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Passivity and Absolute Stability Analyses of Trilateral Haptic Collaborative Systems

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***Abstract*—Trilateral haptic systems can be modeled as three-port networks. Analysis of coupled stability of a three-port network can be accomplished in either the passivity or the absolute stability frameworks assuming all three ports are connected to passive but otherwise unknown terminations. This paper first introduces our recent results in terms of extending Raisbecks passivity criterion and Llewellyns absolute stability criterion to general three-port networks – both criteria are founded on the properties of a positive-real Hermitian matrix. Next, we show that the absolute stability criterion is less conservative than the passivity criterion. Then, to show how the two criteria may be utilized at the system design stage, we apply them to the problem of designing controllers for a dual-user haptic teleoperation system and a triple-user collaborative haptic virtual environment. Using the two criteria, controllers are then designed and compared in terms of conservatism in simulations and experiments.**

***Index Terms*—Three-port network, trilateral haptic system, absolute stability, passivity.**

I. Introduction

New application of multilateral teleoperation systems have recently emerged including collaboration of multiple users to perform a haptic virtual task and shared control of a robot in a remote environment by multiple users. Practical uses of these include tele-rehabilitation [1], surgical training [2], and cooperative multi-robot systems [3]. An interesting class of multilateral haptic systems is the trilateral one, which can be modeled as a three-port network. Two examples of trilateral haptic systems are dual-user haptic teleoperation systems (two master robots and one slave robot) and triple-user collaborative haptic virtual environments (three master robots).

In designing haptic teleoperation controllers, the main goals are performance and stability. For a bilateral teleoperation system consisting of a teleoperator (master, slave and controllers) coupled to terminations (human operator and environment), performance is the ability of a teleoperation system to present the undistorted dynamics of the environment to the human operator. Taking precedence to performance is stability, which is necessary for safe teleoperation. Direct investigation of teleoperation system stability requires not only the teleoperator's *immittance* (z, y, h, g) parameters, but also the models of the human operator and the environment, which are usually unknown, uncertain, and/or time-varying [4], [5]. Consequently, conventional techniques cannot be used to study the stability of teleoperation systems. Methods for analyzing the stability of teleoperation system can be categorized as teleoperator passivity and teleoperator absolute stability criteria.

By definition, a teleoperator is passive if the total energy delivered to it at its ports for all possible passive terminations is non-negative at all time [6]. This means that, on a net basis, the teleoperators terminations are performing work on the teleoperator. Also, by definition, a teleoperator is absolutely stable if the teleoperation system remains stable for all possible passive terminations. For bilateral teleoperation systems comprising one master and one slave, teleoperator passivity and absolute stability can be analyzed via Raisbeck's criterion [7] and Llewellyn's criterion [8], respectively. In this paper, we extend these two criteria to trilateral teleoperators; this is not a trivial task for reasons discussed later. Similar to Raisbecks and Llewellyns criteria for bilateral teleoperators, the proposed criteria for the passivity and absolute stability of trilateral teleoperators are applicable to LTI systems *with or without communication delay*.

For absolute stability analysis of a trilateral teleoperator, in [9], [10], and [11], methods are proposed in which the three-port network model of the teleoperator is reduced to a two-port network by assuming a known termination for the third port, paving the way for the application of Llewellyn's criterion. *Unfortunately, in the above approaches, a degree of freedom is lost when the third port is coupled to a known termination*. In [12], the stability of a nonreciprocal n-port network was studied by finding a reciprocal n-port network with the same stability characteristics. For the reciprocal n-port network, absolute stability can be studied through its equivalence to passivity. This method can be lengthy for general n-port networks; however, the method is tractable for three-port networks.

For passivity analysis of a trilateral teleoperator, Wang *et al.* [13] proposed three different passive architectures based on four-channel shared control. Shahbazi *et al.* [14] performed stability analysis for dual-user teleoperation systems (three-port networks) by using the passivity theory. In [15], Panzirsch *et al.* propose a time-domain passivity-based control approach for a three-port network. In this work, three passivity observers and three passivity controllers have been used. In [16], Mendez *et al.* presented a criterion for passivity of n-port networks with unknown terminations. The criterion gives necessary and sufficient conditions for passivity of an n-port network assuming that the unknown terminations are passive.

In this paper, a comparison on the performance between absolute stability and passivity criteria for three-port networks is provided. In two case studies involving a dual-user haptic teleoperation system and a triple-user collaborative haptic virtual environment system, each of these two criteria is used for the design of stabilizing controllers.

The rest of the paper is organized as follows: The next section gives mathematical definitions and lemmas for analysis of passivity and absolute stability. In Section III, for two-port

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networks, we show the conservatism of the Raisbeck's passivity conditions as compared to Llewellyn's absolute stability conditions. In Section IV, the proposed passivity and absolute stability criteria for three-port networks are derived. We show the conservatism of the passivity conditions compared to the absolute stability conditions. Then, as a case study, in Section V, the passivity and absolute stability criteria are used in designing a trilateral shared control architecture for a dual-user teleoperation system and a triple-user collaborative haptic virtual environment system. The passivity and absolute stability conditions in terms of system parameters including controller gains are found. Finally, simulations and experiments to verify the validity of the calculated conditions are presented in Section VI. Section VII contains concluding remarks and future work.

II. Mathematical Preliminaries

Lemma 1. *[17] Let P_1 and P_2 be the immittance matrices of two n-port networks. Then, if P_1 and P_2 possess identical principal minors of all orders, the two n-port networks are stable (weakly stable) together.* $\square$

Definition 1. *[18] A Hermitian matrix is a complex square matrix that is equal to its conjugate transpose.*

Property 1. *[19] A Hermitian matrix is positive definite (positive semidefinite) if its principal minors are all positive (nonnegative).*

Lemma 2. *[20] A linear time-invariant system with transfer matrix $G(s)$ is passive (strictly passive) if $G(s)$ is positive real (strictly positive real).*

Definition 2. *[20] A $n \times n$ proper rational transfer matrix $G(s)$ is called positive real if*

i) *Poles of all elements of $G(s)$ are in $\mathrm{Re}[s] \leq 0$,*
ii) *Any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s\to j\omega}(s-j\omega)G(s)$ is positive semidefinite Hermitian,*
iii) *For all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega)+G^T(-j\omega)$ is positive semidefinite.*

Property 2. *[21] A gyration operator, which transforms one immittance matrix to another, preserves the passivity property.*

III. Passivity and Absolute Stability of Two-port Networks

For two-port networks, the well-known Raisbeck's passivity criterion [7] and Llewellyn's absolute stability criterion [8] have been developed to investigate the stability of the network when connected to arbitrary passive terminations.

A. Raisbeck's passivity criterion [7]:

Criterion 1. *If $p_{mn} = r_{mn} + jx_{mn}$, $m, n = 1, 2$, represents any of the four immittance parameters (z, y, h, and g) of a two-port network, for all real values of frequencies ω, the network is passive if and only if*

1) *The P matrix have no poles in the right-half plane (RHP).*
2) *Any poles of P matrix on the imaginary axis are simple, and the residues of the P matrix elements at these poles satisfy*

$$
\begin{aligned}
&k_{mm} \geq 0, \quad m = 1, 2 \\
&k_{11}k_{22} - k_{12}k_{21} \geq 0, \quad k_{12} = k_{21}^* \qquad (1)
\end{aligned}
$$

where k_{mn}, $m, n = 1, 2$, denotes the residue of p_{mn} and k_{mn}^* is the complex conjugate of k_{mn}.

3) *The real and imaginary part of the P matrix elements satisfy*

$$r_{11} \geq 0 \tag{2a}$$
$$r_{22} \geq 0 \tag{2b}$$
$$4r_{11}r_{22} - (r_{12}+r_{21})^2 - (x_{12}-x_{21})^2 \geq 0 \tag{2c}$$

$\square$

B. Llewellyn's absolute stability criterion [8]:

Criterion 2. *A two-port network with the immittance parameter P is absolutely stable if and only if Conditions 1) and 2) in Criterion 1 hold and, for all real values of frequencies ω, we have*

3)

$$r_{11} \geq 0 \tag{3a}$$
$$r_{22} \geq 0 \tag{3b}$$
$$r_{11}r_{22} - \frac{|p_{12}p_{21}| + \mathrm{Re}(p_{12}p_{21})}{2} \geq 0 \tag{3c}$$

$\square$

Conditions 1) and 2) of Criterion 1 imply those of Criterion 2. As part of the two conditions 3) in the two criteria, (2a)-(2b) in the passivity criterion are the same as (3a)-(3b) in the absolute stability criterion. Now, based on the relationship

$$\mathrm{Re}(\sqrt{p_{mn}p_{nm}}) = \sqrt{\frac{|p_{mn}p_{nm}| + \mathrm{Re}(p_{mn}p_{nm})}{2}} \tag{4}$$

where $m, n = 1, 2$, Condition (3c) in Criterion 2 can be rewritten as

$$\frac{(\mathrm{Re}(\sqrt{p_{12}p_{21}}))^2}{r_{11}r_{22}} \leq 1 \tag{5}$$

while Condition (2c) in Criterion 1 can be manipulated into the form

$$\frac{(\mathrm{Re}(\sqrt{p_{12}p_{21}}))^2}{r_{11}r_{22}} + \frac{(|p_{12}| - |p_{21}|)^2}{4r_{11}r_{22}} \leq 1 \tag{6}$$

Obviously, the passivity condition (6) (or the equivalent (2c)) is more conservative than the absolute stability condition (5) (or the equivalent (3c)). These two conditions are equivalent if and only if $|p_{12}| = |p_{21}|$.

As shown above, Raisbeck's passivity criteria and Llewellyn's absolute stability criteria are equivalent if and only if the two-port network with immittance matrix have $|p_{12}| = |p_{21}|$. Also, all passive two-port network are absolutely stable but not vice versa [22].

IV. Passivity and Absolute Stability of Three-port Networks

In this paper, we will discuss conditions for the passivity and absolute stability of three-port networks . We will show that these criteria are equivalent conditions, and compare the two in the general case in terms of conservativeness.

Consider a general nonreciprocal three-port network shown in Figure 1 with the immittance matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{7}$$

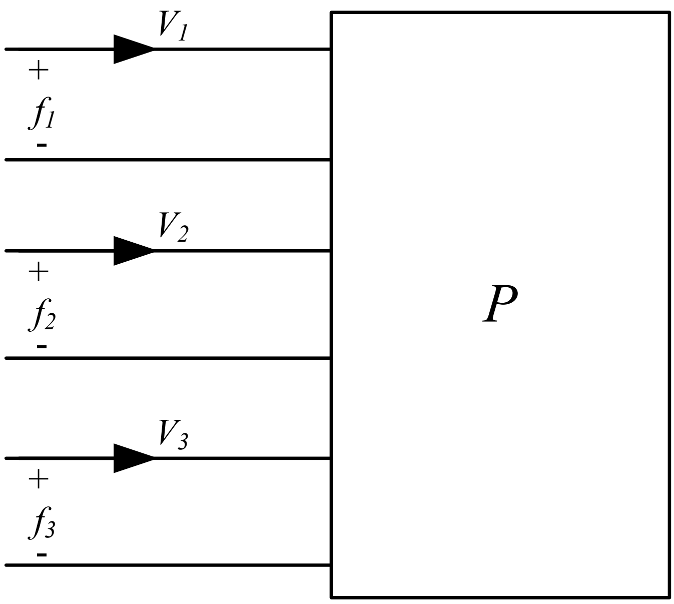

Figure 1. A general three-port network.

Where, f_i and V_i, $i = 1, 2, 3$ are the forces and velocities of each port. Let $p_{mn} = r_{mn} + jx_{mn}$, $m, n = 1, 2, 3$. Later in the paper, we will show through case studies how to calculate such a matrix for a given trilateral haptic system. We propose the following theorems for the passivity and absolute stability of the three-port network modeled by P.

A. Passivity theorem

Theorem 1. *A three-port network with the impedance matrix P in (7) is passive if and only if*

1) *The P matrix elements have no poles in the RHP.*
2) *Any poles of the P matrix elements on the imaginary axis are simple, and the residues of the P matrix elements at these poles satisfy*

$$\begin{aligned}
&k_{mm} \geq 0, \quad m = 1, 2, 3\\
&\frac{k_{11}k_{22} - k_{12}k_{21}}{k_{11}} \geq 0\\
&\frac{k_{11}k_{33} - k_{13}k_{31}}{k_{33}} \geq 0\\
&\frac{k_{22}k_{33} - k_{23}k_{32}}{k_{22}} \geq 0\\
&\frac{k_{11}k_{33} - k_{13}k_{31}}{k_{11}} - \frac{k_{11}k_{23} - k_{21}k_{13}}{k_{11}k_{22} - k_{12}k_{21}} \frac{k_{11}k_{32} - k_{31}k_{12}}{k_{11}} \geq 0\\
&k_{12} = k_{21}^*, \quad k_{13} = k_{31}^*, \quad k_{23} = k_{32}^*
\end{aligned}$$

where k_{mn} $m, n = 1, 2, 3$, denotes the residue of p_{mn} and k_{mn}^* is the complex conjugate of k_{mn}.

3) *The real and imaginary part of the P matrix elements satisfy the following inequalities*

$$r_{11} \geq 0 \tag{8a}$$
$$r_{22} \geq 0 \tag{8b}$$
$$r_{33} \geq 0 \tag{8c}$$
$$4r_{11}r_{22} - (r_{12} + r_{21})^2 - (x_{12} - x_{21})^2 \geq 0 \tag{8d}$$
$$4r_{11}r_{33} - (r_{13} + r_{31})^2 - (x_{13} - x_{31})^2 \geq 0 \tag{8e}$$
$$4r_{22}r_{33} - (r_{23} + r_{32})^2 - (x_{23} - x_{32})^2 \geq 0 \tag{8f}$$
$$\begin{aligned}
&4r_{11}r_{22}r_{33} - r_{33}[(r_{12} + r_{21})^2 + (x_{12} - x_{21})^2]\\
&- r_{22}[(r_{13} + r_{31})^2 + (x_{13} - x_{31})^2]\\
&- r_{11}[(r_{23} + r_{32})^2 + (x_{23} - x_{32})^2]\\
&+ (r_{23} + r_{32})(r_{13} + r_{31})(r_{12} + r_{21})\\
&+ (r_{12} + r_{21})(x_{13} - x_{31})(x_{23} - x_{32})\\
&- (r_{13} + r_{31})(x_{12} - x_{21})(x_{23} - x_{32})\\
&+ (r_{23} + r_{32})(x_{13} - x_{31})(x_{12} - x_{21}) \geq 0
\end{aligned} \tag{8g}$$

□

Proof. According to Lemma 2, the three-port network is passive if and only if its transfer matrix (i.e., the matrix P in (7)) is positive real, which can be verified through Definition 2. It is obvious that Condition 1) in the theorem is the same as Condition i) in Definition 2.

According to Condition ii) in Definition 2, the residue matrix is positive semidefinite Hermitian, so we have

$$k_{12} = k_{21}^*, \quad k_{13} = k_{31}^*, \quad k_{23} = k_{32}^* \tag{9}$$

where k_{mn}^* is the complex conjugate of k_{mn}, $m, n = 1, 2, 3$. Based on Property 1, the residue matrix is positive semidefinite if its principal minors are all nonnegative, i.e.,

$$k_{mm} \geq 0, \quad m = 1, 2, 3 \tag{10}$$
$$k_{11}k_{22} - k_{12}k_{21} \geq 0 \tag{11}$$
$$k_{11}k_{33} - k_{13}k_{31} \geq 0 \tag{12}$$
$$k_{22}k_{33} - k_{23}k_{32} \geq 0 \tag{13}$$
$$\begin{aligned}
&k_{11}k_{22}k_{33} - k_{11}k_{23}k_{32} - k_{22}k_{13}k_{31} - k_{33}k_{12}k_{21}\\
&+ k_{12}k_{23}k_{31} - k_{13}k_{21}k_{32} \geq 0
\end{aligned} \tag{14}$$

The inequalities (9)-(14) are equivalent to those in Condition 2) of Theorem 1.

According to Condition iii) of Definition 2,

$$\begin{aligned}
P(j\omega) + P^T(-j\omega) = &\begin{bmatrix} 2r_{11} & r_{12} + r_{21} & r_{13} + r_{31} \\ r_{12} + r_{21} & 2r_{22} & r_{23} + r_{32} \\ r_{13} + r_{31} & r_{23} + r_{32} & 2r_{33} \end{bmatrix}\\
&+ j \begin{bmatrix} 0 & x_{12} - x_{21} & x_{13} - x_{31} \\ x_{21} - x_{12} & 0 & x_{23} - x_{32} \\ x_{31} - x_{13} & x_{32} - x_{23} & 0 \end{bmatrix}
\end{aligned} \tag{15}$$

needs to be positive semidefinite. Using Property 1, this leads to Conditions (8a)-(8g). This concludes the proof. □

B. Absolute stability theorem

Theorem 2. *A three-port network with impedance matrix P in (7) satisfying the symmetrization condition*

$$p_{13}p_{21}p_{32} - p_{12}p_{23}p_{31} = 0 \tag{16}$$

is absolutely stable if and only if Conditions 1) and 2) in Theorem 1 hold and, for all real values of frequencies ω, we have

3)

$$r_{11} \geq 0 \tag{17a}$$
$$r_{22} \geq 0 \tag{17b}$$
$$r_{33} \geq 0 \tag{17c}$$
$$r_{11}r_{22} - \frac{|p_{12}p_{21}| + \mathrm{Re}(p_{12}p_{21})}{2} \geq 0 \tag{17d}$$
$$r_{11}r_{33} - \frac{|p_{13}p_{31}| + \mathrm{Re}(p_{13}p_{31})}{2} \geq 0 \tag{17e}$$
$$r_{22}r_{33} - \frac{|p_{23}p_{32}| + \mathrm{Re}(p_{23}p_{32})}{2} \geq 0 \tag{17f}$$
$$\begin{aligned} &r_{11}r_{22}r_{33} - r_{11}\frac{|p_{23}p_{32}| + \mathrm{Re}(p_{23}p_{32})}{2} \\ &- r_{22}\frac{|p_{13}p_{31}| + \mathrm{Re}(p_{13}p_{31})}{2} \\ &- r_{33}\frac{|p_{12}p_{21}| + \mathrm{Re}(p_{12}p_{21})}{2} \\ &+ 2\mathrm{Re}(\sqrt{p_{12}p_{21}})\mathrm{Re}(\sqrt{p_{13}p_{31}})\mathrm{Re}(\sqrt{p_{23}p_{32}}) \geq 0 \end{aligned} \tag{17g}$$

□

Proof. [23] According to Lemma 1, if there exists a reciprocal three-port network with impedance matrix P_{eq} that has the same stability (weak stability) characterization as the nonreciprocal three-port network with impedance matrix P, then

$$\det(P_{eq} + P_0) = \det(P + P_0) \tag{18}$$

for any passive (strictly passive) $P_0 = \mathrm{diag}[p_1, p_2, p_3]$. According to (18) in the paper, we have

$$\det \begin{bmatrix} p_a + p_1 & p_b & p_d \\ p_b & p_c + p_2 & p_f \\ p_d & p_f & p_h + p_3 \end{bmatrix} = \det \begin{bmatrix} p_{11} + p_1 & p_{12} & p_{13} \\ p_{21} & p_{22} + p_2 & p_{23} \\ p_{31} & p_{32} & p_{33} + p_3 \end{bmatrix}$$

Calculating the two determinants and equating the coefficients of p_1, p_2, and p_3 (because the above is to hold for *any* passive (or strictly passive) $P_0 = \mathrm{diag}[p_1, p_2, p_3]$), if and only if the symmetrization condition (16) holds, we get

$$P_{eq} = \begin{bmatrix} p_{11} & \gamma_1\sqrt{p_{12}p_{21}} & \gamma_2\sqrt{p_{13}p_{31}} \\ \gamma_1\sqrt{p_{12}p_{21}} & p_{22} & \gamma_3\sqrt{p_{23}p_{32}} \\ \gamma_2\sqrt{p_{13}p_{31}} & \gamma_3\sqrt{p_{23}p_{32}} & p_{33} \end{bmatrix} \tag{19}$$

where $\gamma_i = \pm 1$ for $i = 1, 2, 3$. The three-port network with transfer matrix P_{eq} is absolutely stable if and only if it is passive [24]. According to Lemma 2, P_{eq} is passive if and only if it is nonnegative real, which can be verified through Definition 2. Thus, it is easy to show that Conditions 1) and 2) in Theorem 1 need to be satisfied. Additionally, according to Condition iii) of Definition 2,

$$P_{eq}(j\omega) + P_{eq}^T(-j\omega) =$$
$$\begin{bmatrix} 2r_{11} & 2\gamma_1\mathrm{Re}\sqrt{p_{12}p_{21}} & 2\gamma_2\mathrm{Re}\sqrt{p_{13}p_{31}} \\ 2\gamma_1\mathrm{Re}\sqrt{p_{12}p_{21}} & 2r_{22} & 2\gamma_3\mathrm{Re}\sqrt{p_{23}p_{32}} \\ 2\gamma_2\mathrm{Re}\sqrt{p_{13}p_{31}} & 2\gamma_3\mathrm{Re}\sqrt{p_{23}p_{32}} & 2r_{33} \end{bmatrix} \tag{20}$$

needs to be positive semidefinite. Using Property 1 and equality (4) (where $m, n = 1, 2, 3$) leads us to Conditions (17a)-(17g). This concludes the proof. □

Remark 1. Theorem 1 and Theorem 2 holds not only for the impedance matrix (Z) of a general network but also for its other immittance matrices (Y, H, G). The reason for this is that according to Property 2 a gyration operators transform one immittance matrix to another, and preserves passivity.

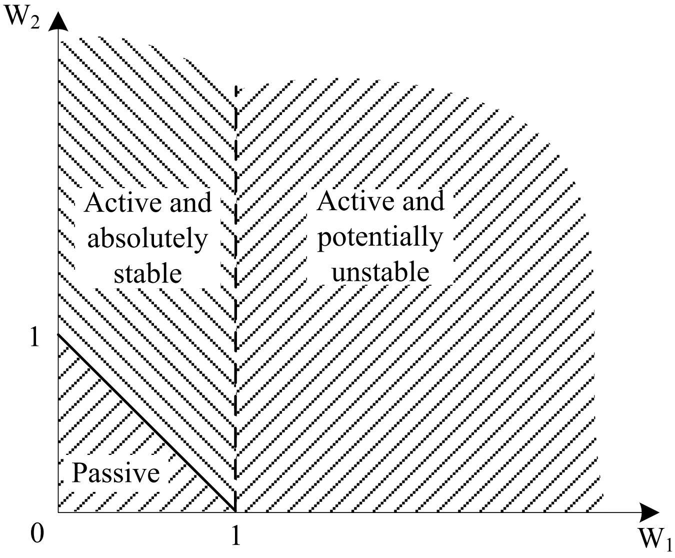

Figure 2. Stability-activity diagram.

C. Comparison of passivity and absolute stability conditions

Conditions 1) and 2) of Theorem 1 imply those of Theorem 2. Also, as part of Condition 3) in Theorem 1, (8a)-(8c) in Theorem 1 are the same as (17a)-(17c) in Theorem 2. As shown in Section III, the passivity condition (8d)-(8f) is more conservative than the absolute stability condition (17d)-(17f), respectively. These conditions are equivalent if and only if $|p_{12}| = |p_{21}|$, $|p_{13}| = |p_{31}|$, and $|p_{23}| = |p_{32}|$.

Furthermore, Condition (17g) in Theorem 2 can be rewritten as

$$\begin{aligned} &W_1 = \\ &\frac{(\mathrm{Re}(\sqrt{p_{23}p_{32}}))^2}{r_{22}r_{33}} + \frac{(\mathrm{Re}(\sqrt{p_{13}p_{31}}))^2}{r_{11}r_{33}} + \frac{(\mathrm{Re}(\sqrt{p_{12}p_{21}}))^2}{r_{11}r_{22}} \\ &- \frac{2\mathrm{Re}(\sqrt{p_{12}p_{21}})\mathrm{Re}(\sqrt{p_{13}p_{31}})\mathrm{Re}(\sqrt{p_{23}p_{32}})}{r_{11}r_{22}r_{33}} \leq 1. \end{aligned} \tag{21}$$

On the other hand, noting that

$$\begin{aligned} &(r_{mn} + r_{nm})^2 + (x_{mn} - x_{nm})^2 \\ &= 4(\mathrm{Re}(\sqrt{p_{mn}p_{nm}}))^2 + (|p_{mn}| - |p_{nm}|)^2 \end{aligned} \tag{22}$$

where $m, n = 1, 2, 3$, Condition (8g) in Theorem 1 can be manipulated into the form

$$W_1 + W_2 \leq 1 \tag{23}$$

In the following, we will show that $W_2 \geq 0$, establishing the fact that the passivity condition (23) (or the equivalent (8g)) is more conservative than the absolute stability condition (21) (or the equivalent (17g)).

In (23), we have $W_2 = W_3 + W_4 - W_5$ where

$$W_3 = \frac{r_{11}(|p_{23}| - |p_{32}|)^2}{4r_{11}r_{22}r_{33}} + \frac{r_{22}(|p_{13}| - |p_{31}|)^2}{4r_{11}r_{22}r_{33}} + \frac{r_{33}(|p_{12}| - |p_{21}|)^2}{4r_{11}r_{22}r_{33}},$$

$$W_4 = \frac{2\mathrm{Re}\sqrt{p_{12}p_{21}}\mathrm{Re}\sqrt{p_{13}p_{31}}\mathrm{Re}\sqrt{p_{23}p_{32}}}{r_{11}r_{22}r_{33}},$$

$$W_5 = \frac{(r_{12}+r_{21})(r_{13}+r_{31})(r_{23}+r_{32})}{4r_{11}r_{22}r_{33}} + \frac{(r_{12}+r_{21})(x_{13}-x_{31})(x_{23}-x_{32})}{4r_{11}r_{22}r_{33}} - \frac{(r_{13}+r_{31})(x_{12}-x_{21})(x_{23}-x_{32})}{4r_{11}r_{22}r_{33}} + \frac{(r_{23}+r_{32})(x_{13}-x_{31})(x_{12}-x_{21})}{4r_{11}r_{22}r_{33}}$$

Obviously, $W_3 \geq 0$. Because of (4), $W_4 \geq 0$. Therefore, whenever $W_5 < 0$, then $W_2 > 0$. When $W_5 > 0$, then $W_2 \geq 0$ if and only if $(W_3 + W_4)^2 - W_5^2 \geq 0$, which is equivalent to

$$\begin{aligned}
&(|p_{12}| - |p_{21}|)^2[16r_{33}r_{123} + 4r_{11}r_{33}(|p_{23}| - |p_{32}|)^2 \\
&- 4(\mathrm{Re}\sqrt{p_{13}p_{31}})^2((r_{23}+r_{32})^2 + (x_{23}-x_{32})^2)] \\
&+ (|p_{13}| - |p_{31}|)^2[16r_{22}r_{123} + 4r_{22}r_{33}(|p_{12}| - |p_{21}|)^2 \\
&- 4(\mathrm{Re}\sqrt{p_{23}p_{32}})^2((r_{12}+r_{21})^2 + (x_{12}-x_{21})^2)] \\
&+ (|p_{23}| - |p_{32}|)^2[16r_{11}r_{123} + 4r_{11}r_{22}(|p_{13}| - |p_{31}|)^2 \\
&- 4(\mathrm{Re}\sqrt{p_{12}p_{21}})^2((r_{13}+r_{31})^2 + (x_{13}-x_{31})^2)] \geq 0 \quad (24)
\end{aligned}$$

where $r_{123} = \mathrm{Re}\sqrt{p_{12}p_{21}}\mathrm{Re}\sqrt{p_{13}p_{31}}\mathrm{Re}\sqrt{p_{23}p_{32}}$. It is easy to show that (24) always holds. Thus, regardless of sign of W_5, we have $W_2 \geq 0$.

In the stability-activity diagram of Figure 2, we have graphically represented (21) and (23) in a two-dimensional space by choosing W_1 and W_2 as the two coordinates. Evidently, all passive three-port networks are absolutely stable, but not all absolutely stable three-port networks are passive.

Remark 2. The passivity criterion of three-port network in Theorem 1 is equivalent to the absolute stability criterion in Theorem 2 if and only if the impedance matrix in (7) have

$$|p_{12}| = |p_{21}|, \quad |p_{13}| = |p_{31}|, \quad |p_{23}| = |p_{32}|. \qquad (25)$$

This is true because $W_2 = 0$ if and only if (25) holds. This holds not only for the impedance parameters of a general three-port network but also for its other immittance parameters.

Remark 3. For teleoperation control systems, using the absolute stability criterion will allow for higher transparency compared to using the passivity theorem. The reason for this is that passivity criterion is more restrictive than the absolute stability criterion, and there is a trade-off between stability and transparency. In the case studies that will follows higher teleoperation transparency under absolute stability conditions compared to passivity conditions will be shown.

V. Case Study: Comparison of Passivity and Absolute Stability for Trilateral Haptic Systems

In this section, the aim is to compare passivity and absolute stability for trilateral haptic systems. A trilateral haptic system may be a dual-user haptic teleoperation system with one slave robot, or a collaborative haptic virtual environment with three users. In the following, we will consider both cases, which happen to the nonreciprocal three-port networks. We begin by reviewing a four-channel, dual-user teleoperation system and specifically investigate the stability of position-position control scheme. Later, we will introduce a triple-user haptic virtual environment and study its stability.

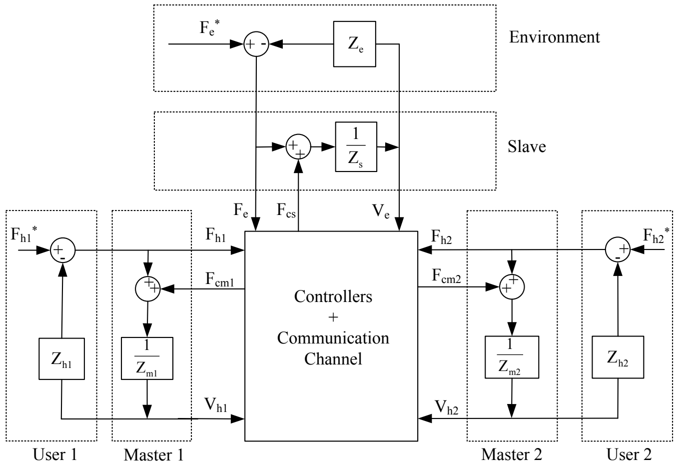

Figure 3. A dual-user haptic teleoperation system under four-channel control.

A. A dual-user teleoperation system with position-position shared control

In a dual-user teleoperation control system, the goal is that two users collaboratively control a robot to perform a desired task on a remote environment. Such a system consists of two master robots for the two users and one slave robot that is in contact with the environment. This configuration has applications in many real-world scenarios such as when the aim is to train a novice trainee (user 1) to do a task in a remote environment under haptic guidance from a mentor (user 2). As elaborated by [2], [11], the desired position and force of each robot are the weighted sum of positions and forces of the other two robots. The weights are determined by a parameter $\alpha \in [0, 1]$ which in practice gives the relative authority that each operator has over the slave robot.

In a dual-user teleoperation system, the dynamics of the two masters and the slave in contact with the two users and the environment, respectively, are

$$Z_{mi}V_{hi} = F_{hi} + F_{cmi} \qquad (26a)$$

$$Z_sV_e = F_e + F_{cs} \qquad (26b)$$

where $i = 1, 2$, and Z_{mi} and Z_s are the impedances of the two masters and the slave, respectively. Also, F_{hi} denotes the interaction force between the two users and the two masters and F_s denotes the interaction force between the slave and the environment. Lastly, V_{hi}, and V_e are the users and the environment velocities.

The four-channel dual-user shared control laws [11], [16]:

$$F_{cmi} = -C_{mi}V_{hi} - C_{4mi}V_{hid} + C_{6mi}F_{hi} - C_{2mi}F_{hid} \qquad (27a)$$

$$F_{cs} = -C_sV_e + C_1V_{ed} - C_5F_e + C_3F_{ed} \qquad (27b)$$

where C_{mi} and C_s are local position controllers, C_{6mi} and C_5 are local force controllers, and C_1, C_{2mi}, C_3, and C_{4mi} are feedforward and feedback compensators. Also, V_{hid} and V_{ed} are the reference velocities and F_{hid} and F_{ed} are the references forces for the two masters and the slave, where using the complementary-linear-combination (CLC) laws for authority sharing are

$$V_{h1d} = \alpha V_e + (1-\alpha)V_{h2} \tag{28a}$$
$$V_{h2d} = (1-\alpha)V_e + \alpha V_{h1} \tag{28b}$$
$$V_{ed} = \alpha V_{h1} + (1-\alpha)V_{h2} \tag{28c}$$
$$F_{h1d} = \alpha F_e + (1-\alpha)F_{h2} \tag{28d}$$
$$F_{h2d} = (1-\alpha)F_e + \alpha F_{h1} \tag{28e}$$
$$F_{ed} = \alpha F_{h1} + (1-\alpha)F_{h2} \tag{28f}$$

Consequently, α determines how the two users collaborate and contribute to the reference position for the slave, and it also determines what share of force feedback they (trainee and mentor) receive. For instance, if $\alpha = 0$, the slave robot will be completely controlled by the mentor and the trainee will receive large force feedback urging him/her to follow the mentor's motions. On the other hand if $\alpha = 1$, the slave robot is completely controlled by the trainee, allowing the mentor to assess the skill level of the trainee by feeling the reflected forces. If $0 < \alpha < 1$, the trainee and the mentor collaborate and each contribute to the slave robot position while receiving some force feedback.

Position-position control is a special case of dual-user shared control in which there is no force sensor measurements and $C_{2m1} = C_{2m2} = C_3 = C_5 = C_{6m1} = C_{6m2} = 0$. For simplicity of deriving stability conditions, we consider this special case of the earlier-described 4-channel teleoperation system. For good position tracking, the common choice is $C_1 = C_s$, $C_{4m1} = -C_{m1}$, and $C_{4m2} = -C_{m2}$. Assume $Z_{m1} = M_{m1}s$, $Z_{m2} = M_{m2}s$, $Z_s = M_s s$, and let us make the following choices for the local position controllers:

$$C_{m1} = \frac{K_{pm1} + K_{vm1}s}{s}, \quad C_{m2} = \frac{K_{pm2} + K_{vm2}s}{s},$$
$$C_s = \frac{K_{ps} + K_{vs}s}{s} \tag{29}$$

To get the impedance matrix of position-position control dual-user teleoperation system, first substitute (28) in (27) and then substitute the result in (26) to get

$$\begin{bmatrix} F_{h1} \\ F_{h2} \\ F_e \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} V_{h1} \\ V_{h2} \\ V_e \end{bmatrix} \tag{30}$$

where

$$\begin{aligned}
z_{11} &= \frac{K_{pm1} + K_{vm1}s}{s} + M_{m1}s \\
z_{12} &= -(1-\alpha)\frac{K_{pm1} + K_{vm1}s}{s} \\
z_{13} &= -\alpha\frac{K_{pm1} + K_{vm1}s}{s} \\
z_{21} &= -\alpha\frac{K_{pm2} + K_{vm2}s}{s} \\
z_{22} &= \frac{K_{pm2} + K_{vm2}s}{s} + M_{m2}s \\
z_{23} &= -(1-\alpha)\frac{K_{pm2} + K_{vm2}s}{s} \\
z_{31} &= -\alpha\frac{K_{ps} + K_{vs}s}{s} \\
z_{32} &= -(1-\alpha)\frac{K_{ps} + K_{vs}s}{s} \\
z_{33} &= \frac{K_{ps} + K_{vs}s}{s} + M_s s
\end{aligned} \tag{31}$$

While for simplicity this example did not involve communication time delay, the teleoperator's impedance matrix can be calculated in a similar manner in the presence of delay. Also, the upcoming passivity and absolute stability analyses can be performed in a similar manner for delayed teleoperators as the proposed criteria apply to general immittance matrices for trilateral teleoperators.

In the following subsections, we will discuss different methods to analyze the system stability. The first method tries to find an (infinite) set of equivalent bilateral teleoperations system for the trilateral teleoperation system by coupling one port to a known termination and then utilizes Llewellyn's criterion for finding the stability conditions; this proves to be a cumbersome and open-ended investigation. The second and third methods are based on Theorem 1 and Theorem 2 for direct stability analysis of a three-port network for three passive but otherwise arbitrary terminations; these methods involve compact, closed-form conditions.

1) Stability analysis via reduction to two-port networks: To reduce the three-port network to an equivalent two-port network between the two users, one can couple the environment port to a known load termination and then absorb the load termination into the network. To find the equivalent two-port impedance matrix, in the simplest case, one can consider the aforementioned load to be a pure known stiffness $K > 0$. Assume $\alpha = \frac{1}{2}$. Then, using $\frac{F_e}{V_e} = K$, the equivalent two-port network for the dual-user teleoperation control system is given by

$$\begin{bmatrix} F_{h1} \\ F_{h2} \end{bmatrix} = \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} V_{h1} \\ V_{h2} \end{bmatrix} \tag{32}$$

For brevity, we do not show the elements of the matrix $Z'(j\omega)$.

Now, the stability of the reduced two-port network (32) must be tested *for all possible choices of* K and *all frequencies* ω. By Llewellyn's criterion, the stability of the dual-user teleoperation system is guaranteed if, for all K and all ω, we have

$$K_{vm1} - \frac{1}{4}\frac{(K_{vs}K_{vm1}\omega^2 - K_{ps}K_{pm1})(K + K_{vs})}{(K+K_{vs})^2\omega^2 + (K_{ps} - M_s\omega^2)^2}$$
$$-\frac{1}{4}\frac{(K_{vs}K_{pm1} + K_{vm1}K_{ps})(K_{ps} - M_s\omega^2))}{(K+K_{vs})^2\omega^2 + (K_{ps} - M_s\omega^2)^2} \geq 0 \tag{33a}$$
$$\frac{1}{4}\frac{(K_{pm2} + K_{vm2})(K_{ps}K + K_{vs}M_s\omega^2)}{(K+K_{vs})^2\omega^2 + (K_{ps} - M_s\omega^2)^2} \geq 0 \tag{33b}$$
$$2\mathrm{Re}(Z'_{11})\mathrm{Re}(Z'_{22}) - \mathrm{Re}(Z'_{12}Z'_{21}) - |Z'_{12}Z'_{21}| \geq 0 \tag{33c}$$

To synthesize controllers based on (33) for all values of K and ω is a daunting task if not impossible. This issue is exacerbated once one considers that the environment port's load may include damping and inertia in addition to stiffness, in which case (33) would have to be satisfied for all values ranging from 0 to ∞ of stiffness, damping, inertia and frequency. As discussed in [11], the computational burden can be alleviated by using the transformation $\Gamma = \frac{Z_e - 1}{Z_e + 1}$, where Z_e is the complex impedance of the load termination, to map the right half of the Z_e plane to the inside of a unit disk in the Γ plane. However, this method still requires to pick a large number of points in the unit disk in the Γ plane, test (33), and then repeat this process for a large number of frequencies ω before one can reasonably be sure that Llewellyn's conditions are met for a large set of points in the right half of the Z_e plane and for a large set of frequencies.

2) Stability analysis via Theorem 1: All the elements of (30) have only a simple pole on the imaginary axis. Analysis of the residues leads to

$$k_{11} = K_{pm1} \geq 0 \tag{34}$$
$$k_{22} = K_{pm2} \geq 0 \tag{35}$$
$$k_{33} = K_{ps} \geq 0 \tag{36}$$
$$\frac{k_{11}k_{22} - k_{12}k_{21}}{k_{11}} = (1 - \alpha + \alpha^2)K_{pm2} \geq 0 \tag{37}$$
$$\frac{k_{11}k_{33} - k_{13}k_{31}}{k_{33}} = (1 - \alpha^2)K_{pm1} \geq 0 \tag{38}$$
$$\frac{k_{22}k_{33} - k_{23}k_{32}}{k_{22}} = (2\alpha - \alpha^2)K_{ps} \geq 0 \tag{39}$$
$$\frac{k_{11}k_{33} - k_{13}k_{31}}{k_{11}} - \frac{k_{11}k_{23} - k_{21}k_{13}}{k_{11}k_{22} - k_{12}k_{21}} \frac{k_{11}k_{32} - k_{31}k_{12}}{k_{11}} = 0 \tag{40}$$

(34)-(36) and (40) are always satisfied. Also, (37) always holds for all $\alpha \in [0, 1]$, thus, Conditions 1) and 2) of Theorem 1 are fulfilled.

Applying (8a)-(8f) to (30) results in

$$K_{vm1} \geq 0 \tag{41}$$
$$K_{vm2} \geq 0 \tag{42}$$
$$K_{vs} \geq 0 \tag{43}$$
$$4K_{vm1}K_{vm2} - (K_{vm1} - \alpha K_{vm1} + \alpha K_{vm2})^2 - \frac{(K_{pm1} - \alpha K_{pm1} + \alpha K_{pm2})^2}{\omega^2} \geq 0 \tag{44}$$
$$4K_{vm1}K_{vs} - (K_{vm1} - \alpha K_{vm1} + \alpha K_{vs})^2 - \frac{(K_{pm1} - \alpha K_{pm1} + \alpha K_{ps})^2}{\omega^2} \geq 0 \tag{45}$$

It is easy to see that, condition (44) and (45) will be fulfilled for *all frequencies* ω if the gains of the PD controllers in (29) satisfy

$$(1 - \alpha)K_{pm1} = \alpha K_{pm2} \tag{46}$$
$$(1 - \alpha)K_{pm1} = \alpha K_{ps} \tag{47}$$
$$4K_{vm1}K_{vm2} - (K_{vm1} - \alpha K_{vm1} + \alpha K_{vm2})^2 \geq 0 \tag{48}$$
$$4K_{vm1}K_{vs} - (K_{vm1} - \alpha K_{vm1} + \alpha K_{vs})^2 \geq 0 \tag{49}$$

On the other hand, under (46), condition (8g) becomes

$$\begin{aligned}
&- \frac{1}{2\alpha\omega^2}(K_{pm1} - K_{ps})^2[K_{vm1}(1 - \alpha)^2(2 - \alpha) \\
&+ K_{vm2}\alpha^2(1 + \alpha^3)] - \frac{1}{\alpha^2\omega^2}(1 - 2\alpha)^2 K_{vm1} \\
&- \frac{1}{2\alpha\omega^2}(1 - 2\alpha)(1 - \alpha)(K_{pm1}^2 - K_{ps}^2)[\alpha^2 K_{vm2} \\
&+ (\alpha + 2)K_{vm1}] + (1 + \alpha)(2 - \alpha)K_{vm1}K_{vm2}K_{vs} \\
&- \alpha^2(2 - \alpha)K_{vm2}K_{vs}(K_{vm2} + K_{vs}) \\
&- (1 - \alpha + \alpha^2)K_{vm1}K_{vm2}[(1 - \alpha)K_{vm1} + \alpha K_{vm2}] \\
&- (1 - \alpha)^2(1 + \alpha)K_{vm1}K_{vs}(K_{vm1} + K_{vs}) \geq 0
\end{aligned} \tag{50}$$

It is easy to see that (46), (47), (48), (49), and (50) will be fulfilled for *all frequencies* ω if

$$\alpha = \frac{1}{2}, \quad K_{pm1} = K_{pm2} = K_{ps}, \quad K_{vm1} = K_{vm2} = K_{vs} \tag{51}$$

So, a sufficient, *frequency-independent*, and compact condition for passivity of the above-described position-position dual-user teleoperation systems is given by (51).

3) Stability analysis via Theorem 2: In this case, it can be shown that the symmetrization condition (16) will hold only if $\alpha = \frac{1}{2}$. It is possible to see that the absolute stability conditions (17a)-(17d) become

$$K_{vm1} \geq 0 \tag{52}$$
$$K_{vm2} \geq 0 \tag{53}$$
$$K_{vs} \geq 0 \tag{54}$$
$$\frac{7}{8}K_{vm1}K_{vm2} + \frac{1}{8\omega^2}K_{pm1}K_{pm2} - \frac{Q_{m1}Q_{m2}}{8\omega^2} \geq 0 \tag{55}$$
$$\frac{7}{8}K_{vm1}K_{vs} + \frac{1}{8\omega^2}K_{pm1}K_{ps} - \frac{Q_{m1}Q_s}{8\omega^2} \geq 0 \tag{56}$$
$$\frac{7}{8}K_{vs}K_{vm2} + \frac{1}{8\omega^2}K_{ps}K_{pm2} - \frac{Q_sQ_{m2}}{8\omega^2} \geq 0 \tag{57}$$

where $Q_{m1} = \sqrt{K_{vm1}^2\omega^2 + K_{pm1}^2}$, $Q_{m2} = \sqrt{K_{vm2}^2\omega^2 + K_{pm2}^2}$, and $Q_s = \sqrt{K_{vs}^2\omega^2 + K_{ps}^2}$. Now, under (52) and (53), condition (55) will be fulfilled for all frequencies ω if the gains of the PD controllers C_{m1} and C_{m2} satisfy

$$\frac{K_{pm1}}{K_{vm1}} = \frac{K_{pm2}}{K_{vm2}}, 7 - 4\sqrt{3} \leq \frac{K_{pm1}}{K_{vm1}}\frac{K_{vs}}{K_{ps}} \leq 7 + 4\sqrt{3}. \tag{58}$$

On the other hand, condition (17g) will be fulfilled for all frequencies ω if the gains of the PD controllers in (29) satisfy

$$\frac{K_{pm1}}{K_{vm1}} = \frac{K_{pm2}}{K_{vm2}}, \quad 5 - 2\sqrt{6} \leq \frac{K_{pm1}}{K_{vm1}}\frac{K_{vs}}{K_{ps}} \leq 5 + 2\sqrt{6} \tag{59}$$

Clearly, (58) holds if (59) holds. So, a sufficient, *frequency-independent*, and compact condition for absolute stability of the above-described position-position dual-user teleoperation systems is given by (59), where all control gains are nonnegative. Evidently, (59) is less restrictive than (51).

At the first glance, the constraint $\alpha = \frac{1}{2}$ imposed by the symmetrization condition (16) seems very limiting. However, one must note that various combinations of authority sharing and teleoperation control laws exist and $\alpha = \frac{1}{2}$ is only an artifact of using CLC authority sharing laws in conjunction with position-position teleoperation control laws. For instance, by changing the authority sharing laws (28) to the masters-correspondence-with-environment-transfer (MCET) law proposed in [11], for the same dynamics for the master and the slave and the same position-position control laws as in Section IV.A, the symmetrization condition (16) holds for any α because $Z_{13}Z_{21}Z_{32} - Z_{12}Z_{23}Z_{31}$ is identical to zero. **We compare the effect of CLC and MCET authority sharing laws on the conditions that results from the absolute stability and passivity criteria.**

B. A triple-user collaborative haptic virtual environment

In a one degree of freedom triple-user collaborative haptic virtual environment, the goal is that three users cooperate with one another in a virtual environment to perform a task while receiving haptic feedback. This corresponds to multi-point-of-contact interaction with a virtual environment. The system consists of three master robots, each of which operate on a specific point (grid mesh node) on the virtual object as shown in Figure 4 [25], [26], [27]. The virtual object computes the dynamic response (in terms of force feedback) at each of these points by using the positions of the three masters. The virtual object's mechanical properties such as mass, stiffness, and

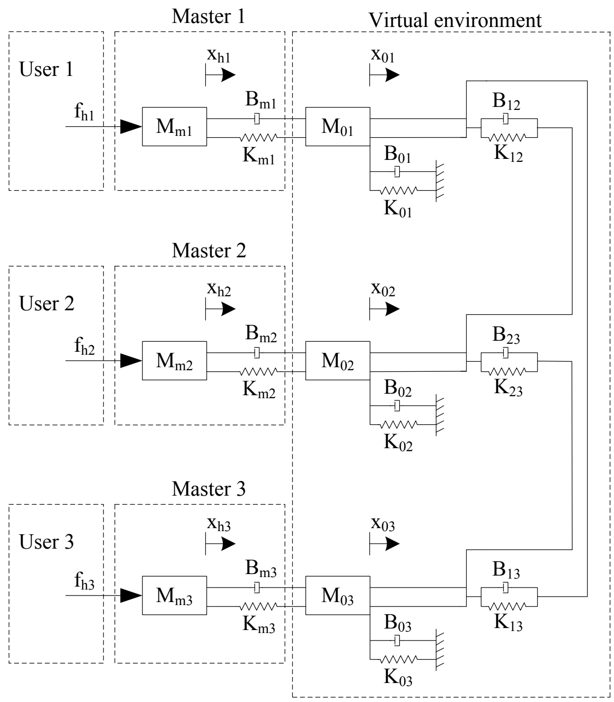

Figure 4. A triple-user collaborative haptic virtual environment system.

damping can be adjusted to correspond to real life objects. One application of such a trilateral system is in tele-rehabilitation, in which two master robots are operated by two patients and the third master robot is operated by a therapist. The therapist interacts with the patients in a virtual environment designed for rehabilitation exercises and monitoring the patients' progress through the received force feedback.

Dynamic modeling of the entire system based on a mass-spring-damper mesh model for the virtual environment follows. Consider the triple-user collaborative haptic virtual environment system shown in Figure 4. In this figure, M_{mi}, K_{mi} and B_{mi}, $i = 1, 2, 3$, are the mass, stiffness, and damping of the three masters. Also, M_{0i} represents the mass of a node of the virtual object mesh that is in contact with master i. We assume M_{0i} is connected to a stationary ground via spring K_{0i} and damper B_{0i}. We also assume K_{12}, K_{13}, K_{23} are the stiffness of springs connecting the three nodes of the mesh of the virtual object. Similarly, B_{12}, B_{13}, B_{23} are the dampers connecting the same three nodes. Lastly, f_{hi} denotes the interaction force between each user and the corresponding master.

The dynamics of the three masters are

$$M_{m1}\ddot{x}_{h1} = f_{h1} + K_{m1}(x_{01} - x_{h1}) + B_{m1}(\dot{x}_{01} - \dot{x}_{h1}) \quad \text{(60a)}$$
$$M_{m2}\ddot{x}_{h2} = f_{h2} + K_{m2}(x_{02} - x_{h2}) + B_{m2}(\dot{x}_{02} - \dot{x}_{h2}) \quad \text{(60b)}$$
$$M_{m3}\ddot{x}_{h3} = f_{h3} + K_{m3}(x_{03} - x_{h3}) + B_{m3}(\dot{x}_{03} - \dot{x}_{h3}) \quad \text{(60c)}$$

Also, the dynamics of the three nodes on the mesh of the virtual object are

$$\begin{aligned}
M_{01}\ddot{x}_{01} &= K_{m1}(x_{h1} - x_{01}) + B_{m1}(\dot{x}_{h1} - \dot{x}_{01}) \\
&+ K_{12}(x_{02} - x_{01}) + B_{12}(\dot{x}_{02} - \dot{x}_{01}) \\
&+ K_{13}(x_{03} - x_{01}) + B_{13}(\dot{x}_{03} - \dot{x}_{01}) \\
&+ K_{01}(0 - x_{01}) + B_{01}(0 - \dot{x}_{01}) \qquad \text{(61a)} \\
M_{02}\ddot{x}_{02} &= K_{m2}(x_{h2} - x_{02}) + B_{m2}(\dot{x}_{h2} - \dot{x}_{02}) \\
&+ K_{12}(x_{01} - x_{02}) + B_{12}(\dot{x}_{01} - \dot{x}_{02}) \\
&+ K_{23}(x_{03} - x_{02}) + B_{23}(\dot{x}_{03} - \dot{x}_{02}) \\
&+ K_{02}(0 - x_{02}) + B_{02}(0 - \dot{x}_{02}) \qquad \text{(61b)} \\
M_{03}\ddot{x}_{03} &= K_{m3}(x_{h3} - x_{03}) + B_{m3}(\dot{x}_{h3} - \dot{x}_{03}) \\
&+ K_{13}(x_{01} - x_{03}) + B_{13}(\dot{x}_{01} - \dot{x}_{03}) \\
&+ K_{23}(x_{02} - x_{03}) + B_{23}(\dot{x}_{02} - \dot{x}_{03}) \\
&+ K_{03}(0 - x_{03}) + B_{03}(0 - \dot{x}_{03}) \qquad \text{(61c)}
\end{aligned}$$

For simplicity, let us choose $B_{m1} = B_{m2} = B_{m3} = B_{01} = B_{02} = B_{03} = B_{12} = B_{13} = B_{23} = 0$. Thus, the impedance matrix representation of the closed-loop triple-user haptic virtual environment system is

$$\begin{bmatrix} f_{h1} \\ f_{h2} \\ f_{h3} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_{h1} \\ \dot{x}_{h2} \\ \dot{x}_{h3} \end{bmatrix} \qquad (62)$$

where $Z = A^{-1}B$ and

$$A = \begin{bmatrix} a_1 s & -\frac{K_{12}}{K_{m2}} & -\frac{K_{13}}{K_{m3}} \\ -\frac{K_{12}}{K_{m1}} & a_2 s & -\frac{K_{23}}{K_{m3}} \\ -\frac{K_{13}}{K_{m1}} & -\frac{K_{23}}{K_{m2}} & a_3 s \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{a_1 q_1 s - K_{m1}}{s} & -\frac{q_2 K_{12}}{s} & -\frac{q_3 K_{13}}{s} \\ -\frac{q_1 K_{12}}{s} & \frac{a_2 q_2 s - K_{m2}}{s} & -\frac{q_3 K_{23}}{s} \\ -\frac{q_1 K_{13}}{s} & -\frac{q_2 K_{23}}{s} & \frac{a_3 q_3 s - K_{m3}}{s} \end{bmatrix}$$

In the above,

$$\begin{aligned}
a_1 &= M_{01}s + \frac{K_{m1} + K_{12} + K_{13} + K_{01}}{s}, \\
a_2 &= M_{02}s + \frac{K_{m2} + K_{12} + k_{23} + K_{02}}{s}, \\
a_3 &= M_{03}s + \frac{K_{m3} + K_{13} + K_{23} + K_{03}}{s}, \\
q_1 &= \frac{M_{m1}s^2 + K_{m1}}{K_{m1}}, \quad q_2 = \frac{M_{m2}s^2 + K_{m2}}{K_{m2}}, \\
q_3 &= \frac{M_{m3}s^2 + K_{m3}}{K_{m3}}.
\end{aligned}$$

Next, we will consider this triple-user collaborative haptic virtual environment system and analyze its stability based on both Theorem 1 and Theorem 2.

1) Stability analysis via Theorem 1: One can see that all the elements of impedance matrix (62) have only a simple pole on the imaginary axis, thus, Conditions 1) and 2) of Theorem 1 are fulfilled. For the impedance matrix (62), the passivity conditions (8a)-(8c) will always equal zero. Condition (8d)-(8f) turns out to be

$$-\frac{1}{\omega^2 K_{m1}^4 K_{m2}^4 K_{m3}^2} Q_1^2 Q_2^2 \geq 0 \qquad (63)$$

where, the

$$\begin{aligned}Q_1 &= K_{12}K_{m3}M_{03}\omega^2 - K_{12}K_{m3}K_{23} - K_{12}K_{m3}K_{13}\\&\quad - K_{12}K_{m3}K_{03} - K_{23}K_{13} - K_{12}K_{m3}^2\\Q_2 &= (1-K_{m1})K_{m2}^2M_{01}M_{m1}(\omega^2+1)^2\\&\quad + (K_{m2}-1)K_{m1}^2M_{02}M_{m2}(\omega^2+1)^2\\&\quad + \omega^2(K_{m1}-1)K_{m2}^2(K_{m1}M_{m1}+K_{m1}M_{01}+K_{01}M_{m1}\\&\quad + K_{12}M_{m1}+K_{13}M_{m1}+2M_{m1}M_{01})\\&\quad + \omega^2(1-K_{m2})K_{m1}^2(K_{m2}M_{m2}+K_{m2}M_{02}+K_{02}M_{m2}\\&\quad + K_{23}M_{m2}+K_{12}M_{m2}+2M_{m2}M_{02})\\&\quad + K_{m2}^2(1-K_{m1})(K_{01}K_{m1}+K_{13}K_{m1}-M_{m1}M_{01})\\&\quad + K_{m1}^2(K_{m2}-1)(K_{02}K_{m2}+K_{23}K_{m2}-M_{m2}M_{02})\\&\quad + (K_{m1}-K_{m2})(K_{m1}^2K_{m2}^2-K_{12})\end{aligned}$$

Obviously, (63) will be fulfilled for all frequencies ω if and only if $Q_2 = 0$, which happens when $K_{m1} = K_{m2} = 1$. This will also make the left side of (8g) equal to zero, and stability is ensured.

Given the symmetry between the three ports in a trilateral system, a necessary and sufficient, frequency-independent, and compact condition for passivity of the above-described triple-user collaborative haptic virtual environment system is

$$K_{m1} = K_{m2} = K_{m3} = 1. \tag{64}$$

2) Stability analysis via Theorem 2: It can be shown that the symmetrization condition (16) will always holds once we find the elements of the impedance matrix (62). It is easy to see that the left side of stability conditions (17a)-(17c) will always equal zero. Condition (17d)-(17f) turns out to be

$$\frac{1}{2}Q_1^2Q_3Q_4(-1+\operatorname{sign}(Q_3)\operatorname{sign}(Q_4))\operatorname{sign}(Q_3)\operatorname{sign}(Q_4) \geq 0 \tag{65}$$

where

$$\begin{aligned}Q_3 &= \omega^4M_{m1}M_{01}(K_{m1}-1) - \omega^2(K_{m1}-1)(M_{m1}K_{m1}\\&\quad + K_{m1}K_{01}+K_{12}M_{m1}+K_{13}M_{m1})\\&\quad + K_{m1}(K_{m1}-1)(K_{12}+K_{13}+K_{01}) + K_{m1}^3\\Q_4 &= \omega^4M_{m2}M_{02}(K_{m2}-1) - \omega^2(K_{m2}-1)(M_{m2}K_{m2}\\&\quad + K_{m2}K_{02}+K_{12}M_{m2}+K_{23}M_{m2})\\&\quad + K_{m2}(K_{m2}-1)(K_{12}+K_{23}+K_{02}) + K_{m2}^3\end{aligned}$$

Obviously, (65) will be fulfilled for all frequencies ω if $K_{m1} = K_{m2} = 1$. Also, if Q_3 and Q_4 have the same sign, (65) will be fulfilled.

Given that Q_3 and Q_4 are quadratic polynomials in ω, there is a total of 4 possibilities as shown in Figure 5 for the signs of Q_3 and Q_4. Here, we only consider the case (b) in Figure 5 and the other 3 cases can be analyzed on a similar basis. In this case, a sufficient condition for stability is

$$\begin{aligned}&K_{m1} > 1, \quad K_{m2} > 1, \quad M_{m1} > M_{o1}, \quad M_{m2} > M_{02}\\&4K_{m1}^3 > (K_{m1}-1)(K_{12}+K_{13}+K_{01})^2\\&4K_{m2}^3 > (K_{m2}-1)(K_{12}+K_{23}+K_{02})^2\end{aligned} \tag{66}$$

These conditions will make the left hand side of (65) equal to zero. Since the left side of (17a)-(17d) have become identical to zero, the left side of condition (17g) will also equal zero and stability is ensured.

Given the symmetry between the three ports in a trilateral system, a sufficient, frequency-independent, and compact condition for stability of the above-described triple-user collaborative haptic virtual environment system is either

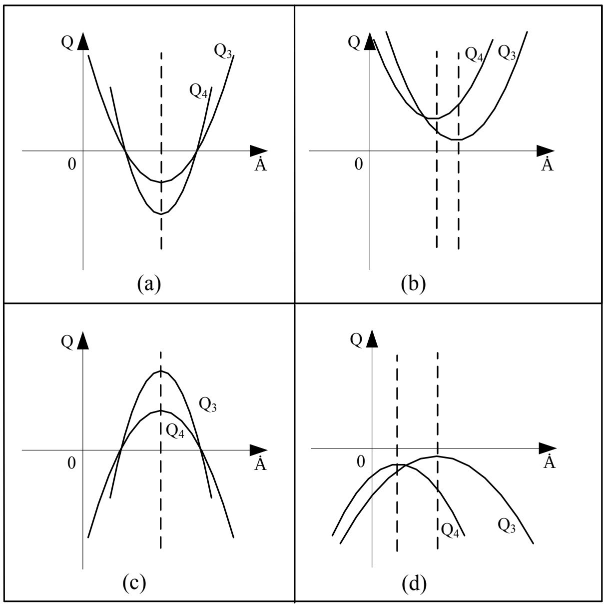

Figure 5. The four cases for Q_3 and Q_4 have same sign.

$$K_{m1} = K_{m2} = K_{m3} = 1 \tag{67}$$

or

$$\begin{cases}K_{m1} > 1, \quad M_{m1} > M_{01},\\K_{m2} > 1, \quad M_{m2} > M_{02},\\K_{m3} > 1, \quad M_{m3} > M_{03},\\4K_{m1}^3 > (K_{m1}-1)(K_{12}+K_{13}+K_{01})^2,\\4K_{m2}^3 > (K_{m2}-1)(K_{12}+K_{23}+K_{02})^2,\\4K_{m3}^3 > (K_{m3}-1)(K_{13}+K_{23}+K_{03})^2.\end{cases} \tag{68}$$

Evidently, (67)-(68) is less restrictive than (64).

VI. Simulations and Experiments

In this section, the passivity and absolute stability conditions for the dual-user teleoperation system found in the previous sections will be verified via simulations and experiments. For brevity, we do not report the experimental results of a similar exercise for the triple-user collaborative haptic virtual environment. For checking the passivity of trilateral haptic teleoperator, a passivity observer that calculates the dissipated energy in the system has been incorporated into the simulations and experiments. The dissipated energy is given by the input-output energy integral

$$\begin{aligned}E_p(t) &= \int_0^t f_{h1}(\tau)V_{h1}(\tau)\,\mathrm{d}\tau + \int_0^t f_{h2}(\tau)V_{h2}(\tau)\,\mathrm{d}\tau\\&\quad + \int_0^t f_e(\tau)V_e(\tau)\,\mathrm{d}\tau \geq 0\end{aligned} \tag{69}$$

The teleoperator is passive if $E_p(t)$ is non-negative at all time [6].

For checking the absolute stability of the trilateral haptic teleoperator, the ports #2 and #3 were connected to passive terminations while the input energy at the port #1 was measured. The three-port network teleoperator is absolute stable

Table I
THE CONTROLLERS GAINS OF THE POSITION-POSITION DUAL-USER TELEOPERATION SYSTEM USED IN SIMULATIONS. (A) PASSIVE AND ABSOLUTELY STABLE, (B) ABSOLUTELY STABLE BUT NON-PASSIVE, (C) POTENTIALLY UNSTABLE (I.E., NOT ABSOLUTELY STABLE) AND NON-PASSIVE.

	Master #1		Master #2		Slave	
(A)	K_{pm1}	30	K_{pm2}	30	K_{ps}	30
	K_{vm1}	5	K_{vm2}	5	K_{vs}	5
(B)	K_{pm1}	3	K_{pm2}	30	K_{ps}	150
	K_{vm1}	5	K_{vm2}	50	K_{vs}	150
(C)	K_{pm1}	3	K_{pm2}	80	K_{ps}	15
	K_{vm1}	5	K_{vm2}	20	K_{vs}	60

if and only if, at all times $t > 0$, we have [28]:

$$E_s(t) = \int_0^t f_{h1}(\tau) V_{h1}(\tau)\, \mathrm{d}\tau \geq 0. \tag{70}$$

A. Simulations

The position-position dual-user teleoperation system has been simulated in MATLAB/Simulink. There is no time delay in the communication channel between the masters and the slave. Three 1-DOF robots as the two masters and the slave are modeled by masses $M_{m1} = 0.7$, $M_{m2} = 0.9$, and $M_s = 0.5$, respectively. In simulations for both passivity and absolute stability, the master #2 and the slave are connected to LTI terminations with transfer functions $\frac{1}{s+1}$, which are passive as, for $s = j\omega$, we have $\mathrm{Re}(\frac{1}{s+1}) = \frac{1}{\omega^2+1} > 0$. In passivity simulations, the master #1 is also connected to another passive termination with transfer function $\frac{1}{s+1}$, and a sine-wave exogenous input F^*_{h1} is applied. In absolute stability simulations, port 1 is open and a sine-wave input F_{h1} is applied to the master #1.

The triple-user collaborative haptic virtual environment system has been simulated in MATLAB/Simulink. There is no time delay in the communication channel between the masters and the virtual objects. Three 1-DOF robots as the three masters are modeled by masses $M_{m1} = M_{m2} = M_s = 1.6$, respectively. In simulations for both passivity and absolute stability, the master #2 and #3 are connected to LTI terminations with transfer functions $\frac{1}{s+1}$. In passivity simulations, The master #1 connected to another passive termination with transfer functions $\frac{1}{s+1}$, and a sine-wave input f^*_{h1} is applied. In absolute stability simulations, port 1 is open and a sine-wave input f_{h1} is applied to the master #1.

1) Dual-user teleoperation system: According to (51) and (59), the stability of the position-position dual-user teleoperation system should depend on the controllers gains. In the simulations, the controllers gains K_{pm1}, K_{vm1}, K_{pm2}, K_{vm2}, K_{ps}, and K_{vs} were chosen according to Table I. Gains in Table I(A) satisfy (51) and (59), in Table I(B) satisfy (59) but not (51), and in Table I(C) satisfy neither (51) nor (59). Also, $\alpha = \frac{1}{2}$.

The input energy (70) profiles (E_s) are plotted in Figure 6(a). As it can be seen, if the controllers gains are selected according to (59), e.g., as listed in Table I(A) and (B), then the input energy at port 1 is non-negative at all times, indicating the absolute stability of the trilateral haptic system. However, when the controllers gains violate (59), e.g., as listed in Table I(C), the input energy E_s will become negative at least for a period of time, indicating potential instability of the trilateral haptic system.

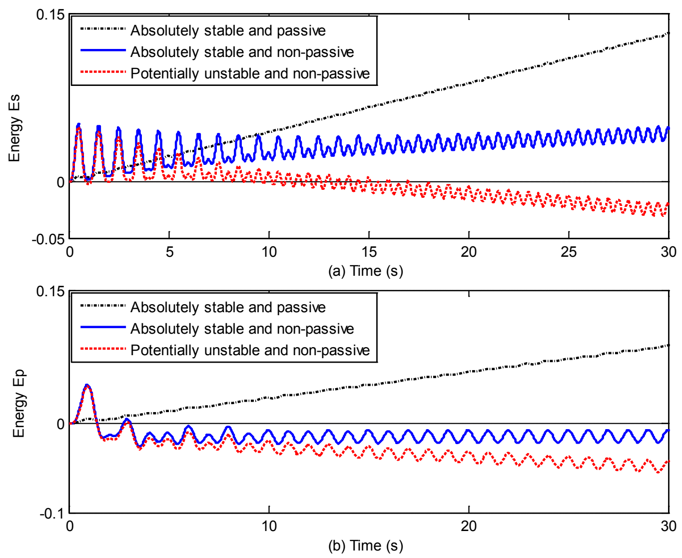

Figure 6. Simulation results for the dual-user teleoperation system. (a) Input energy E_s at the master #1's port for absolute stability analysis, and (b) passivity observer energy E_p used for passivity analysis. Simulation parameters are listed in Table I: parameters (A) for the absolutely stable and passive, parameters (B) for the absolutely stable and non-passive, and parameters (C) for the potentially unstable and non-passive.

Table II
THE CONTROLLERS GAINS OF THE TRIPLE-USER COLLABORATIVE HAPTIC VIRTUAL ENVIRONMENT SYSTEM USED IN SIMULATIONS. (A) PASSIVE AND ABSOLUTELY STABLE, (B) ABSOLUTELY STABLE BUT NON-PASSIVE, (C) POTENTIALLY UNSTABLE AND NON-PASSIVE.

(A)	M_{01}	0.4	K_{m1}	1	K_{01}	15	K_{12}	6
	M_{02}	0.4	K_{m2}	1	K_{02}	15	K_{23}	6
	M_{03}	0.4	K_{m3}	1	K_{03}	15	K_{13}	6
(B)	M_{01}	0.4	K_{m1}	260	K_{01}	15	K_{12}	6
	M_{02}	0.4	K_{m2}	260	K_{02}	15	K_{23}	6
	M_{03}	0.4	K_{m3}	260	K_{03}	15	K_{13}	6
(C)	M_{01}	0.4	K_{m1}	2	K_{01}	6	K_{12}	3
	M_{02}	0.4	K_{m2}	1	K_{02}	6	K_{23}	3
	M_{03}	0.4	K_{m3}	3	K_{03}	6	K_{13}	3

The dissipated energy (69) profiles (E_p) are plotted in Figure 6(b). As it can be seen, if the controllers gains are selected according to (51), e.g., as listed in Table I(A), then the passivity observer output E_p is non-negative at all times. However, when the controllers gains violate (51), e.g., as listed in Table I(B) and (C), then the passivity observer output E_p is not always positive, indicating the loss of passivity of the haptic teleoperator. Evidently, passivity is more restrictive than absolute stability.

2) Triple-user collaborative haptic virtual environment system: According to (64) and (68), the stability of the triple-user collaborative haptic virtual environment system should depend on the controllers gains and robots parameters. In the simulations, the parameters M_{mi}, M_{0i}, K_{mi}, K_{01}, K_{12}, K_{13}, and K_{23}, where $i = 1, 2, 3$, were chosen according to Table II. Gains in Table II(A) satisfy (64) and (68), in Table II(B) satisfy (68) but not (64), and in Table II(C) satisfy neither (64) nor (68). Also, $\alpha = \frac{1}{2}$.

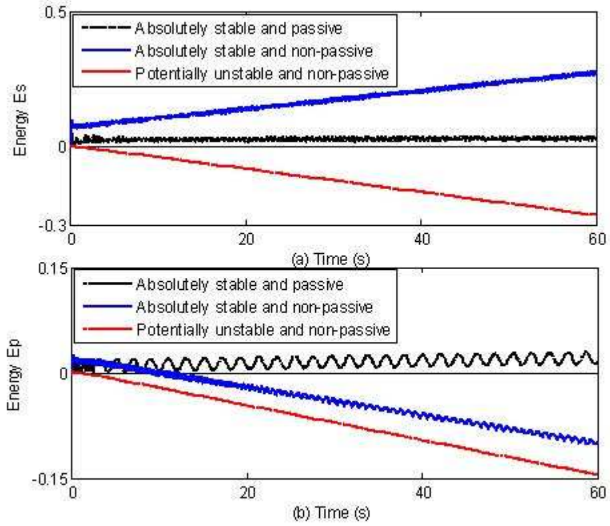

Figure 7. Simulation results for triple-user haptic virtual environment system. (a) Input energy E_s at the master #1's port for absolute stability analysis, and (b) passivity observer energy E_p used for passivity analysis. Simulation parameters are listed in Table II: parameters (A) for the absolutely stable and passive, parameters (B) for the absolutely stable and non-passive, and parameters (C) for the potentially unstable and non-passive.

The input energy (70) profiles (E_s) are plotted in Figure 7(a). As it can be seen, if the parameters are selected according to (67)-(68), e.g., as listed in Table II(A) and (B), then the input energy at port 1 E_s is non-negative at all times, indicating the absolute stability of the trilateral haptic system. However, when the parameters violate (67)-(68), e.g., as listed in Table II(C), the input energy E_s will become negative at least for a period of time, indicating potential instability of the trilateral haptic system.

The dissipated energy (69) profiles E_p are plotted in Figure 7(b). As it can be seen, if the parameters are selected according to (64), e.g., as listed in Table II(A), then the passivity observer output E_p is non-negative at all times. However, when the parameters violate (64), e.g., as listed in Table II(B) and (C), then the passivity energy observer output E_p is not always positive, indicating non-passivity of the trilateral haptic system. Evidently, passivity is more restrictive than absolute stability, as far as the stability of the overall teleoperation system is concerned.

B. Experiments

For experiments with a dual-user haptic teleoperation system, we use an Phantom Omni robot (Sensable Technologies/Geomagic, Wilmington, MA, USA) as the master #2, and two Phantom Premium 1.5A robots as the master #1 and as the slave. Out of the three actuated joints of each robot, the first joint, which rotates about the vertical, is considered in the experiments while the second and the third joints, which form a parallel mechanism, are locked using high-gain controllers. The Phantom Premium robot for master #1 is equipped with a JR3 6-DOF force/torque sensor (JR3, Inc., Woodland, CA, USA) for measuring the external contact forces.

The experimental setup is shown in Figure 8; this figure shows the exact arrangement for the passivity experiments. For the absolute stability experiments, the only difference is that the master #1 is controlled by a human user rather than being physically clamped. In both passivity and absolute stability experiments, the master #2 is connected via a pair of passive springs to a stiff wall and the slave is in free motion. Each of the passivity and the absolute stability experiments are done under three different set of position-position teleoperation control gains according to Table III. Gains in Table III(A) satisfy (51) and (59), in Table III(B) satisfy (59) but not (51), and in Table III(C) satisfy neither (51) nor (59).

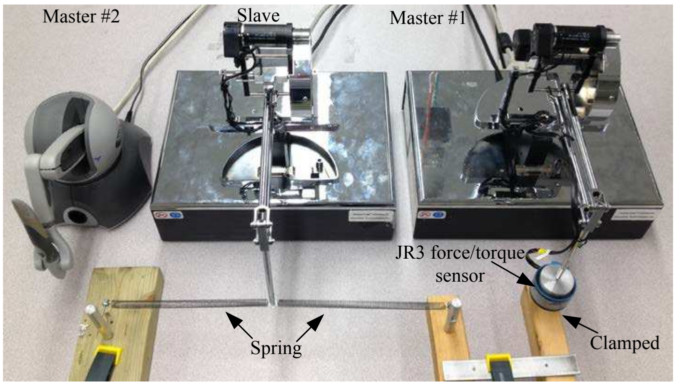

Figure 8. Experimental setup where the master #2 is connected via passive spring to stiff wall, the slave is in free motion. In absolute stability experiment, the master #1 is controlled by a human user. In passivity experiment, the master #1 is physically clamped.

Table III
THE CONTROLLERS GAINS OF THE POSITION-POSITION DUAL-USER TELEOPERATION SYSTEM USED IN EXPERIMENTS. (A) PASSIVE AND ABSOLUTELY STABLE, (B) ABSOLUTELY STABLE BUT NON-PASSIVE, (C) POTENTIALLY UNSTABLE AND NON-PASSIVE.

	Master #1		Master #2		Slave	
(A)	K_{pm1}	580	K_{pm2}	580	K_{ps}	580
	K_{vm1}	360	K_{vm2}	360	K_{vs}	360
(B)	K_{pm1}	580	K_{pm2}	580	K_{ps}	1740
	K_{vm1}	360	K_{vm2}	360	K_{vs}	1080
(C)	K_{pm1}	30	K_{pm2}	800	K_{ps}	150
	K_{vm1}	50	K_{vm2}	200	K_{vs}	600

As far as the absolute stability experiments, the input energy (70) profiles E_s are plotted in Figure 9(a). As it can be seen, if the controllers gains are selected according to (59), e.g., as listed in Table III(A) and (B), then the input energy at port 1 E_s is non-negative at all times, indicating the absolute stability of the trilateral haptic teleoperator. However, when the controllers gains violate (59), e.g., as listed in Table III(C), the input energy E_s will become negative at least for a period of time, indicating potential instability of the trilateral haptic teleoperator.

As far as the passivity experiments, the dissipated energy (69) profiles E_p are plotted in Figure 9(b). As it can be seen, if the controllers gains are selected according to (51), e.g., as listed in Table III(A), then the passivity observer output E_p is non-negative at all times. However, when the controllers gains violate (51), e.g., as listed in Table III(B) and (C), then the passivity observer output E_p is not always positive, indicating the loss of passivity of the haptic teleoperator. This again reaffirms that the teleoperator passivity requirement is too restrictive and conservative (for the teleoperation system stability) compared to teleoperator absolute stability.

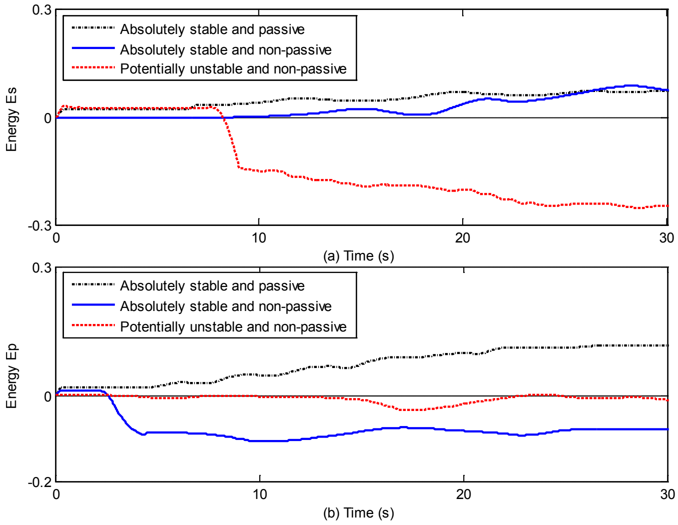

Figure 9. Experiment results for the dual-user teleoperation system. (a) Input energy E_s at the master #1's port for absolute stability analysis, and (b) passivity observer energy E_p used for passivity analysis. Both plots pertain to a position-position dual-user teleoperator. Experimental parameters are listed in Table III: parameters (A) for the absolutely stable and passive, parameters (B) for the absolutely stable and non-passive, and parameters (C) for the potentially unstable and non-passive.

VII. Conclusions and Future Works

In this paper, we showed how the absolute stability criterion is less conservative than the passivity criterion for both bilateral and trilateral haptic teleoperators and that the two criteria become the same when a bilateral or trilateral haptic system is modeled by a immittance matrix have $|p_{12}| = |p_{21}|$, $|p_{13}| = |p_{31}|$, and $|p_{23}| = |p_{32}|$. Both analytically and through simulations/experiments involving dual-user haptic teleoperation of one slave robot and triple-user collaborative haptic teleoperation in a virtual environment, the two criteria were compared. It was concluded that the absolute stability criterion is less conservative compared to the passivity criterion for position tracking trilateral haptic teleoperators. In the future, the absolute stability criterion can be used to investigate the stability of trilateral haptic systems that experience time delays in their communication channels.

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