note that $Z_i > 0$, $i = 1, 2$, $M > 0$, thus Ξ_i, $i = 7, \ldots, 10$, are all non-positive. By the Schur complement, (9) guarantees $\Xi_1 + \Xi_2 + \Xi_2^{\mathrm{T}} + \Xi_3 + \Xi_6 < 0$. Therefore, from (18) we have $\dot{V}(t) < -\varepsilon\|x(t)\|^2$ for a sufficiently small $\varepsilon > 0$ and $x(t) \neq 0$, and the asymptotic stability is established (Hale & Lunel, 1993). □

Remark 1. Theorem 1 presents a new stability criterion for system Σ with two successive time-varying delay components. This criterion is derived by defining the new Lyapunov–Krasovskii functional in (11), which makes full use of the information about $d_1(t)$ and $d_2(t)$. It is also worth mentioning that some novel techniques have been exploited in the calculation of the time derivative of $V(t)$. On one hand, no system transformation has been performed to the original system and thus there is no need to seek upper bounds of the inner product between two vectors, which has the potential to yield less conservative results; on the other hand, when deriving $\dot{V}_4(t)$ in (15), we keep the last term $\int_{t-\bar{d}}^{t-d(t)} \dot{x}^{\mathrm{T}}(\alpha)Z_2\dot{x}(\alpha)\,\mathrm{d}\alpha$, which was often ignored in the derivation of stability condition for time-delay systems.

In the following, we further extend the above idea to an important case. More specifically, we assume the two successive delay components $d_1(t)$ and $d_2(t)$ have very different properties in that $d_1(t)$ and $d_2(t)$ are assumed to be constant and non-differentiable, respectively. Thus the assumption in (6) reads

$$d_1(t) \equiv \bar{d}_1 < \infty, \quad 0 \leqslant d_2(t) \leqslant \bar{d}_2 < \infty. \tag{19}$$

As can be seen in the next section, this case is much related to the model we use for network-based control. Then, we have the following corollary.

Corollary 1. *System Σ in (5) with delays $d_1(t)$ and $d_2(t)$ satisfying (19) is asymptotically stable if there exist matrices $P > 0$, $Q \geqslant 0$, $R \geqslant 0$, $Z_i > 0$, $i = 1, 2$, $M > 0$, and S, T, U, V satisfying*

$$\begin{bmatrix} \bar{\Xi}_1 + \Xi_2 + \Xi_2^{\mathrm{T}} + \Xi_3 & \Xi_4 \\ * & \Xi_5 \end{bmatrix} < 0, \tag{20}$$

where Ξ_i, $i = 2, \ldots, 5$, are given in (10) and

$$\bar{\Xi}_1 = \begin{bmatrix} PA + A^{\mathrm{T}}P + Q + R & 0 & PA_d & 0 \\ * & -Q & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & -R \end{bmatrix}. \tag{21}$$

Proof. Define the Lyapunov–Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t),$$

where $V_1(t)$, $V_3(t)$ and $V_4(t)$ are given in (11) and

$$V_2(t) = \int_{t-\bar{d}_1}^{t} x^{\mathrm{T}}(s)Qx(s)\,\mathrm{d}s,$$

with $P > 0$, $Q \geqslant 0$, $R \geqslant 0$, $Z_i > 0$, $i = 1, 2$, $M > 0$ being matrices to be determined. Then, the corollary can be proved along similar lines as in the proof of Theorem 1. □

2.2. $\mathscr{H}_\infty$ performance analysis

In this subsection, we investigate the problem of $\mathscr{H}_\infty$ performance analysis for systems with two successive delay components in the state. Consider the following system:

$$\begin{aligned}
\bar{\Sigma}:\quad & \dot{x}(t) = Ax(t) + A_d x(t - d_1(t) - d_2(t)) + Ew(t), \\
& y(t) = Cx(t) + C_d x(t - d_1(t) - d_2(t)) + Fw(t), \\
& x(t) = \phi(t), \qquad t \in [-\bar{d}, 0].
\end{aligned} \tag{22}$$

Here $x(t)$, $\phi(t)$, $d_1(t)$ and $d_2(t)$ are the same as those in the above subsection; $w(t) \in \mathbb{R}^l$ is the disturbance input which belongs to $L_2[0, \infty)$; A, A_d, E, C, C_d, F are system matrices with appropriate dimensions. Our objective is to investigate under what condition system $\bar{\Sigma}$ in (22) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ, that is, $\|y\|_2 < \gamma\|w\|_2$ for all nonzero $w \in L_2[0, \infty)$ under zero initial condition. We first consider the assumption in (6).

Theorem 2. *System $\bar{\Sigma}$ in (22) with delays $d_1(t)$ and $d_2(t)$ satisfying (6) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P>0$, $Q_1 \geqslant Q_2 \geqslant 0$, $R \geqslant 0$, $Z_1 \geqslant Z_2 > 0$, $M>0$, and S, T, U, V satisfying*

$$\begin{bmatrix} \Psi_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_3+\Psi_4+\Psi_5 & \Xi_4 \\ * & \Xi_5 \end{bmatrix} < 0, \tag{23}$$

where Ξ_4 and Ξ_5 are given in (10), and

$$\Psi_1 = \begin{bmatrix} \Xi_{11} & 0 & PA_d & 0 & PE \\ * & \Xi_{12} & 0 & 0 & 0 \\ * & * & -(1-\tau)Q_2 & 0 & 0 \\ * & * & * & -R & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Psi_2 = [S+V \quad T-S \quad U-T \quad -U-V \quad 0],$$

$$\Psi_3 = \Psi_{31}^{\mathrm{T}}[\bar{d}_1 Z_1 + \bar{d}_2 Z_2 + \bar{d} M]\Psi_{31},$$

$$\Psi_{31} = [A \quad 0 \quad A_d \quad 0 \quad E], \qquad \Psi_{41} = [C \quad 0 \quad C_d \quad 0 \quad F],$$

$$\Psi_4 = \Psi_{41}^{\mathrm{T}}\Psi_{41}, \; \Psi_5 = \mathrm{diag}\{0,0,0,0,-\gamma^2 I\}, \tag{24}$$

and Ξ_{11}, Ξ_{12} are given in (10).

Proof. First, (23) implies (9), thus $\bar{\Sigma}$ in (22) with delays $d_1(t)$ and $d_2(t)$ satisfying (6) is asymptotically stable. Now, define the Lyapunov–Krasovskii functional as in (11). Then, by following similar lines as in the proof of Theorem 1, along the solution of system $\bar{\Sigma}$ in (22), the time derivative of $V(t)$ is given by

$$\dot{V}(t) \leqslant \bar{\zeta}^{\mathrm{T}}(t)[\Psi_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_3+\Xi_6]\bar{\zeta}(t) + \sum_{i=7}^{10} \Xi_i,$$

where Ψ_i, $i=1,2,3$, are given in (24), Ξ_i, $i=6,\ldots,10$, are given in (18) and $\bar{\zeta}(t) = [\zeta^{\mathrm{T}}(t) \;\; w^{\mathrm{T}}(t)]^{\mathrm{T}}$. Thus, we have

$$\begin{aligned} & y^{\mathrm{T}}(t)y(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(t) \\ & \quad \leqslant \bar{\zeta}^{\mathrm{T}}(t)[\Psi_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_3+\Xi_6+\Psi_4+\Psi_5]\bar{\zeta}(t) \\ & \qquad + \sum_{i=7}^{10} \Xi_i. \end{aligned} \tag{25}$$

Note that $Z_i > 0$, $i=1,2$, $M>0$, thus Ξ_i, $i=7,\ldots,10$, are all non-positive. Since (23) guarantees $\Psi_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_3+\Xi_6+\Psi_4+\Psi_5<0$, we have

$$y^{\mathrm{T}}(t)y(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(t) < 0 \tag{26}$$

for all nonzero $w \in L_2[0,\infty)$. Under zero initial condition, we have $V(0)=0$ and $V(\infty) \geqslant 0$. Integrating both sides of (26) yields $\|y\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0,\infty)$, and the proof is completed. □

For the assumption in (19), we have the following corollary (the proof follows similar lines as in the proofs of Theorem 2 and Corollary 1 and is thus omitted).

Corollary 2. *System $\bar{\Sigma}$ in (22) with delays $d_1(t)$ and $d_2(t)$ satisfying (19) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P>0$, $Q \geqslant 0$, $R \geqslant 0$, $Z_i > 0$, $i=1,2$, $M>0$, and S, T, U, V satisfying*

$$\begin{bmatrix} \bar{\Psi}_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_3+\Psi_4+\Psi_5 & \Xi_4 \\ * & \Xi_5 \end{bmatrix} < 0,$$

where Ξ_4 and Ξ_5 are given in (10), Ψ_i, $i=2,\ldots,5$, are given in (24), and

$$\bar{\Psi}_1 = \begin{bmatrix} PA + A^{\mathrm{T}}P + Q + R & 0 & PA_d & 0 & PE \\ * & -Q & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -R & 0 \\ * & * & * & * & 0 \end{bmatrix}.$$

2.3. Illustrative example

In this subsection, we use a numerical example to illustrate the advantage of the proposed new model and the developed stability condition.

Example 1. Consider system Σ in (5) with the following parameters, borrowed from Fridman and Shaked (2003), and Jing, Tan, and Wang (2004):

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Suppose we know that

$$\dot{d}_1(t) \leqslant 0.1, \quad \dot{d}_2(t) \leqslant 0.8.$$

Our purpose is to find the upper bound $\bar{d}_1$ of delay $d_1(t)$, or $\bar{d}_2$ of $d_2(t)$, when the other is known, below which the system is asymptotically stable. By combining the two delay components together, some existing stability results can be applied to this system. The calculation results obtained by Theorem 1 in this

Table 1
Calculated delay bounds for different cases

	Delay bound $\bar{d}_2$ for given $\bar{d}_1$			Delay bound $\bar{d}_1$ for given $\bar{d}_2$		
	$\bar{d}_1=1$	$\bar{d}_1=1.2$	$\bar{d}_1=1.5$	$\bar{d}_2=0.1$	$\bar{d}_2=0.2$	$\bar{d}_2=0.3$
Theorem 1	0.512	0.406	0.283	2.300	1.779	1.453
Lam et al. (2007)	0.415	0.376	0.248	2.263	1.696	1.324
Wu et al. (2004), Jing et al. (2004), Fridman and Shaked (2003)	0.180	0.080	Infeasible	1.080	0.980	0.880
Lee et al. (2001)	Infeasible	Infeasible	Infeasible	0.098	Infeasible	Infeasible
Kim (2001)	Infeasible	Infeasible	Infeasible	0.074	Infeasible	Infeasible

paper, Theorem 1 in Lam et al. (2007), Theorem 2 in Wu, He, She, and Liu (2004), Theorem 1 in Jing et al. (2004), Theorem 3.2 in Lee, Moon, Kwon, and Lee (2001), Corollary 1 in Kim (2001) and Theorem 1 in Fridman and Shaked (2003) for different cases are listed in Table 1. It can be seen from the table that Theorem 1 in this paper yields the least conservative stability test than other approaches, showing the advantage of the stability result in this paper.

3. Application to network-based control

3.1. Problem formulation

In this section, we apply the results obtained above to the problem of $\mathscr{H}_\infty$ control for NCSs. Consider a typical NCS shown in Fig. 1. Suppose the physical plant is given by the following linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t),$$
$$y(t) = Cx(t) + Du(t) + Fw(t). \tag{27}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^p$ is the control input; $w(t) \in \mathbb{R}^l$ is the disturbance input which belongs to $L_2[0,\infty)$; $y(t) \in \mathbb{R}^q$ is the output; and A, B, C, D, E, F are system matrices with appropriate dimensions.

In Fig. 1, it is assumed that the sampler is clock-driven, while the quantizer, controller and zero-order hold (ZOH) are event-driven. The sampling period is assumed to be h where h is a positive real constant and we denote the *sampling instant* of the sampler as s_k, $k = 1, \ldots, \infty$. In addition, it is assumed that the state variable $x(t)$ is measurable, and the measurements of $x(t)$ are firstly quantized via a quantizer, and then transmitted with a single packet. The quantizer is denoted as $f(\cdot) = [f_1(\cdot)\ f_2(\cdot)\ \cdots\ f_n(\cdot)]^{\mathrm{T}}$, which is assumed to be symmetric, that is, $f_j(-v) = -f_j(v)$, $j = 1, \ldots, n$. In this paper, we are interested in the logarithmic static and time-invariant quantizer. For each $f_j(\cdot)$, the set of quantized levels is described by

$$\mathscr{U}_j = \{\pm u_i^{(j)},\ i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\}. \tag{28}$$

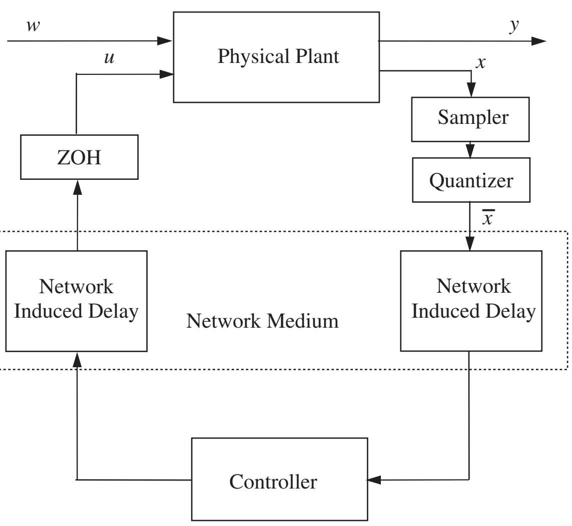

Fig. 1. A typical networked control system.

According to Elia and Mitter (2001), and Fu and Xie (2005), a quantizer is called *logarithmic* if the set of quantized levels is characterized by

$$\mathscr{U}_j = \{\pm u_i^{(j)}, u_i^{(j)} = \rho_j^i u_0^{(j)}, i = \pm 1, \pm 2, \ldots\},$$
$$\cup \{\pm u_0^{(j)}\} \cup \{0\}, \quad 0 < \rho_j < 1, \quad u_0^{(j)} > 0. \tag{29}$$

Each of the quantization level $u_i^{(j)}$ corresponds to a segment such that the quantizer maps the whole segment to this quantization level. In addition, these segments form a partition of $\mathbb{R}$, that is, they are disjoint and their union equals to $\mathbb{R}$. For the logarithmic quantizer, the associated quantizer $f_j(\cdot)$ is defined as

$$f_j(v) = \begin{cases} u_i^{(j)} & \text{if } \dfrac{1}{1+\sigma_j} u_i^{(j)} < v \leqslant \dfrac{1}{1-\sigma_j} u_i^{(j)}, \quad v > 0, \\ 0 & \text{if } v = 0, \\ -f_j(-v) & \text{if } v < 0, \end{cases}$$

where

$$\sigma_j = \frac{1-\rho_j}{1+\rho_j}. \tag{30}$$

Then, at the sampling instant s_k, we have

$$\bar{x}(s_k) = f(x(s_k)) = [f_1(x_1(s_k))\ f_2(x_2(s_k))\ \cdots\ f_n(x_n(s_k))]^{\mathrm{T}}.$$

Now denote the *updating instants* of the ZOH as t_k, $k = 1, \ldots, \infty$, and suppose that the updating signal (successfully transmitted signal from the sampler to the controller and to the ZOH) at the instant t_k has experienced signal transmission delays η_k ($\eta_k = \tau_k + d_k$ where τ_k is the delay from the quantizer to the controller and d_k is the delay from the controller to the ZOH. It is assumed that there is no delay between the sensor and quantizer). Therefore, the state-feedback controller takes the following form:

$$u(t_k) = Kf(x(t_k - \eta_k)), \tag{31}$$

where K is the state-feedback control gain. Thus, considering the behavior of the ZOH, we have

$$u(t) = Kf(x(t_k - \eta_k)), \quad t_k \leqslant t < t_{k+1}, \tag{32}$$

with t_{k+1} being the next updating instant of the ZOH after t_k.

A natural assumption on the network induced delays η_k can be made as

$$\eta_{\mathrm{m}} \leqslant \eta_k \leqslant \eta_{\mathrm{M}}, \tag{33}$$

where η_{m} and η_{M} denote the minimum and the maximum delays, respectively. In addition, at the updating instant t_{k+1} the number of accumulated data packet dropouts since the last updating instant t_k is denoted as δ_{k+1}. We assume that the maximum number of data packet dropouts is $\bar{\delta}$, that is,

$$\delta_{k+1} \leqslant \bar{\delta}. \tag{34}$$

Then, it can be seen from (33) and (34) that

$$t_{k+1} - t_k = (\delta_{k+1} + 1)h + \eta_{k+1} - \eta_k. \tag{35}$$

Remark 2. It is worth noting that the assumption on the network induced delays η_k made in (33) is more general than those in Yu et al. (2005a, 2005b), and Yue et al. (2005). The main difference lies in the lower bound we introduced. By assuming $\eta_m = 0$, (33) is the same as those in Yu et al. (2005a, 2005b), and Yue et al. (2005). The introduction of the lower bound η_m will be shown later, via a numerical example, to be advantageous for reducing conservativeness by utilizing the idea of successive delay components developed in the above section.

Therefore, from (27)–(32) we obtain the following closed-loop system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKf(x(t_k - \eta_k)) + Ew(t),\\ y(t) &= Cx(t) + DKf(x(t_k - \eta_k)) + Fw(t),\\ &\quad t_k \leqslant t < t_{k+1}.\end{aligned} \tag{36}$$

Remark 3. It is important to note that in (32), t_k refers to the updating instant of the ZOH. While in Yu et al. (2005a), the controller is expressed as

$$u(t) = K\bar{x}(t_k), \quad t_k \leqslant t < t_{k+1}, \tag{37}$$

with t_k standing for the *sampling instant*. It should be noted that when the controller and actuator are event-driven, we cannot use the sampling instant to model the behavior of the ZOH. The reason is that the signal transmission delays may not necessarily be integer multiples of the sampling period, and thus the ZOH may be updated between sampling instants. By using the updating instant in this paper, we do not need to synchronize the ZOH and the sampler, and thus the networked control model formulated here is essentially different from that in Yu et al. (2005a) and is more general, though they appear to be similar.

3.2. Key idea

It is noted that the closed-loop system in (36) is in the form of a sampled-data system. As the time sequence $\{t_k\}$ depends on both the network induced delays and data packet dropouts, the period $t_{k+1} - t_k$ for the sampled-data system in (36) is variable and uncertain. Now, let us represent $t_k - \eta_k$ in (36) as

$$t_k - \eta_k = t - \eta_m - t + t_k + \eta_m - \eta_k = t - \eta_m - \eta(t), \tag{38}$$

where

$$\eta(t) = t - t_k + \eta_k - \eta_m. \tag{39}$$

Then, from (35) we have

$$0 \leqslant \eta(t) \leqslant \kappa, \tag{40}$$

where

$$\kappa = \eta_M - \eta_m + (\bar{\delta} + 1)h. \tag{41}$$

By substituting (38) into (36) we obtain

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKf(x(t - \eta_m - \eta(t))) + Ew(t),\\ y(t) &= Cx(t) + DKf(x(t - \eta_m - \eta(t))) + Fw(t).\end{aligned} \tag{42}$$

Remark 4. It is worth noting that in the above transformed system, η_m is a constant delay, and $\eta(t)$ is a non-differentiable time-varying delay with bound κ. Our main idea in the above transformation is to represent the sampled-data system in (36) as a continuous time system with two successive delay components in the state, which takes a very similar form as system Σ in (5) with the assumption in (19). The $\mathscr{H}_\infty$ control problem will be solved based on this new model.

Remark 5. If the lower bound of the network induced delays is assumed to be zero, that is, $\eta_m = 0$, (42) takes the following form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKf(x(t - \eta(t))) + Ew(t),\\ z(t) &= Cx(t) + DKf(x(t - \eta(t))) + Fw(t),\end{aligned} \tag{43}$$

with

$$0 \leqslant \eta(t) \leqslant \bar{\kappa}, \tag{44}$$

where $\bar{\kappa} = \eta_M + \left(\bar{\delta} + 1\right)h$. Compared with (40), the upper bound of $\eta(t)$ in (44) is increased by η_m. In other words, without taking the lower bound of the transmission delays into consideration, η_m will be treated as a non-differentiable time-varying delay instead of a constant one when it is nonzero. Therefore, the introduction of the lower bound η_m will naturally reduce conservativeness, which will be shown, via a numerical example later. However, existing results on networked control systems, such as Yu et al. (2005a, 2005b), Yue et al. (2005), and Yue, Han, and Peng (2004),did not offer to take the lower bound η_m into consideration.

3.3. $\mathscr{H}_\infty$ performance analysis

This subsection is concerned with the problem of $\mathscr{H}_\infty$ performance analysis. More specifically, assuming that the matrices A, B, C, D, E, F in (27) and the controller gain matrix K in (31) are known, we shall study the conditions under which the closed-loop NCS in (36) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ. The following theorem shows that the closed-loop $\mathscr{H}_\infty$ performance can be guaranteed if there exist some matrices satisfying certain LMIs. This theorem will play an instrumental role in the problem of $\mathscr{H}_\infty$ network-based control.

Before proceeding further, we give the following lemma which will be used later.

Lemma 1. *Given appropriately dimensioned matrices Σ_1, Σ_2, Σ_3, with $\Sigma_1^{\mathrm{T}} = \Sigma_1$. Then,*

$$\Sigma_1 + \Sigma_3\Sigma_2 + \Sigma_2^{\mathrm{T}}\Sigma_3^{\mathrm{T}} < 0 \tag{45}$$

holds if for some matrix $W > 0$

$$\Sigma_1 + \Sigma_3 W^{-1}\Sigma_3^{\mathrm{T}} + \Sigma_2^{\mathrm{T}} W \Sigma_2 < 0. \tag{46}$$

Theorem 3. *Consider the NCS in Fig. 1. Given the controller gain matrix K and a positive constant γ, the closed-loop system in (36) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P > 0$, $Q \geqslant 0$, $R \geqslant 0$, $Z_i > 0$,*

$i=1,2$, $M>0$, S, T, U, V, and a diagonal matrix $W>0$ satisfying

$$\begin{bmatrix} \Omega_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5+\Omega_2 & \Xi_4 & \Omega_3 & \Omega_5 & \Omega_6 \\ * & \Omega_8 & 0 & 0 & 0 \\ * & * & \Omega_4 & 0 & \Omega_7 \\ * & * & * & -I & DK \\ * & * & * & * & -W \end{bmatrix} < 0, \tag{47}$$

where Ψ_2 and Ψ_5 are given in (24), *Ξ_4 is given in* (10), *and*

$$\Omega_1 = \begin{bmatrix} PA+A^{\mathrm{T}}P+Q+R & 0 & PBK & 0 & PE \\ * & -Q & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -R & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Omega_2=\mathrm{diag}\{0,0,\Lambda^2W,0,0\}, \quad \Omega_3=\Omega_{31}^{\mathrm{T}}\Omega_{32},$$

$$\Omega_{31}=[A\ 0\ BK\ 0\ E], \quad \Omega_{32}=[Z_1\ Z_2\ M],$$

$$\Omega_4=\mathrm{diag}\{-\eta_{\mathrm{m}}^{-1}Z_1, -\kappa^{-1}Z_2, -v^{-1}M\}, \quad \Omega_7=\Omega_{32}^{\mathrm{T}}BK,$$

$$\Omega_5=[C\ 0\ KD\ 0\ F]^{\mathrm{T}}, \quad \Omega_6=[K^{\mathrm{T}}B^{\mathrm{T}}P\ 0\ 0\ 0\ 0]^{\mathrm{T}},$$

$$\Omega_8=\mathrm{diag}\{-\eta_{\mathrm{m}}^{-1}Z_1, -\kappa^{-1}Z_2, -v^{-1}Z_2, -v^{-1}M\},$$

$$\Lambda=\mathrm{diag}\{\sigma_1,\ldots,\sigma_n\}, \quad v=\eta_{\mathrm{m}}+\kappa. \tag{48}$$

Proof. First, considering the quantization behavior shown in (28)–(30) and according to Elia and Mitter (2001), and Fu and Xie (2005), (42) can be expressed as

$$\begin{aligned} \dot{x}(t) &= Ax(t)+BK(I+\Lambda(t))(x(t-\eta_{\mathrm{m}}-\eta(t)))+Ew(t), \\ y(t) &= Cx(t)+DK(I+\Lambda(t))(x(t-\eta_{\mathrm{m}}-\eta(t)))+Fw(t), \end{aligned} \tag{49}$$

where

$$\Lambda(t)=\mathrm{diag}\{\Lambda_1(t),\Lambda_2(t),\ldots,\Lambda_n(t)\}, \tag{50}$$

with

$$\Lambda_j(t)\in[-\sigma_j,\sigma_j], \quad j=1,\ldots,n. \tag{51}$$

By comparing system (49) and system (22) with the assumption in (19), according to Corollary 2, we know that system (49) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P>0$, $Q\geqslant 0$, $R\geqslant 0$, $Z_i>0$, $i=1,2$, $M>0$, and S, T, U, V satisfying

$$\begin{bmatrix} \Gamma_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5+\Gamma_4\Gamma_4^{\mathrm{T}}+\Gamma_2 & \Xi_4 \\ * & \Omega_8 \end{bmatrix} < 0, \tag{52}$$

where Ψ_2, Ψ_5 are given in (24), Ξ_4 is given in (10) and

$$\Gamma_1 = \begin{bmatrix} PA+A^{\mathrm{T}}P+Q+R & 0 & PBK(I+\Lambda(t)) & 0 & PE \\ * & -Q & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -R & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Gamma_2=\Gamma_3[\eta_{\mathrm{m}}Z_1+\kappa Z_2+vM]\Gamma_3^{\mathrm{T}},$$

$$\Gamma_3=[A\ \ 0\ \ BK(I+\Lambda(t))\ \ 0\ \ E]^{\mathrm{T}},$$

$$\Gamma_4=[C\ \ 0\ \ DK(I+\Lambda(t))\ \ 0\ \ F]^{\mathrm{T}}.$$

By the Schur complement, (52) is equivalent to

$$\begin{bmatrix} \Gamma_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5 & \Xi_4 & \Gamma_5 & \Gamma_4 \\ * & \Omega_8 & 0 & 0 \\ * & * & \Omega_4 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{53}$$

where

$$\Gamma_5=[A\ \ 0\ \ BK(I+\Lambda(t))\ \ 0\ \ E]^{\mathrm{T}}[Z_1\ \ Z_2\ \ M].$$

Rewrite (53) in the form of (45) with

$$\Sigma_1 = \begin{bmatrix} \Omega_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5 & \Xi_4 & \Omega_3 & \Omega_5 \\ * & \Omega_8 & 0 & 0 \\ * & * & \Omega_4 & 0 \\ * & * & * & -I \end{bmatrix},$$

$$\Sigma_2=[\Gamma_6\ \ 0\ \ 0\ \ 0], \quad \Gamma_6=[0\ \ 0\ \ \Lambda(t)\ \ 0\ \ 0],$$

$$\Sigma_3=[\Omega_6^{\mathrm{T}}\ \ 0\ \ \Omega_7^{\mathrm{T}}\ \ K^{\mathrm{T}}D^{\mathrm{T}}]^{\mathrm{T}}.$$

Then, according to Lemma 1, (53) holds if for some matrix $W>0$

$$\begin{aligned} &\begin{bmatrix} \Omega_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5 & \Xi_4 & \Omega_3 & \Omega_5 \\ * & \Xi_5 & 0 & 0 \\ * & * & \Omega_4 & 0 \\ * & * & * & -I \end{bmatrix} \\ &+\begin{bmatrix} \Omega_6 \\ 0 \\ \Omega_7 \\ DK \end{bmatrix} W^{-1} \begin{bmatrix} \Omega_6 \\ 0 \\ \Omega_7 \\ DK \end{bmatrix}^{\mathrm{T}} \\ &+[\Gamma_6\ \ 0\ \ 0\ \ 0]^{\mathrm{T}}W[\Gamma_6\ \ 0\ \ 0\ \ 0]<0. \end{aligned} \tag{54}$$

Note that W is required to be diagonal and positive definite. Then, by using a Schur complement operation and by considering (51), (47) guarantees (54), and the proof is completed. $\square$

If there is no quantizer in the NCS shown in Fig. 1, the closed-loop system in (36) reads

$$\begin{aligned} \dot{x}(t) &= Ax(t)+BKx(t_k-\eta_k)+Ew(t), \\ y(t) &= Cx(t)+DKx(t_k-\eta_k)+Fw(t), \quad t_k\leqslant t<t_{k+1}. \end{aligned} \tag{55}$$

Then, we have the following corollary, which can be proved by following similar lines as in the proof of Theorem 3.

Corollary 3. *Consider the NCS in Fig.* 1, *but without the quantizer. Given the controller gain matrix K and a positive constant γ, the closed-loop system in* (55) *is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P>0$, $Q\geqslant 0$, $R\geqslant 0$, $Z_i>0$, $i=1,2$, $M>0$, and S, T, U, V satisfying*

$$\begin{bmatrix} \Omega_1+\Psi_2+\Psi_2^{\mathrm{T}}+\Psi_5 & \Xi_4 & \Omega_3 & \Omega_5 \\ * & \Omega_8 & 0 & 0 \\ * & * & \Omega_4 & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$

where Ψ_2 and Ψ_5 are given in (24), *Ξ_4 is given in* (10), *and Ω_i, $i=1,3,4,5,8$, are given in* (48).

3.4. $\mathscr{H}_\infty$ controller design

This subsection is devoted to solving the problem of $\mathscr{H}_\infty$ controller design for NCSs.

Proposition 1. *Consider the NCS in Fig. 1. Given a positive constant γ, there exists a state-feedback controller in the form of* (31) *such that the closed-loop system in* (36) *is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $\bar{P}>0$, $\bar{Q}\geqslant 0$, $\bar{R}\geqslant 0$, $\bar{Z}_i>0$, $i=1,2$, $\bar{M}>0$, $\bar{K}$, $\bar{S}$, $\bar{T}$, $\bar{U}$, $\bar{V}$, and a diagonal matrix $\bar{W}>0$ satisfying*

$$\begin{bmatrix} \Pi_1+\Pi_2+\Pi_2^{\mathrm{T}}+\Psi_5 & \Pi_3 & \Pi_5 H & \Pi_7 & \Pi_8 & \Pi_{10} \\ * & \Pi_4 & 0 & 0 & 0 & 0 \\ * & * & \Pi_6 & 0 & \Pi_9 & 0 \\ * & * & * & -I & D\bar{K} & 0 \\ * & * & * & * & -\bar{P}\bar{W}^{-1}\bar{P} & 0 \\ * & * & * & * & * & -\Lambda^{-2}\bar{W} \end{bmatrix}<0, \tag{56}$$

where Ψ_5 is given in (24) *and*

$$\Pi_1=\begin{bmatrix} A\bar{P}+\bar{P}A^{\mathrm{T}}+\bar{Q}+\bar{R} & 0 & B\bar{K} & 0 & E \\ * & -\bar{Q} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -\bar{R} & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Pi_2=[\bar{S}+\bar{V}\quad \bar{T}-\bar{S}\quad \bar{U}-\bar{T}\quad -\bar{U}-\bar{V}\quad 0],$$

$$\Pi_3=[\bar{S}\quad \bar{T}\quad \bar{U}\quad \bar{V}],\qquad H=[I\quad I\quad I],$$

$$\Pi_4=\mathrm{diag}\{-\eta_{\mathrm{m}}^{-1}\bar{P}\bar{Z}_1^{-1}\bar{P},-\kappa^{-1}\bar{P}\bar{Z}_2^{-1}\bar{P},\\ -v^{-1}\bar{P}\bar{Z}_2^{-1}\bar{P},-v^{-1}\bar{P}\bar{M}^{-1}\bar{P}\},$$

$$\Pi_5=[A\bar{P}\quad 0\quad B\bar{K}\quad 0\quad E]^{\mathrm{T}},\qquad \Pi_7=[C\bar{P}\quad 0\quad D\bar{K}\quad 0\quad F]^{\mathrm{T}},$$

$$\Pi_8=[\bar{K}^{\mathrm{T}}B^{\mathrm{T}}\quad 0\quad 0\quad 0\quad 0]^{\mathrm{T}},\qquad \Pi_{10}=[0\quad 0\quad \bar{P}\quad 0\quad 0]^{\mathrm{T}},$$

$$\Pi_6=\mathrm{diag}\{-\eta_{\mathrm{m}}^{-1}\bar{Z}_1,-\kappa^{-1}\bar{Z}_2,-v^{-1}\bar{M}\},\qquad \Pi_9=H^{\mathrm{T}}B\bar{K}. \tag{57}$$

Moreover, if the above condition is feasible, a desired controller gain matrix in the form of (31) *is given by*

$$K=\bar{K}\bar{P}^{-1}. \tag{58}$$

Proof. From Theorem 3, we know that there exists a state-feedback controller in the form of (31) such that the closed-loop NCS in (36) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P>0$, $Q\geqslant 0$, $R\geqslant 0$, $Z_i>0$, $i=1,2$, $M>0$, K, S, T, U, V, and a diagonal matrix $W>0$ satisfying (47). Define

$$J=\mathrm{diag}\{J_1,J_2,J_3,I,P^{-1}\},\\ J_1=\mathrm{diag}\{P^{-1},P^{-1},P^{-1},P^{-1},I\},$$

$$J_2=\mathrm{diag}\{P^{-1},P^{-1},P^{-1},P^{-1}\},\qquad J_3=\mathrm{diag}\{Z_1^{-1},Z_2^{-1},M^{-1}\}.$$

Performing a congruence transformation to (47) by J, and a Schur complement operation to the term $\Lambda^2P^{-1}WP^{-1}$ in the (3, 3) block, together with the change of matrix variables defined by

$$\bar{P}=P^{-1},\quad \bar{M}=M^{-1},\quad \bar{W}=W^{-1},\quad \bar{Z}_1=Z_1^{-1},\quad \bar{Z}_2=Z_2^{-1},$$

$$\bar{K}=KP^{-1},\quad \bar{Q}=P^{-1}QP^{-1},\quad \bar{R}=P^{-1}RP^{-1},$$

$$[\bar{S}\quad \bar{T}\quad \bar{U}\quad \bar{V}]=J_1[S\quad T\quad U\quad V]J_2,$$

we obtain (56), and the proposition is proved. □

The condition in Proposition 1 cannot be implemented by using standard numerical software due to the existence of the terms $\bar{P}\bar{Z}_i^{-1}\bar{P}$, $\bar{P}\bar{W}^{-1}\bar{P}$ and $\bar{P}\bar{M}^{-1}\bar{P}$. By noticing $\bar{Z}_i>0$ and $\bar{M}>0$, we have $(\bar{Z}_i-\bar{P})\bar{Z}_i^{-1}(\bar{Z}_i-\bar{P})\geqslant 0$, $(\bar{M}-\bar{P})\bar{M}^{-1}(\bar{M}-\bar{P})\geqslant 0$, $(\bar{W}-\bar{P})\bar{W}^{-1}(\bar{W}-\bar{P})\geqslant 0$, which are equivalent to, respectively,

$$-\bar{P}\bar{Z}_i^{-1}\bar{P}\leqslant\bar{Z}_i-2\bar{P},\qquad -\bar{P}\bar{M}^{-1}\bar{P}\leqslant\bar{M}-2\bar{P},\\ -\bar{P}\bar{W}^{-1}\bar{P}\leqslant\bar{W}-2\bar{P}. \tag{59}$$

By combining (56) and (59), we readily obtain the following theorem.

Theorem 4. *Consider the NCS in Fig. 1. Given a positive constant γ, there exists a state-feedback controller in the form of* (31) *such that the closed-loop system in* (36) *is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $\bar{P}>0$, $\bar{Q}\geqslant 0$, $\bar{R}\geqslant 0$, $\bar{Z}_i>0$, $i=1,2$, $\bar{M}>0$, $\bar{K}$, $\bar{S}$, $\bar{T}$, $\bar{U}$, $\bar{V}$, and a diagonal matrix $\bar{W}>0$ satisfying*

$$\begin{bmatrix} \Pi_1+\Pi_2+\Pi_2^{\mathrm{T}}+\Psi_5 & \Pi_3 & \Pi_5 H & \Pi_7 & \Pi_8 & \Pi_{10} \\ * & \bar{\Pi}_4 & 0 & 0 & 0 & 0 \\ * & * & \Pi_6 & 0 & \Pi_9 & 0 \\ * & * & * & -I & D\bar{K} & 0 \\ * & * & * & * & \bar{W}-2\bar{P} & 0 \\ * & * & * & * & * & -\Lambda^{-2}\bar{W} \end{bmatrix}<0, \tag{60}$$

where Π_i, H are given in (57) *and $\bar{\Pi}_4=\mathrm{diag}\{\eta_{\mathrm{m}}^{-1}(\bar{Z}_1-2\bar{P}),\kappa^{-1}(\bar{Z}_2-2\bar{P}),v^{-1}(\bar{Z}_2-2\bar{P}),v^{-1}(\bar{M}-2\bar{P})\}$. Moreover, if the above condition is feasible, a desired controller gain matrix in the form of* (31) *is given by* (58).

Remark 6. Note that (60) is an LMI not only over the matrix variables, but also over the scalar γ. This implies that the scalar γ can be included as an optimization variable to obtain a reduction of the $\mathscr{H}_\infty$ disturbance attention level bound. Then, the minimum (in terms of the feasibility of (60)) $\mathscr{H}_\infty$ disturbance attention level bound with admissible controllers can be readily found by solving the following convex optimization problem:

Minimize γ subject to (60) over $\bar{P}>0$, $\bar{Q}\geqslant 0$, $\bar{R}\geqslant 0$, $\bar{Z}_i>0$, $i=1,2$, $\bar{M}>0$, $\bar{K}$, $\bar{S}$, $\bar{T}$, $\bar{U}$, $\bar{V}$, and diagonal matrix $\bar{W}>0$.

Theorem 4 presents an LMI condition for the existence of desired state-feedback controllers based on the inequalities in (59). In the following, we present another approach to check the condition in Proposition 1.

Now introduce additional matrix variables $\bar{N}_i > 0$, $\bar{X} > 0$, and $\bar{Y} > 0$ and replace (56) with

$$\begin{bmatrix} \Pi_1 + \Pi_2 + \Pi_2^{\mathrm{T}} + \Psi_5 & \Pi_3 & \Pi_5 H & \Pi_7 & \Pi_8 & \Pi_{10} \\ * & \tilde{\Pi}_4 & 0 & 0 & 0 & 0 \\ * & * & \Pi_6 & 0 & \Pi_9 & 0 \\ * & * & * & -I & D\bar{K} & 0 \\ * & * & * & * & -\bar{Y} & 0 \\ * & * & * & * & * & -\Lambda^{-2}\bar{W} \end{bmatrix} < 0, \tag{61}$$

$$\bar{X} - \bar{P}\bar{M}^{-1}\bar{P} \leqslant 0, \quad \bar{Y} - \bar{P}\bar{W}^{-1}\bar{P} \leqslant 0,$$
$$\bar{N}_i - \bar{P}\bar{Z}_i^{-1}\bar{P} \leqslant 0, \quad i = 1, 2, \tag{62}$$

where

$$\tilde{\Pi}_4 = \mathrm{diag}\{-\eta_{\mathrm{m}}^{-1}\bar{N}_1, -\kappa^{-1}\bar{N}_2, -v^{-1}\bar{N}_2, -v^{-1}\bar{X}\}.$$

Then, we readily obtain the following theorem.

Theorem 5. *Consider the NCS in Fig. 1. Given a positive constant γ, there exists a state-feedback controller in the form of* (31) *such that the closed-loop system in* (36) *is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ if there exist matrices $P > 0$, $\bar{P} > 0$, $W > 0$, $\bar{Q} \geqslant 0$, $\bar{R} \geqslant 0$, $X > 0$, $\bar{X} > 0$, $Y > 0$, $\bar{Y} > 0$, $N_i > 0$, $\bar{N}_i > 0$, $Z_i > 0$, $\bar{Z}_i > 0$, $i = 1, 2$, $\bar{M} > 0$, $\bar{K}$, $\bar{S}$, $\bar{T}$, $\bar{U}$, $\bar{V}$, and a diagonal matrix $\bar{W} > 0$ satisfying* (61) *and*

$$\begin{bmatrix} -X & P \\ * & -M \end{bmatrix} \leqslant 0, \quad \begin{bmatrix} -Y & P \\ * & -W \end{bmatrix} \leqslant 0,$$
$$\begin{bmatrix} -N_i & P \\ * & -Z_i \end{bmatrix} \leqslant 0, \tag{63}$$

$$\bar{P}P = I, \quad \bar{X}X = I, \quad \bar{Y}Y = I, \quad \bar{W}W = I,$$
$$\bar{Z}_i Z_i = I, \quad \bar{N}_i N_i = I, \quad i = 1, 2. \tag{64}$$

Moreover, if the above condition is feasible, a desired controller gain matrix in the form of (31) *is given by* (58).

The condition presented in Theorem 5 is equivalent to that in Proposition 1. It is noted that this condition is not a convex set due to the matrix equality constraints in (64). Several approaches have been proposed to solve such nonconvex feasibility problems, among which the cone complementarity linearization (CCL) method (El Ghaoui, Oustry, & Rami, 1997) is the most commonly used one (for instance, the CCL algorithm has been used for solving the controller design problems as well as model reduction problems (Gao & Wang, 2003; Gao et al., 2004)). The basic idea in CCL algorithm is that if the LMI $\left[\begin{smallmatrix} P & I \\ I & L \end{smallmatrix}\right] \geqslant 0$ is feasible in the $n \times n$ matrix variables $L > 0$ and $P > 0$, then $\mathrm{tr}(PL) \geqslant n$, and $\mathrm{tr}(PL) = n$ if and only if $PL = I$.

Now using a cone complementarity approach (El Ghaoui et al., 1997), we suggest the following nonlinear minimization problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 5.

Problem NBCD (network-based controller design):

$$\min\ \mathrm{tr}\left(\bar{P}P + \bar{X}X + \bar{Y}Y + \bar{W}W + \sum_{i=1}^{2}(\bar{Z}_i Z_i + \bar{N}_i N_i)\right)$$

subject to (61), (63) and

$$\begin{bmatrix} \bar{P} & I \\ I & P \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} \bar{X} & I \\ I & X \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} \bar{Y} & I \\ I & Y \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} \bar{W} & I \\ I & W \end{bmatrix} \geqslant 0,$$
$$\begin{bmatrix} \bar{Z}_i & I \\ I & Z_i \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} \bar{N}_i & I \\ I & N_i \end{bmatrix} \geqslant 0, \quad i = 1, 2.$$

According to El Ghaoui et al. (1997), if the solution of the above minimization problem is $8n$, that is,

$$\min\ \mathrm{tr}\left(\bar{P}P + \bar{X}X + \bar{Y}Y + \bar{W}W + \sum_{i=1}^{2}(\bar{Z}_i Z_i + \bar{N}_i N_i)\right) = 8n,$$

then the conditions in Theorem 5 are solvable. Algorithm 1 in El Ghaoui et al. (1997) can be easily adapted to solve Problem NBCD.

3.5. Illustrative examples

In this subsection, two examples are provided to illustrate the results developed above. We first use a numerical example to show the advantage by introducing the lower bound of transmission delays. The second example is utilized to show the applicability of the proposed controller design methods.

Example 2. Suppose the system matrices A, B, C, D, E, F in (27) and the controller gain K in (31) are given

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix},$$

$$C = [1 \ \ 0], \quad D = 0.3, \quad F = 0.5, \quad K = [-1 \ \ 1].$$

The parameters for the quantizer $f(\cdot)$ are given by $\rho_1 = 0.9$ and $\rho_2 = 0.8$, thus we have $\sigma_1 = 0.0526$ and $\sigma_2 = 0.1111$. It is assumed that the network induced delays η_k satisfy $\eta_{\mathrm{m}} \leqslant \eta_k \leqslant \eta_{\mathrm{M}}$, the maximum number of data packet dropouts is 2, and the sampling period is 10 ms. Our purpose is to determine the minimum guaranteed closed-loop $\mathscr{H}_\infty$ performances for different values of lower delay bound η_{m}.

Firstly, we assume $\eta_{\mathrm{M}} = 0.4$ s. When we do not consider the lower bound of the network induced delays, that is, $\eta_{\mathrm{m}} = 0$, by using Theorem 3 (assuming that η_{m} is sufficiently small), the minimum guaranteed closed-loop $\mathscr{H}_\infty$ performance obtained is $\gamma_{\min} = 3.1207$. However, if we assume $\eta_{\mathrm{m}} = 0.1$ s, the minimum guaranteed closed-loop $\mathscr{H}_\infty$ performance obtained is $\gamma_{\min} = 2.8113$. Secondly, we assume $\eta_{\mathrm{M}} = 0.6$ s. When $\eta_{\mathrm{m}} = 0.3$ s, we get $\gamma_{\min} = 6.0780$. However, when $\eta_{\mathrm{m}} = 0.05$, the condition in Theorem 3 is found infeasible. A more detailed comparison for different cases is provided in Table 2, which shows that considering the lower bound of the signal transmission delay gives rise to less conservative results.

Table 2
Minimum feasible γ for different cases

η_M (s)	0.4		0.6		
η_m (s)	0	0.2	0.05	0.2	0.3
$\gamma_{\min}$	3.1207	2.5233	Infeasible	11.2314	6.0780

Example 3. Suppose the physical plant in Fig. 1 is the satellite system, considered in Biernacki, Hwang, and Battacharyya (1987). The satellite system consists of two rigid bodies joined by a flexible link. This link is modelled as a spring with torque constant k and viscous damping f. Denoting the yaw angles for the two bodies (the main body and the instrumentation module) by θ_1 and θ_2, the control torque by $u(t)$, the moments of inertia of the two bodies by J_1 and J_2, the dynamic equations are given by

$$J_1\ddot{\theta}_1(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k((\theta_1(t) - \theta_2(t))) = u(t),$$
$$J_2\ddot{\theta}_2(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k((\theta_1(t) - \theta_2(t))) = 0.1w(t),$$

where $w(t)$ denotes the disturbance. Assume the output is the angular positions $\theta_2(t)$. Thus, a state-space representation of the above equation is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} + [0\ \ 0\ \ 1\ \ 0]^{\mathrm{T}} u(t) + [0\ \ 0\ \ 0\ \ 0.1]^{\mathrm{T}} w(t),$$

$$y(t) = [0\ \ 1\ \ 0\ \ 0][\theta_1(t)\ \ \theta_2(t)\ \ \dot{\theta}_1(t)\ \ \dot{\theta}_2(t)]^{\mathrm{T}}.$$

Here we choose $J_1 = J_2 = 1$, $k = 0.09$ and $f = 0.04$ (the values of k and f are chosen within their respective ranges). Then, the corresponding matrices described in Section 2 are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3 & 0.3 & -0.004 & 0.004 \\ 0.3 & -0.3 & 0.004 & -0.004 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix},$$

$$C = [0\ \ 1\ \ 0\ \ 0], \qquad D = 0, \qquad F = 0.$$

It is assumed that: the sampling period $h = 10$ ms; the network induced delay bound in (33) are given by $\eta_m = 10$ ms and $\eta_M = 40$ ms ms; the maximum number of data packet dropouts $\bar{\delta} = 2$. Then, from (41) we have $\kappa = 60$ ms. In addition, the parameters for the quantizer $f(\cdot)$ are assumed to be $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.9$.

The eigenvalues of A are $-0.04 + 0.4224j$, $-0.0400 - 0.4224j$, 0, 0; thus the above system is not stable. Our purpose is to design a state-feedback controller in the form of (31) such that the closed-loop system is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attention level γ. By using Theorem 4 (minimizing γ in (60)), we obtain the following matrices (for space consideration we do not list all the obtained matrices here):

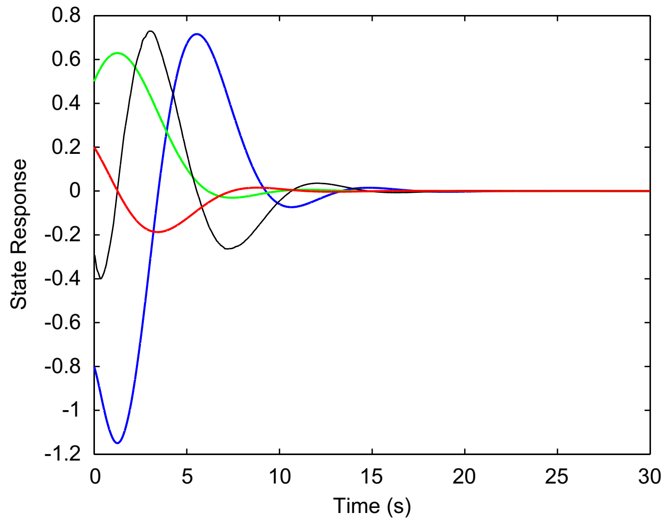

Fig. 2. State response.

$$\bar{P} = \begin{bmatrix} 6.2203 & 0.2400 & -0.6033 & -0.6113 \\ 0.2400 & 0.6331 & 0.3561 & -0.2321 \\ -0.6033 & 0.3561 & 2.6993 & -0.3402 \\ -0.6113 & -0.2321 & -0.3402 & 0.2044 \end{bmatrix},$$

$$\bar{K} = [-2.1220\ \ 0.0125\ \ -1.7269\ \ 0.0781].$$

Thus, according to (58), the gain matrix for the state-feedback controller in (31) is given by

$$K = [-1.1789\ \ -1.3096\ \ -1.6629\ \ -7.3974],$$

and the obtained minimum guaranteed $\mathscr{H}_\infty$ performance in terms of the feasibility of (60) is $\gamma^* = 0.7864$.

We first illustrate that the closed-loop system is asymptotically stable under the above obtained controller. The initial condition is assumed to be $[-0.8\ 0.5\ \ -0.3\ 0.2]^{\mathrm{T}}$. The state responses are depicted in Fig. 2, from which we can see that all four state components converge to zero. In the simulation, the network induced delays and the data packet dropouts are generated randomly (meanly distributed within their ranges) according to the above assumption, and shown in Figs. 3 and 4. The computed control inputs arriving at the ZOH are shown in Fig. 5, where we can see the discontinuous holding behavior of the control inputs.

Next, we illustrate the $\mathscr{H}_\infty$ performance of the closed-loop system. To this end, let us assume zero initial conditions, and select a set of input signals as follows:

$$w(t) = \begin{cases} \sin 0.2t, & 5 \leqslant t \leqslant 15 \text{ s}, \\ 0 & \text{otherwise}. \end{cases} \tag{65}$$

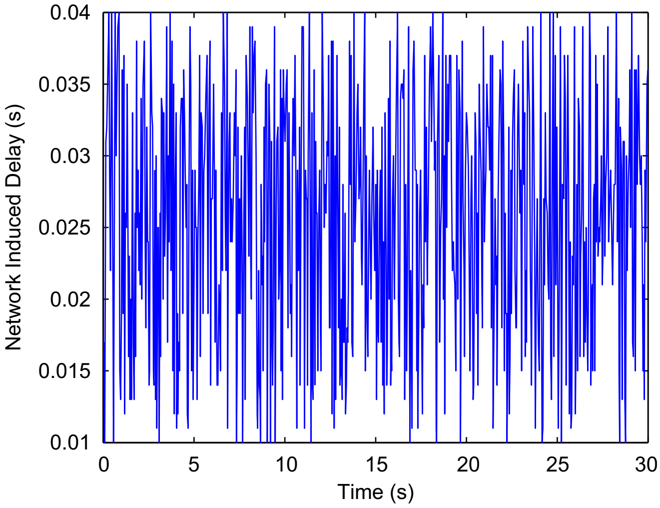

Fig. 3. Network induced delays.

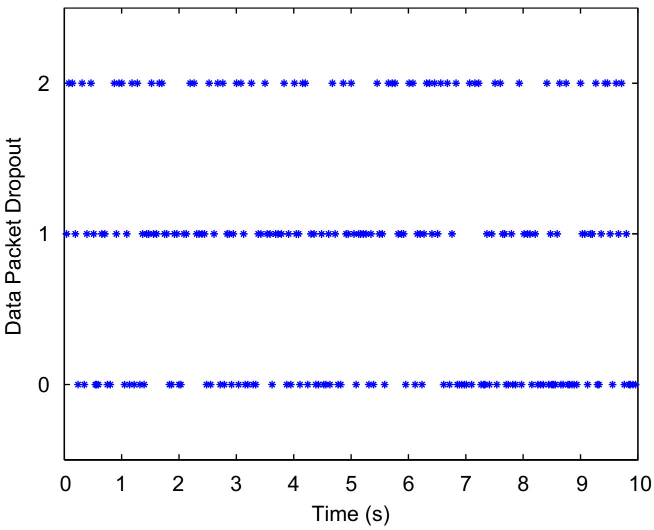

Fig. 4. Data packet dropouts.

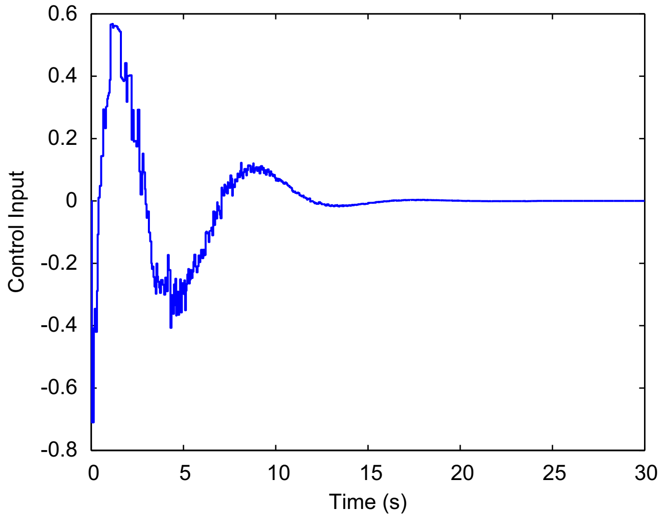

Fig. 5. Control input signal.

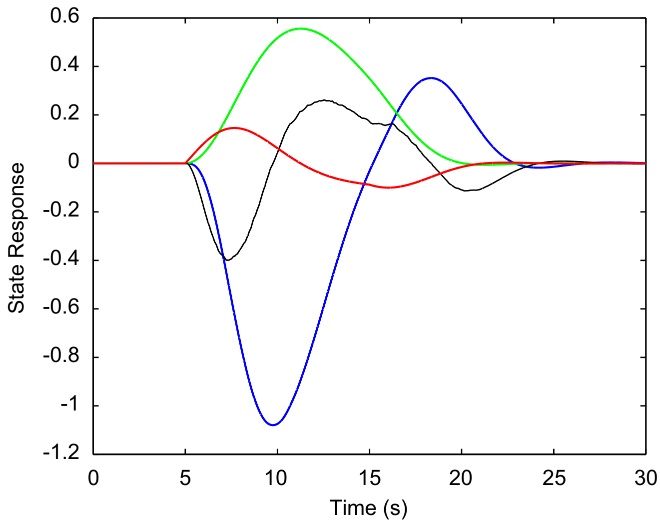

Fig. 6. State response under (65).

Fig. 6 depicts the state responses. By calculation, $\|w\|_2 = 2.5468$, $\|y\|_2 = 1.3537$, which yields

$$\frac{\|y\|_2}{\|w\|_2} = 0.5315 < \gamma^* = 0.7864,$$

showing the effectiveness of the $\mathscr{H}_\infty$ controller design.

4. Conclusions

This paper has presented new results on stability and $\mathscr{H}_\infty$ performance for systems with two successive delay components in the state by exploiting a new Lyapunov–Krasovskii functional and by making use of novel techniques for time-delay systems. An illustrative example is provided to show the significant advantage of these results. Moreover, the proposed new results have been utilized to investigate networked control systems with simultaneous consideration of network induced delays, data packet dropouts and measurement quantization. Less conservative and easily verifiable conditions for the existence of admissible $\mathscr{H}_\infty$ controllers have been obtained. Illustrative examples have been presented to show the advantage and applicability of the proposed network-based controller design methods.

References

Antsaklis, P., & Baillieul, J. (2004). Guest editorial Special issue on networked control systems. *IEEE Transactions on Automatic Control*, *31*(9), 1421–1423.

Biernacki, R. M., Hwang, H., & Battacharyya, S. P. (1987). Robust stability with structured real parameter perturbations. *IEEE Transactions on Automatic Control*, *32*, 495–506.

El Ghaoui, L., Oustry, F., & Rami, M. A. (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, *42*(8), 1171–1176.

Elia, N., & Mitter, S. K. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, *46*(9), 1384–1400.

Fridman, E., & Shaked, U. (2003). Delay-dependent stability and H_∞ control: constant and time-varying delays. *International Journal of Control*, *76*(1), 48–60.

Fu, M., & Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic Control*, *50*(11), 1698–1711.

Gao, H., Lam, J., Wang, C., & Xu, S. (2004). H_∞ model reduction for discrete time-delay systems: Delay independent and dependent approaches. *International Journal of Control*, *77*, 321–335.

Gao, H., & Wang, C. (2003). Comments and further results on A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, *48*(3), 520–525.

Gao, H., & Wang, C. (2004). A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems. *IEEE Transactions of Signal Processing*, *52*(6), 1631–1640.

Goodwin, G. C., Haimovich, H., Quevedo, D. E., & Welsh, J. S. (2004). A moving horizon approach to networked control system design. *IEEE Transactions on Automatic Control*, *49*(9), 1427–1445.

Gu, K., Kharitonov, V. L., & Chen, J. (2003). *Stability of time-delay systems*. Berlin: Springer.

Hale, J., & Lunel, S. M. V. (1993). *Introduction to functional differential equations*. Berlin: Springer.

He, Y., Wang, Q.-G., Lin, C., & Wu, M. (2007a). Delay-range-dependent stability for systems with time-varying delay. *Automatica*, *43*(2), 371–376.

He, Y., Wang, Q. G., Xie, L. H., & Lin, C. (2007b). Further improvement of free-weighting matrices technique for systems with time-varying delay. *IEEE Transactions on Automatic Control*, *52*(2), 293–299.

He, Y., Wu, M., She, J. H., & Liu, G. P. (2004). Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties. *IEEE Transactions on Automatic Control*, *49*(5), 828–832.

Hua, C., Guan, X., & Shi, P. (2005). Robust backstepping control for a class of time delayed systems. *IEEE Transactions on Automatic Control*, *50*(6), 894–899.

Ishii, H., & Francis, B. A. (2002). *Limited data rate in control systems with networks* (vol. 275), *Lecture Notes in Control and Information Sciences*. Berlin: Springer.

Jing, X. J., Tan, D. L., & Wang, Y. C. (2004). An LMI approach to stability of systems with severe time-delay. *IEEE Transactions on Automatic Control*, *49*(7), 1192–1195.

Kim, J.-H. (2001). Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty. *IEEE Transactions on Automatic Control*, *46*(5), 789–792.

Lam, J., Gao, H., & Wang, C. (2007). Stability analysis for continuous systems with two additive time-varying delay components. *Systems Control Letters*, *56*(1), 16–24.

Lee, Y. S., Moon, Y. S., Kwon, W. H., & Lee, K. H. (2001). Delay-dependent robust H_∞ control for uncertain systems with time-varying state-delay. In *Proceedings of the 40th conference on decision control* (pp. 3208–3213), Orlando, FL.

Lian, F.-L., Moyne, J., & Tilbury, D. (2003). Modelling and optimal controller design of networked control systems with multiple delays. *International Journal of Control*, *76*(6), 591–606.

Lin, C., Wang, Q.-G., & Lee, T. H. (2006). A less conservative robust stability test for linear uncertain time-delay systems. *IEEE Transactions on Automatic Control*, *51*(1), 87–91.

Liu, H., Sun, F., He, K., & Sun, Z. (2004). Design of reduced-order H_∞ filter for Markovian jumping systems with time delay. *IEEE Transactions on Circuits and Systems (II)*, *51*, 607–612.

Montestruque, L. A., & Antsaklis, P. (2004). Stability of model-based networked control systems with time-varying transmission times. *IEEE Transaction on Automatic Control*, *49*(9), 1562–1572.

Niculescu, S. I. (2001). *Delay effects on stability: a robust control approach*. Germany, Heidelberg: Springer.

Richard, J. P. (2003). Time-delay systems: An overview of some recent advances and open problems. *Automatica*, *39*(10), 1667–1694.

Seiler, P., & Sengupta, R. (2005). An H_∞ approach to networked control. *IEEE Transactions on Automatic Control*, *50*(3), 356–364.

Walsh, G. C., Ye, H., & Bushnell, L. (2002). Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, *10*(3), 438–446.

Wang, Z., & Burnham, K. J. (2001). Robust filtering for a class of stochastic uncertain nonlinear time-delay systems via exponential state estimation. *IEEE Transactions on Signal Processing*, *49*(4), 794–804.

Wang, Z., Huang, B., & Unbehauen, H. (1999). Robust H_∞ observer design of linear state delayed systems with parametric uncertainty: the discrete-time case. *Automatica*, *35*, 1161–1167.

Wu, M., He, Y., She, J. H., & Liu, G. P. (2004). Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, *40*, 1435–1439.

Xia, Y., & Jia, Y. (2002). Robust stability functionals of state delayed systems with polytopic type uncertainties via parameter-dependent Lyapunov functions. *International Journal of Control*, *75*, 1427–1434.

Xu, S., Lam, J., Huang, S., & Yang, C. (2001). H_∞ model reduction for linear time-delay systems: Continuous-time case. *International Journal of Control*, *74*(11), 1062–1074.

Yang, F. W., Wang, Z. D., Hung, Y. S., & Gani, M. (2006). H_∞ control for networked systems with random communication delays. *IEEE Transactions on Automatic Control*, *51*(3), 511–518.

Yu, M., Wang, L., & Chu, T. (2005a). Sampled-data stabilization of networked control systems with nonlinearity. *IEE Proceedings Part D: Control Theory and Applications*, *152*(6), 609–614.

Yu, M., Wang, L., Chu, T., & Hao, F. (2005b). Stabilization of networked control systems with data packet dropout and transmission delays: Continuous-time case. *European Journal of Control*, *11*(1), 40–55.

Yue, D., Han, Q.-L., & Lam, J. (2005). Network-based robust H_∞ control of systems with uncertainty. *Automatica*, *41*(6), 999–1007.

Yue, D., Han, Q.-L., & Peng, C. (2004). State feedback controller design of networked control systems. *IEEE Transactions on Circuits and Systems (II)*, *51*(11), 640–644.

Zhang, L., Shi, Y., Chen, T., & Huang, B. (2005). A new method for stabilization of networked control systems with random delays. *IEEE Transactions on Automatic Control*, *50*(8), 1177–1181.

Zhang, W., Branicky, M., & Phillips, S. (2001). Stability of networked control systems. *IEEE Control Systems Magazine*, *21*, 84–99.

Zhang, X.-M., Wu, M., She, J.-H., & He, Y. (2005). Delay-dependent stabilization of linear systems with time-varying state and input delays. *Automatica*, *41*(8), 1405–1412.

Zhivoglyadov, P. V., & Middleton, R. H. (2003). Networked control design for linear systems. *Automatica*, *39*(4), 743–750.

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