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An Adaptive Order/State Estimator for Linear Systems with Non-integer Time-Varying Order

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Abstract

This paper proposes the design of a simultaneous order estimator and state observer for non-integer time-varying order linear systems. Several lemmas and theorems pertaining to the stability of variable order systems are provided first. Next, a theorem proposes an order/state estimator for linear variable order systems. Then, a simulation study is presented to verify the theoretical results.

Key words: Adaptive Estimation, Order Estimator, State Observer, Non-integer Order Calculus, Time-Varying Order Systems.

1 Introduction

Differential equations involving derivatives of quantities are the most common tools for describing dynamic events. A generalized version of calculus involves non-integer order for differentiation or integration, namely the fractional order or non-integer order calculus [3]. Fractional order calculus, mostly developed in the 19^{th} century, was considered a theoretical topic until recently when the increased flexibility, generality, and degrees of freedom in non-integer order differential equations proved them powerful in better describing certain real-world events as compared to integer-order differential equations. Interesting applications of such dynamics with real or even complex order have been reported in modeling, physics, and engineering [17,14]. The non-integer order dynamics have been used to model the memory in electronic devices [11], viscoelastic damping [35], the human's ability to forget and remember [30], and properties of tissue [28].

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1.1 Prior Art in Variable Order Dynamics

The order of a non-integer order system is not limited to be a constant. It can vary depending on time, the states of the system, or even has its own dynamics. The concept of variable-order calculus was first developed in [20]. Afterwards, the topic was studied from different aspects. In [8], the variable and distributed order operators and their properties are studied. Variable order derivative and its numerical approximation are investigated in [33]. The variable order derivation operator is redefined in [23] for better accuracy in cases where the variable order is not a continuous function of time. This operator is used in [24] to form a switching order derivative operator. The problem of existence and uniqueness of the response of variable order differential equations has been studied in [16] and [37]. Extensions for the concepts previously introduced for traditional dynamics, such as variational calculus [12,31] and the Noether's theorem [13] have been also proposed for variable order dynamics. From the application sight of view, some evidences have been reported to relate the order of derivation to some physical quantities [21,25]. The main interpretation of the order is the memory. Such memory can belong to a physical system. In [26], a study has been done to compare the effect of constant order versus variable order on the ability of a system to be influenced from the past, which is in fact, relating the memory of a system to its order. Order is used to measure the memory of a system in [5]. The word memory can also be related to the human's mental ability to remember. It is shown in [32] that the order of an emotional dynamical system and the human's memory are strongly related. In [29], it is proven that the variable order of derivation is consistent with variable-length memory. Variable-order dynamical systems are used in [29] to explain emotional behaviors of human caused by the effect of memory on context. There are some other applications and interpretations for the non-integer order. In [36], the non-integer order derivation operator is used to improve the image quality by means of using spatial non-integer order derivative instead of integer order one for edge preserving and varying the order with respect to the location of each pixel. Describing state dependent viscoelasticity by the variable order [28], describing two-dimensional cable equations [2], describing diffusion and sub-diffusion in different cases [4,27], and explaining the behavior of ladders and nested ladders [25] are some other application of variable order dynamic systems.

1.2 Summary of Contributions

Since the order is a key characteristic of a non-integer order system, for having a precise model describing specific dynamics, it is vital to estimate it with an acceptable level of precision. In most previous works involving variable order dynamics, the order is assumed to be known [4,2]. Some researchers have tried to propose an estimation scheme for the order of a non-integer order system, assuming a **constant order**. In [7], the order is treated just like other system constant parameters in the estimator which is not a real time approach. A numerical method is used in [34] to find the constant order of a non-integer system. In [22], a discrete technique is used to estimate the constant order of a system using Kalman filter. In [15], an adaptive estimation process is introduced to estimate the constant order of non-integer commensurate order systems. The approach proposed in the current paper differs from the above in the following aspects:
1. While the order is supposed to be constant in [15], the order is allowed to be varying with time in this paper. This is a significant useful generalization. In fact, even when the order is a function of the states, it can be considered as an implicit function of time. Hence, the method proposed in this paper can be utilized in all cases involving varying order dynamics.
2. A definite convergence proof is established in this paper, guaranteeing that the estimation error for both order and states can be made arbitrarily small.
3. While the prior art only deal with the order and other system parameters, here, a state observer is designed for the cases where the states of the systems are not available and only the output can be measured.
4. Theorem 1-3, which act as intermediate results to prove the main result in Theorem 4, are useful for stability analysis of variable order systems and introduced for the first time in this paper.
5. Based on item 4 above, the order/state estimator proposed in Theorem 4 for variable-order systems includes a Luenberger-type state observer for regular integer-order systems. This means that the wealth of existing methods on observer gain selection, noise attenuation, etc. can be easily extended to the state observer part of the order/state estimator of a variable-order system.
6. The method presented in [15] is designed for the single input single output systems, described in frequency domain. However, the novel method proposed here can be used in multi-input multi-output cases, as well.

The main achievement of this paper is a powerful estimator for the variable order of any linear system in a compact temporal interval $[0 \quad T]$, even when its states are not available. The estimator can be used to determine if a system is of constant or variable order or even, whether it is of integer or non-integer order. The estimation process is done in a real-time manner, so it can be used in adaptive control or model predictive control based approaches. Moreover, for the design of the proposed estimator, some new lemmas and theorems are proposed that are very useful in determining some of the most important properties of variable order systems including their stability. It is noteworthy to mention that the algorithm works for any given $T < \infty$. Hence, depending on the studied problem, when the final time is free, T can be set large enough to provide the required time for the convergence.

The rest of the paper is organized as follows: In Section II, several definitions related to the non-integer order field and the problem statement will be presented. Section III is dedicated to proposing the adaptation rules for estimating the order and the states. After presenting some new and useful lemmas and theorems, the state observer will be designed and combined with the order estimator to build a comprehensive order/state estimator. In this Section, a case study is considered through a simulation study to show the effectiveness of the suggested methods. Finally, Section IV concludes the paper.

2 Preliminaries

Definition 1 *The definition of modified initialized left non-integer variable order derivation with respect to time in the sense of Caputo is*

$$ {}_0^C D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-\tau)^{-\alpha(t)} \frac{d}{d\tau} x(\tau) d\tau + \Psi_c^x(t) \\ 0 < \alpha(t) < 1, \forall t \geq 0 \tag{1} $$

where $\Gamma(.)$ *is the extension of the factorial function to the non-integer arguments:*

$$ \Gamma(z) = \int_0^\infty r^{z-1} e^{-r} dr \tag{2} $$

$\Psi_c^x(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{-c}^{0}(t-\tau)^{-\alpha(t)}\frac{d}{d\tau}x(\tau)d\tau$ is a decaying function capturing the effect of the values of the signal x before $t=0$, supposing that x begins from $-c<0$ [9,18]. The use of this time varying initializing function is necessary to avoid discontinuity.

Definition 2 *The inverse of the variable derivation operator is the variable order integration operator of the same order, could be interpreted as*

$$
\begin{gathered}
{}_0I_t^{\alpha(t)}x(t) = \tfrac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}x(\tau)d\tau \\
0<\alpha(t)<1, \forall t \geq 0
\end{gathered} \tag{3}
$$

In fact;

$$ {}_0^CD_t^{\alpha(t)}I_t^{\alpha(t)}x(t) = x(t) \tag{4} $$

Definition 3 *Generally speaking, a non-integer order dynamic system is a set of coupled non-integer order differential equations in which the derivation operators could be of different orders [1].*

$$
\begin{gathered}
\begin{cases}
{}_0^CD_t^{\alpha_1(t)}x_1(t) = f_1\big(x_1(t),\dots,x_n(t),u(t)\big) \\
\vdots \\
{}_0^CD_t^{\alpha_n(t)}x_n(t) = f_n\big(x_1(t),\dots,x_n(t),u(t)\big)
\end{cases} \\
y(t) = g\big(x_1(t),\dots,x_n(t)\big)
\end{gathered} \tag{5}
$$

Here, $0<\alpha_i<1, i=1,2,\dots,n$ are the orders and $u=[u_1(t)\ u_2(t)\ \dots\ u_p(t)]^T \in R^p$ is the exogenous input vector, and $y=[y_1(t)\ y_2(t)\ \dots\ y_q(t)]^T \in R^q$ is the output vector. The set of differential equations described in (5) is equivalent to the following set of Volterra integral equations [6,10,19]:

$$
\begin{aligned}
x_i(t) &= x_i(0) \\
&+ \frac{1}{\Gamma(\alpha_i(t))}\int_0^t (t-\tau)^{\alpha_i(t)-1}f_i(x(\tau),u(\tau))d\tau \\
&+ \frac{1}{\Gamma(\alpha_i(t))}\int_0^t (t-\tau)^{\alpha_i(t)-1}\Psi_c^{x_i}(\tau)d\tau,\ i=1,2,\dots,n
\end{aligned} \tag{6}
$$

System (5) can be written as

$$ {}_0^CD^{\bar{\alpha}(t)}x(t) = F\big(x(t),u(t)\big),\ y(t) = g\big(x(t)\big) \tag{7} $$

where $x=[x_1\ x_2\ \dots\ x_n]^T \in R^n$ is the state vector, $F(x,u)=[f_1(x,u)\ f_2(x,u)\ \dots\ f_n(x,u)]^T$, and $\bar{\alpha}(t) = [\alpha_1(t)\ \alpha_2(t)\ \dots\ \alpha_n(t)]^T$ is the order vector. When all the orders are the same function of time (i.e., $\alpha_1(t)=\alpha_2(t)=\dots=\alpha_n(t)=\alpha(t)$), the system is said to be commensurate order. Additionally, when F is linear with respect to x and u, and g is linear with respect to x, we have a Commensurate Linear variable order system described as

$$ {}_0^CD^{\alpha(t)}x(t) = Ax(t)+Bu(t),\ y(t) = Cx(t) \tag{8} $$

In this paper, we will focus on systems of the form (8) and suppose that the order in (8) is unknown. Also, we will assume that the states of the system (8) are not available. The system is supposed to be at rest in $t<0$. By the phrase "at rest" we mean that when $t<0$; 1. all the states and their derivatives are equal to zero. 2. Since we are dealing with a variable order system, the order is considered to be equal to zero. Considering these hypotheses $\Psi_c^x(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{-c}^{0}(t-\tau)^{-\alpha(t)}\frac{d}{d\tau}x(\tau)d\tau = \frac{1}{\Gamma(1)}\int_{-c}^{0}\frac{d}{d\tau}x(\tau)d\tau|_{\frac{dx(\tau)}{d\tau}=0} = 0$. We will design an order/state estimator for such a system.

3 Adaptive Order/State Estimation

3.1 Main Results

Although the order can be considered a parameter of the system (8), it can be deduced from (6) that the relationship between the response of the system and the order is complicated. Hence, the order estimation is not as easy as the estimation of the other parameters of the system. Fortunately, we can compute the partial derivative of the system response with respect to the order. Consider the following integral equation as a definition for $z_f(t,\beta)$:

$$ z_f(t,\beta) = z_0 + \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}f(\tau)d\tau \tag{9} $$

Then,

$$
\begin{aligned}
\frac{\partial z_f(t,\beta)}{\partial\beta} &= -\frac{\psi(\beta(t))}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}f(\tau)d\tau \\
&+ \frac{1}{\Gamma(\beta(t))}\int_0^t ln(t-\tau)(t-\tau)^{\beta(t)-1}f(\tau)d\tau
\end{aligned} \tag{10}
$$

where

$$ \psi(\beta) = \frac{d}{d\beta}ln(\Gamma(\beta)) = \frac{\Gamma'(\beta(t)}{\Gamma(\beta(t))} \tag{11} $$

Hence, when the history of f in $[0\ \ t)$ and the value of β at t is known, the above derivation can be computed.

Lemma 1 *Suppose that x and $\hat{x}$ are defined as*

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))}f(\tau)d\tau \\
\hat{x}(t) &= \int_0^t \frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))}f(\tau)d\tau
\end{aligned} \tag{12}
$$

Then, the value of $x-\hat{x}$ could be evaluated according to the following equation:

$$x-\hat{x}=\frac{\partial z_f(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}(\alpha-\hat{\alpha})+K(t)(\alpha-\hat{\alpha})^2 \quad (13)$$

where z_f is defined according to (9).

Proof *Define $H(t,\tau,\beta)=\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}$. Since H is continuous and analytical with respect to β when $0<\beta<1$ and $0<\tau<t$, and $\frac{\partial H}{\partial \beta}$, $\frac{\partial^2 H}{\partial \beta^2}$ are both bounded in this interval, the Taylor Series Expansion Theorem can be used to rewrite the above formula as:*

$$\begin{aligned}x-\hat{x}=&\int_0^t\Big(\frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))}-\frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))}\Big)f(\tau)d\tau\\ =&\int_0^t\Big(\frac{\partial H(t,\tau,\beta)}{\partial\beta}|_{\beta=\hat{\alpha}}(\alpha-\hat{\alpha})\\ &+\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}(\alpha-\hat{\alpha})^2\Big)f(\tau)d\tau\end{aligned} \quad (14)$$

for some $\alpha_0\in[min(\alpha,\hat{\alpha})\ \ max(\alpha,\hat{\alpha})]$. Since the values of α and $\hat{\alpha}$ are not related to the integration variable, τ, it can be deduced that:

$$\begin{aligned}x-\hat{x}=&(\alpha-\hat{\alpha})\int_0^t\frac{\partial H(t,\tau,\beta)}{\partial\beta}|_{\beta=\hat{\alpha}}f(\tau)d\tau\\ &+(\alpha-\hat{\alpha})^2\int_0^t\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}f(\tau)d\tau\end{aligned} \quad (15)$$

Consequently,

$$x-\hat{x}=\frac{\partial z_f(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}(\alpha-\hat{\alpha})+K(t)(\alpha-\hat{\alpha})^2 \quad (16)$$

where $\frac{\partial z_f(t,\beta)}{\partial\beta}|_{\beta=\hat{\alpha}}$ can be computed through (10) and $K(t)=\int_0^t\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}f(\tau)d\tau$.

Lemma 2 *Suppose that t belongs to the closed bounded interval $[0\ \ T]$ and f is continuous. Then, in (16) K is bounded.*

Proof *Since f is continuous (and so is $|f|$), according to the Extreme Value Theorem, $|f|$ attains a maximum in this interval, here defined as M_f. The functions $\Gamma(\beta),\psi(\beta),\psi'(\beta)=\frac{d\psi(\beta)}{d\beta}$ are all bounded in $0<\beta(t)<1$ and $H\geq 0$. Furthermore, the integrals $\int_0^t|H|d\tau$, $\int_0^t|ln(t-\tau)H|d\tau$ and $\int_0^t|ln^2(t-\tau)H|d\tau$ all converge for $\forall\beta(t),0<\beta(t)<1$:*

$$\int_0^t|H|d\tau=\int_0^t Hd\tau=\frac{t^{\beta(t)}}{\beta(t)\Gamma\big(\beta(t)\big)}$$

$$\int_0^t|ln(t-\tau)H|d\tau=\begin{cases}\frac{t^{\beta(t)}(1-\beta(t)ln(t))}{\beta^2(t)\Gamma\big(\beta(t)\big)},t\leq 1\\ \frac{2+t^{\beta(t)}(\beta(t)ln(t)-1)}{\beta^2(t)\Gamma\big(\beta(t)\big)},t>1\end{cases}$$

$$\int_0^t|ln^2(t-\tau)H|d\tau=\frac{t^{\beta(t)}\big(1+\beta^2(t)ln^2(t)-2\beta(t)ln(t)\big)}{\beta^3(t)\Gamma(\beta(t))}$$

Now, it should be noted that based on the definition of H,

$$\frac{\partial H}{\partial\beta}=(-\psi(\beta)+ln(t-\tau))H$$

$$\begin{aligned}\frac{\partial^2 H}{\partial\beta^2}&=-\psi' H+(-\psi(\beta)+ln(t-\tau))\frac{\partial H}{\partial\beta}\\ &=-\psi' H+\psi^2(\beta)H-2\psi(\beta)ln(t-\tau)H+ln^2(t-\tau)H\end{aligned}$$

Hence, for a given T, there is a bound $M_H(T)$ such that $\int_0^t|\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}|d\tau<M_H$. Therefore,

$$\begin{aligned}|K|&=\Big|\int_0^t\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}f(\tau)d\tau\Big|\\ &\leq\int_0^t\Big|\frac{\partial^2 H(t,\tau,\beta)}{\partial\beta^2}|_{\beta=\alpha_0}\Big||f(\tau)|d\tau<M_H M_f\equiv M_K\end{aligned}$$

implying that K is bounded.

Lemma 3 *For any signal x, for which the non-integer order derivative ${}_0^C D^{\beta(t)}x$ exists, as long as that the signal is at rest in $t<0$, we have:*

$${}_0^C D^{\beta(t)}(x^T(t)x(t))\leq 2x^T(t){}_0^C D^{\beta(t)}x(t) \quad (17)$$

Proof *We have to show that*

$$2x^T(t){}_0^C D^{\beta(t)}x(t)-{}_0^C D^{\beta(t)}(x^T(t)x(t))\geq 0 \quad (18)$$

It is straightforward to compute the left hand side term using integration by parts:

$$\begin{aligned}&2x^T(t){}_0^C D^{\beta(t)}x(t)-{}_0^C D^{\beta(t)}(x^T(t)x(t))\\ &=\frac{2}{\Gamma(1-\beta(t))}\int_0^t(t-\tau)^{-\beta(t)}\big(x(t)-x(\tau)\big)\frac{dx(\tau)}{d\tau}d\tau\\ &=\frac{1}{\Gamma(1-\beta(t))}\left(\frac{\big(x(t)-x(\tau)\big)^2}{(t-\tau)^{-\beta(t)}}\right)|_{\tau=0}^{\tau=t}\\ &+\frac{\beta(t)}{\Gamma(1-\beta(t))}\int_0^t\frac{\big(x(t)-x(\tau)\big)^2}{(t-\tau)^{\beta(t)+1}}d\tau\\ &+2x^T\Psi_c^x(t)-\Psi_c^{x^Tx}(t)\end{aligned} \quad (19)$$

Since x is supposed to be at rest in $t<0$, so is x^Tx and the initializing functions both vanishes. The term $\frac{\beta(t)}{\Gamma(1-\beta(t))}\int_0^t\frac{\big(x(t)-x(\tau)\big)^2}{(t-\tau)^{\beta(t)+1}}d\tau$ is always non-negative, because $\Gamma(1-\beta(t))$ and $(t-\tau)^{\beta(t)+1}$ are non-negative when $0<\beta(t)<1$ and $0\leq\tau\leq t$. At $\tau=0$, the first term $\frac{1}{\Gamma(1-\beta(t))}\left(\frac{\big(x(t)-x(\tau)\big)^2}{(t-\tau)^{-\beta(t)}}\right)$ is equal to

$\frac{1}{\Gamma(1-\beta(t))}\left(\frac{\big(x(t)-x(0)\big)^2}{t^{-\beta(t)}}\right)$ *which is also non-negative because* $0 < \beta(t) < 1$. *However, there is a singularity in this term at* $\tau = t$, *where H'opital method is used:*

$$\lim_{\tau\to t}\left(\frac{\big(x(t)-x(\tau)\big)^2}{(t-\tau)^{-\beta(t)}}\right) = \lim_{\tau\to t}\left(\frac{\frac{d}{d\tau}\big(x(t)-x(\tau)\big)^2}{\frac{d}{d\tau}(t-\tau)^{-\beta(t)}}\right)$$
$$= \left(2\big(x(t)-x(\tau)\big)\frac{dx(\tau)}{d\tau}\beta(t)(t-\tau)^{\beta(t)+1}\right) = 0 \tag{20}$$

Hence, (19) is non-negative and proof is completed.

Corollary 1 *For the positive definite function* $V = x^T P x$ *with a Hermitian positive definite matrix* P, *the inequality turns to* ${}_0^C D^{\beta(t)}(x^T P x) \le 2x^T P {}_0^C D^{\beta(t)} x$.

Proof *The proof is straightforward using the result of the recent Lemma and the change of variable* $z = P^{\frac{1}{2}}x$, *where* $P^{\frac{1}{2}}$ *is a Hermitian positive definite matrix such that* $P^{\frac{1}{2}}P^{\frac{1}{2}} = P$.

$$x^T P x = x^T P^{\frac{1}{2}} P^{\frac{1}{2}} x = z^T z$$
$$\Rightarrow {}_0^C D^{\beta(t)}(x^T P x) = {}_0^C D^{\beta(t)}(z^T z)$$
$$\le 2z^T {}_0^C D^{\beta(t)} z = 2x^T P^{\frac{1}{2}} {}_0^C D^{\beta(t)}(P^{\frac{1}{2}}x)$$
$$= 2x^T P^{\frac{1}{2}} P^{\frac{1}{2}} {}_0^C D^{\beta(t)} x$$

Thus, ${}_0^C D^{\beta(t)}(x^T P x) \le 2x^T P {}_0^C D^{\beta(t)} x$ *and this completes the proof.*

Theorem 1 *(Lyapunov Stability of Variable Order Systems) Consider a variable order system* ${}_0^C D^{\beta(t)} x = f(x), 0 < \beta(t) < 1, \forall t$ *and the system is at rest in* $t < 0$. *Suppose that* $V \ge 0$ *is a positive definite scalar continuous function of* x *such that* $V(x) = 0 \iff x = 0$. *Then, the origin of the system is stable in the sense of Lyapunov if* ${}_0^C D^{\beta(t)}(V) \le 0$ *in a non-empty domain* $\Delta \subset R^n$ *containing the origin. Additionally, if* ${}_0^C D^{\beta(t)}(V) < 0$, *the origin is asymptotically stable.*

Proof *The condition* $V(x) = 0 \iff x = 0$ *implies that because* x *is at rest in* $t < 0$, V *is also at rest and* $\Psi_c^V(t) = 0$. *Since* ${}_0^C D^{\beta(t)}(V) \le 0$, *we can write* ${}_0^C D^{\beta(t)}(V) = -w(t), w(t) \ge 0, \forall t$. *Hence, using Volterra integral equation, the values of* V *at any time* t *can be calculated:*

$$V(x(t)) = V(x(0)) - \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} w(\tau)d\tau$$
$$+ \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}\Psi_c^V(\tau)d\tau$$
$$= V(x(0)) - \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} w(\tau)d\tau \tag{21}$$

The terms $\frac{1}{\Gamma(\beta(t))}, (t-\tau)^{\beta(t)-1}$ *and* $w(\tau)$ *are all non-negative for* $0 \le \tau \le t, 0 < \beta(t) < 1, \forall t$. *So,* $V(x(t)) \le V(x(0)), \forall t$. *The rest of the proof is similar to the proof of Lyapunov stability for traditional integer order systems:*
Suppose that $r_1 > 0$ *is given. Pick* $r \in (0, r_1]$ *such that* $B_r = \{x, \|x\| \le r\} \subset \Delta$. *Define* $V_{min} = \min\limits_{\|x\|=r} V(x)$. *Choose* $\rho, 0 < \rho < V_{min}$. *Because of the continuity of* V, *the set* $\Phi_\rho = \{x \in B_r, V(x) \le \rho\}$ *is not empty. If* r_0 *is chosen in a way that* $\|x(0)\| < r_0$ *implies that* $x(0) \in \Phi_\rho$ *(which is possible due to the continuity of* V *and the fact that* $V(x=0) = 0$*), since* $V(x(t)) \le V(x(0)) \le \rho$, *then,* $x(t) \in \Phi_\rho$. *So,* $V(x(t)) < V_{min}$, *i.e., always* $x(t) \in B_r$. *As a result,* x *cannot escape from the ball* B_r. *So, for any given* r_1, $(\|x(0)\| < r_0) \Rightarrow (\|x(t)\| < r \le r_1)$ *and the Lyapunov stability is proven.*
For the asymptotic stability, it should be shown that for any $r_2 > 0$, *there is a* T *such that for* $t > T$, $x \in B_{r_2} = \{x, \|x\| \le r_2\}$. *For the sake of contradiction, suppose that* $x \notin B_{r_2}, \forall t$. *Hence, since the sufficient conditions for Lyapunov stability hold,* x *belongs to the compact set* $B_{r_1} - B_{r_2}$. *In the case that* ${}_0^C D^{\beta(t)}(V)$ *is strictly negative, there is a positive constant* μ *where* $-w(t) < -\mu, \forall t$ *implying that:*

$$V = V(x(0)) - \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1} w(\tau)d\tau$$
$$< V(x(0)) - \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}\mu d\tau$$
$$= V(x(0)) + \frac{\mu}{\Gamma(\beta(t))}\left(\frac{(t-\tau)^{\beta(t)}}{\beta(t)}\right)\Bigg|_{\tau=0}^{\tau=t}$$
$$= V(x(0)) - \frac{\mu}{\Gamma(\beta(t))}\frac{t^{\beta(t)}}{\beta(t)}$$

Letting $t \to \infty$, *we will face a contradiction.* V *is supposed to be positive inside* B_{r_1}, *however,* $V(x(0)) - \frac{\mu}{\Gamma(\beta(t))}\frac{t^{\beta(t)}}{\beta(t)} \to -\infty$ *as* $t \to \infty$. *Hence, it can be deduced that there is a positive value* T *for which* $x \in B_{r_2}, t > T$, *i.e., the origin is asymptotically stable.*

Remark: Other properties such as ultimate boundedness, inverse Lyapunov test for instability, etc. hold because in traditional integer order case, these properties are all based on the fact that $\dot{V}(t) \le 0$ implies that $V(x(t)) \le V(x(0))$. Here, as shown during the proof, ${}_0^C D^{\beta(t)} V \le 0$ implies the same.

Theorem 2 *(Stability of Linear Variable Order Commensurate Systems) A sufficient condition for the stability of the linear system* ${}_0^C D^{\beta(t)} x = Ax$ *is that* A *is Hurwitz. In this case* $x \to 0$ *as* $t \to \infty$.

Proof *Consider the Lyapunov function* $V(x) = \frac{1}{2}x^T P x, P > 0$. *According to Lemma 3,*

$${}_0^C D^{\beta(t)} V = \frac{1}{2} {}_0^C D^{\beta(t)}(x^T P x) \le x^T {}_0^C P D^{\beta(t)} x \tag{22}$$

where

$$\begin{aligned} x^T P {}_0^C D^{\beta(t)} x &= x^T P A x \\ &= \frac{1}{2}\big(x^T P A x + x^T A^T P x\big) = \frac{1}{2} x^T \big(PA + A^T P\big) x \end{aligned} \tag{23}$$

If A is Hurwitz, the algebraic Lyapunov equation $PA + A^T P = -Q$ holds for a positive definite matrix Q. So,

$$ {}_0^C D^{\beta(t)} V \leq -x^T Q x < 0 \tag{24}$$

Now, according to Theorem 3, $V > 0$ and ${}_0^C D^{\beta(t)} V < 0$, the system is stable in the sense of Lyapunov. Also, since ${}_0^C D^{\beta(t)} V$ is strictly negative, the system is asymptotically stable and $x \to 0$ as $t \to \infty$.

Theorem 3 *When A is Hurwitz, the system ${}_0^C D^{\beta(t)} x = Ax + v, x(0) = x_0$ is BIBO stable and as $t \to \infty$ this system is equivalent to ${}_0^C D^{\beta(t)} x = v, x(0) = 0$.*

Proof *Consider the Lyapunov function $V(x) = \frac{1}{2} x^T x$. Obviously,*

$$\begin{aligned} {}_0^C D^{\beta(t)} V &\leq x^T {}_0^C D^{\beta(t)} x = x^T (Ax + v) \\ &\leq \frac{1}{2} x^T \big(A + A^T\big) x + \|x\| \|v\| = -\frac{1}{2} x^T Q x + \|x\| \|v\| \\ &\leq \sigma_{min}(Q) \|x\|^2 + \|x\| \|v\| \end{aligned} \tag{25}$$

where $\sigma_{min}(Q)$ is the smallest singular value of the matrix Q and Q relaxes the algebraic Lyapunov equation ($PA + A^T P = -Q$) with $P = I$. Since $Q = -(A + A^T)$, $\sigma_{min}(Q) = 2\sigma_{min}(A)$. Therefore, for $\|x\| > \frac{\|v\|}{2\sigma_{min}(A)}$, ${}_0^C D^{\beta(t)} V$ is negative. So, the states will remain in the bound $\|x\| \leq \frac{\|v\|}{2\sigma_{min}(A)}$. Also, as long as $\|v\|$ is bounded, so is $\|x\|$ and this implies BIBO stability.
Moreover, the variable order derivation operator is linear. So, superposition condition holds and the response can be divided to the zero state and zero input responses. Accordingly, the response of the main system is the sum of the responses of two systems ${}_0^C D^{\beta(t)} x = Ax, x(0) = x_0$ and ${}_0^C D^{\beta(t)} x = v, x(0) = 0$. Based on Theorem 2, the response of the former goes to zero as $t \to \infty$ implying that in the steady state, the main system is equivalent to ${}_0^C D^{\beta(t)} x = v, x(0) = 0$ and this completes the proof.

Corollary 2 *Suppose that $x = {}_0 I_t^{\beta(t)}(Ax) + w, w(0) = 0$ and A is Hurwitz. Then, as $t \to \infty$, $x \to w$.*

Proof *Based on the Volterra equivalent integral, after applying the operator ${}_0^C D^{\beta(t)}$ on both sides of the above equation, using (4) we have ${}_0^C D^{\beta(t)} x = Ax + v, x_0 = 0$, where $v = {}_0^C D^{\beta(t)} w$. Now, since A is Hurwiz, the previous Theorem implies that as $t \to \infty$, ${}_0^C D^{\beta(t)} x = v = {}_0^C D^{\beta(t)} w, x(0) = w(0) = 0$, or, $x \to w$.*

As the final result of this paper, the next Theorem presents the order/state estimation of a commensurate variable order linear system with unavailable states in the compact interval $[0 \;\; T]$ for any given T.

Theorem 4 *Consider the linear commensurate variable order system ${}_0^C D^{\alpha(t)} x = Ax + Bu$ with unknown order $0 < \alpha(t) < 1$ in the compact interval $t \in [0 \;\; T]$ with $\|\dot{\alpha}(t)\| \leq M$ and only the output y is available. Suppose that the pair of (A, C) is observable and L is chosen such that $A - LC$ is Hurwitz. Then, the following estimation approach guarantees that the estimation error is ultimately bounded and could be made arbitrarily small.*

$$\begin{aligned} &{}_0^C D^{\hat{\alpha}(t)} \hat{x} = A\hat{x} + Bu + L(y - C\hat{x}), \hat{y} = C\hat{x} \\ &\dot{\hat{\alpha}} = \gamma \; sgn\Big(\frac{\partial z_{\hat{\phi}}(t, \beta)}{\partial \beta}\big|_{\beta = \hat{\alpha}}\Big)^T C^T (y - \hat{y}), \gamma > 0 \\ &\hat{\phi}(\tau) = A\hat{x}(\tau) + Bu(\tau) + L(y(\tau) - C\hat{x}(\tau)) \end{aligned} \tag{26}$$

where $sgn(.)$ is the sign function.

Proof *According to the Volterra integral we can write:*

$$\begin{aligned} x - \hat{x} = &\big(x(0) - \hat{x}(0)\big) \\ &+ \int_0^t \Bigg(\frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))} \Big((A - LC)(x - \hat{x})\Big) \\ &+ \Big(\frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))} - \frac{(t-\tau)^{\hat{\alpha}(t)-1}}{\Gamma(\hat{\alpha}(t))}\Big) \hat{\phi} \Bigg) d\tau \end{aligned} \tag{27}$$

with an approach similar to the one used to prove Lemma 1:

$$\begin{aligned} x - \hat{x} = &\big(x(0) - \hat{x}(0)\big) \\ &+ \int_0^t \big(\frac{(t-\tau)^{\alpha(t)-1}}{\Gamma(\alpha(t))} \Big((A - LC)(x - \hat{x})\Big) d\tau \\ &+ \frac{\partial z_{\hat{\phi}}(t, \beta)}{\partial \beta}\big|_{\beta = \hat{\alpha}} (\alpha - \hat{\alpha}) + \hat{K}(t)(\alpha - \hat{\alpha})^2 \end{aligned} \tag{28}$$

where $\hat{K}(t) = [\hat{K}_1(t) \dots \hat{K}_n(t)]^T$ and $\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}\big|_{\beta=\hat{\alpha}}$ are both $n \times 1$ vectors. According to the solution existence of variable order systems [16], $\hat{x}$ is continuous. Therefore, if u is continuous, so is $\hat{\phi}$ and according to Lemma 2, $\hat{K}_i(t) < \infty$ for $i = 1, ..., n$ and $0 \leq t \leq T$. Using the Volterra integral and defining $w = \frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}\big|_{\beta=\hat{\alpha}} (\alpha - \hat{\alpha}) + \hat{K}(t)(\alpha - \hat{\alpha})^2$,

$$ x - \hat{x} = w + {}_0 I_t^{\alpha} \Big((A - LC)\big(x - \hat{x}\big)\Big) \tag{29}$$

Now, define $V(\hat{\alpha}) = \frac{1}{2}(\alpha - \hat{\alpha})^2$. Since $y - \hat{y} = C(x - \hat{x})$

we have:

$$\dot{V}(\hat{\alpha}) = \dot{\alpha}(\alpha - \hat{\alpha}) - \gamma sgn\Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)^T C^T(y - \hat{y})$$
$$= \dot{\alpha}(\alpha - \hat{\alpha}) - \gamma sgn\Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)^T C^T C(x - \hat{x})(\alpha - \hat{\alpha}) \tag{30}$$

Substituting $x - \hat{x}$ from (29):

$$\dot{V} \leq \Bigg(\gamma\|C\|^2 \Big\|{}_0I_t^{\alpha}\Big((A - LC)(x - \hat{x})\Big)\Big\| + M\Bigg)\|\alpha - \hat{\alpha}\|$$
$$- \gamma C^T sgn\Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)^T \Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)C(\alpha - \hat{\alpha})^2$$
$$+ \gamma\|C^T C\|\|\hat{K}(t)\|\|\alpha - \hat{\alpha}\|^3 \tag{31}$$

Define $\eta(t) = sgn\Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)^T \Big(\frac{\partial z_{\hat{\phi}}(t,\beta)}{\partial \beta}|_{\beta=\hat{\alpha}}\Big)$ which is non-negative for all t. Now, the negative term is split and all the terms are grouped as follows:

$$\dot{V} \leq \Bigg\{\Bigg(\gamma\|C\|^2 \Big\|{}_0I_t^{\alpha}\Big((A - LC)(x - \hat{x})\Big)\Big\| + M\Bigg)\|\alpha - \hat{\alpha}\|$$
$$- \frac{\gamma}{2}\eta(t)\|C\|^2\|\alpha - \hat{\alpha}\|^2\Bigg\}$$
$$+ \Bigg\{\gamma\|C\|^2\|\hat{K}(t)\|\|\alpha - \hat{\alpha}\|^3 - \frac{\gamma}{2}\eta(t)\|C\|^2\|\alpha - \hat{\alpha}\|^2\Bigg\} \tag{32}$$

In the above, the groups are identified by $\{\}$.
1. The first group is negative in the following interval:

$$\|\alpha - \hat{\alpha}\| > \frac{2M}{\gamma\eta\|C\|^2} + \frac{2\Big\|{}_0I_t^{\alpha}\Big((A-LC)\big(x-\hat{x}\big)\Big)\Big\|}{\eta}$$

According to Corollary 2, as $t \to \infty$, in (29) $x \to w$ and ${}_0I_t^{\alpha}\Big((A - LC)(x - \hat{x})\Big) \to 0$, i.e., for any given $\epsilon_0 > 0$ there is a value T_0 such that $\Big\|{}_0I_t^{\alpha}\Big((A - LC)(x - \hat{x})\Big)\Big\| < \epsilon_0$ for $t > T_0$. The value of T_0 is dependent to the poles of $(A - LC)$. If L is designed in a way that $T_0 << T$, then $\Big\|{}_0I_t^{\alpha}\Big((A - LC)(x - \hat{x})\Big)\Big\| \approx 0$ almost everywhere on $t \in (0 \;\; T]$.
2. The second group is negative when $\|\alpha - \hat{\alpha}\| < \frac{\eta}{2\|\hat{K}\|}$. As long as $\eta > 0$ (i.e. the positive value η_0 exists such that $\eta > \eta_0$), since $\|\hat{K}\|$ is bounded there is a positive real number $M_{\hat{K}}$ such that $\|\alpha - \hat{\alpha}\| < \frac{\eta_0}{2M_{\hat{K}}}$ implies that $\|\alpha - \hat{\alpha}\| < \frac{\eta}{2\|\hat{K}\|}$.
Hence, for $t > T_0$, $\dot{V}$ is negative inside the interval $\frac{2M}{\gamma\eta_0\|C\|^2} < \|\alpha - \hat{\alpha}\| < \frac{\eta_0}{2M_{\hat{K}}}$ almost everywhere on $t \in [0 \;\; T]$.
This implies the locally ultimate boundedness of the error. In fact, since $\dot{V} < 0$ in $\frac{2M}{\gamma\eta_0\|C\|^2} < \|\alpha - \hat{\alpha}\|$, as soon as $\alpha - \hat{\alpha}$ tends to leave the area $\|\alpha - \hat{\alpha}\| \leq \frac{2M}{\gamma\eta_0\|C\|^2}$, it is pulled back inside, thus, the error remains bounded with maximum bound of $\frac{2M}{\gamma\eta_0\|C\|^2}$.
Accordingly, if the initial condition is properly chosen, the estimation error can be made arbitrarily small. The convergence speed is related to the matrix $A - LC$. Also, according to (28) and (29), for a given $\epsilon > 0$, there is an $\epsilon_1 > 0$ such that if $\|\alpha - \hat{\alpha}\| < \epsilon_1$ then $\|x - \hat{x}\| < \epsilon$. It means that γ can be chosen in a way that $\|x - \hat{x}\| < \epsilon$. So, the state vector and the order can both be estimated with desired levels of precision.
Consider the above interval with the lower bound $B_L = \frac{2M}{\gamma\eta_0\|C\|^2}$ and the upper bound $B_U = \frac{2M}{\gamma\eta_0\|C\|^2}$. It should be noted that as long as $B_L < B_U$ the aforementioned interval is never empty i.e. for $\gamma > \frac{4MM_{\hat{K}}}{\eta_0^2\|C\|^2}$ there exist a region such that if the initial condition is chosen inside it, the adaptation rules converge to an estimation with bounded error. Also, larger γ leads to smaller ultimate bound. So, large enough γ ensures the convergence with an acceptable error. Moreover,
1. For the constant order case where $M = 0$, if the initial value is properly chosen, the estimation error asymptotically converges to zero.
2. As $\hat{K} \to 0$, the upper bound tends to infinity i.e. the estimation error will be globally bounded.
3. Since $\{\hat{\alpha}, \eta(t) = 0\}$ is not an invariant set for the system (i.e. it does not yield $\dot{\hat{\alpha}} = 0, \forall t$), so the trajectory will leave this set. Hence, this case can be neglected in stability analysis.
4. Although the proposed method requires temporal compactness, the final time T can be arbitrarily chosen. Actually, for any given value of $T < \infty$, the proposed algorithm works in the interval $[0 \;\; T]$. Hence, the only constraint we are dealing with is $T \neq \infty$, which from the practical sight of view does not cause a problem.

Theorem 4 implies that for simultaneous estimation of the order and the states, a set of two combined systems should be used, one for estimating the order and the other for estimating the states. These two systems are in a high level of interaction with each other. The block diagram shown in Figure 1 illustrates this concept.

3.2 Simulation Study

As the first simulation study, consider the following case study (33) which is an unstable multi-input multi-output system. We aim to estimate the order and the states while only the input and output vectors are available. Figure 2 and 3 show the simulation results. Clearly, the figures show that although the main system is unstable method works properly and the ultimate errors for estimating both the order the states are small enough. This verifies the effectiveness of the method proposed in The-

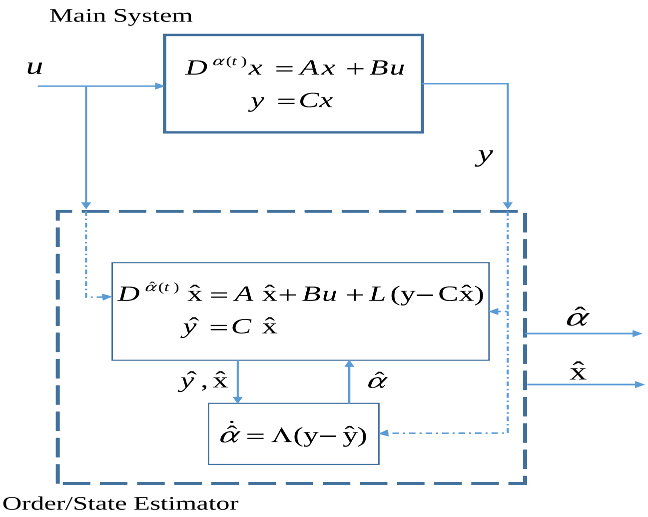

Fig. 1. The simultaneous order/state estimator block diagram

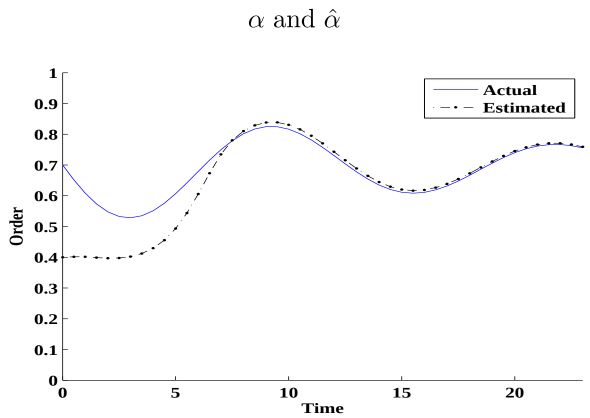

Fig. 2. The actual and estimated order.

orem 4.

$$
\begin{aligned}
&u = \begin{bmatrix} 1 - e^{-0.1t} \\ tanh(0.1(t-16)) \end{bmatrix} \\
&\alpha(t) = 0.7 - 0.2e^{-0.05t} sin(0.5t) \\
&A = \begin{bmatrix} 0.3 & 0.1 & 0.1 \\ -0.1 & -0.2 & 0.4 \\ -0.3 & -0.5 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -2 & -1 \end{bmatrix} \\
&C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, L = \begin{bmatrix} 0.30 & 0.20 \\ -0.10 & 0.08 \\ -0.30 & -0.67 \end{bmatrix}
\end{aligned} \tag{33}
$$

In the second simulation study, an additive measurement noise is also taken into account. Consider the following system.

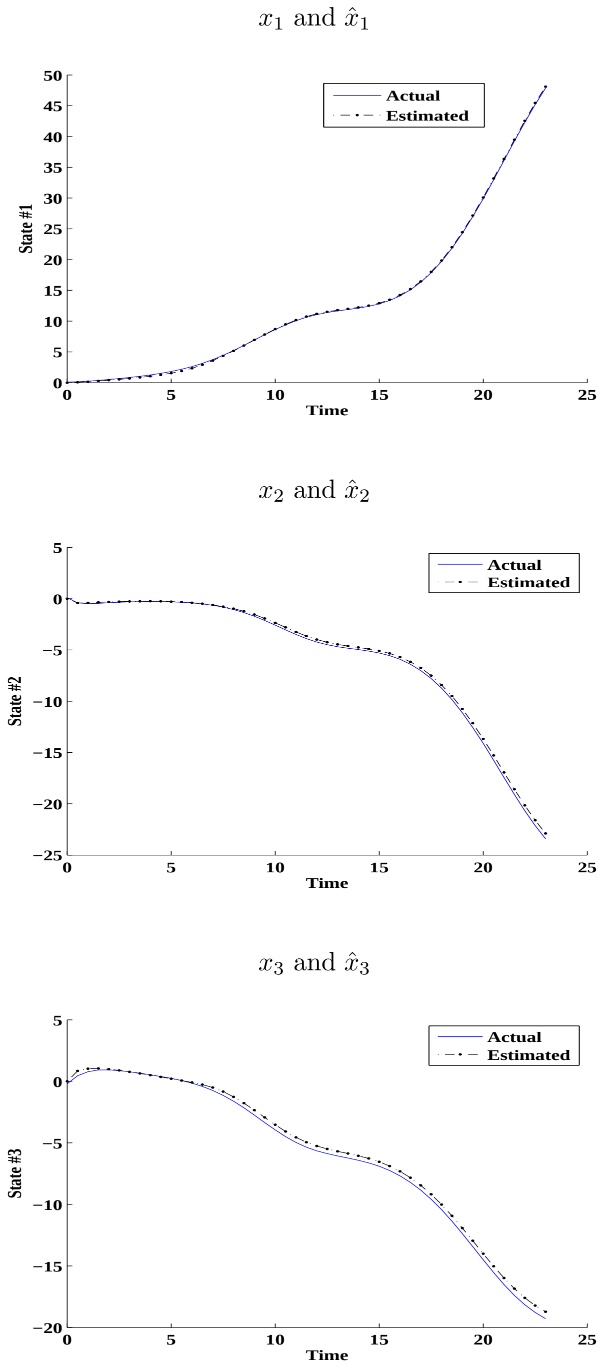

Fig. 3. The actual and auxiliary states. top: x_1 vs. $\hat{x}_1$, middle: x_2 vs. $\hat{x}_2$, bottom: x_3 vs. $\hat{x}_3$.

$$
\begin{aligned}
&{}_0^C D^{\alpha(t)} x(t) = Ax(t) + Bu(t) \\
&y(t) = Cx(t) + \nu, \ \nu \sim N(0, 0.01) \\
&u = 1 - e^{-0.1t} sin(0.6t) \\
&\alpha(t) = 0.7e^{-0.1t}(0.5cos(0.5t) + 0.7) \\
&A = \begin{bmatrix} 0.3 & -0.1 \\ -0.2 & -0.4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&C = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, L = \begin{bmatrix} 0.1138 \\ -0.4234 \end{bmatrix}
\end{aligned} \tag{34}
$$

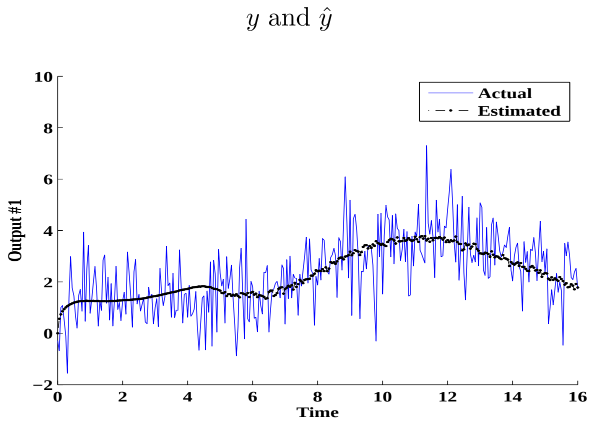

Fig. 4. The actual and estimated output for a noisy system.

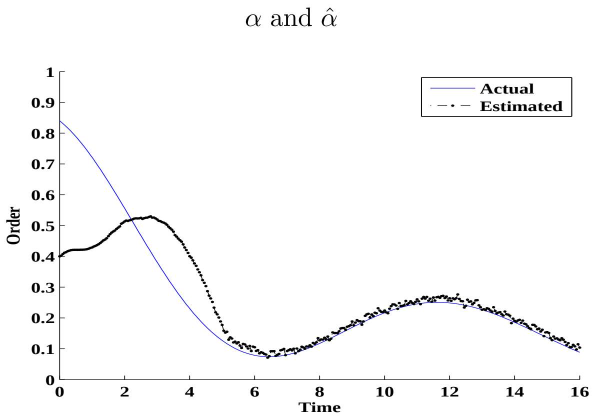

Fig. 5. The actual and estimated order for a noisy system.

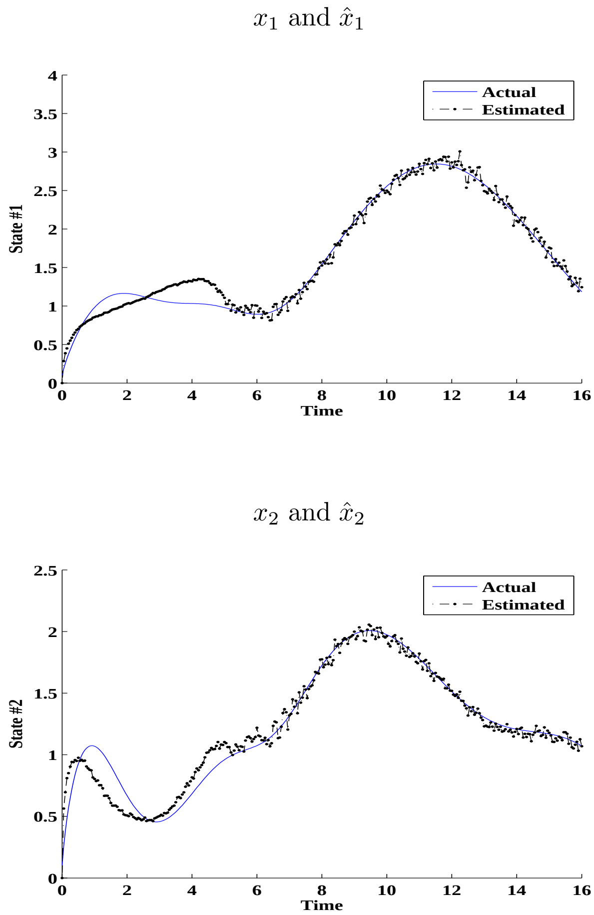

Fig. 6. The actual and auxiliary states for a noisy system. top: x_1 vs. $\hat{x}_1$, bottom: x_2 vs. $\hat{x}_2$.

where ν is a white Gaussian noise. The only available signals are the input and the noisy output. Figures 4-6 show that the proposed algorithm effectively estimates all unavailable signals even in the presence of noise.

4 Conclusion

This paper solved the unknown order estimation problem for non-integer order systems, in combination with designing a state observer for such systems. The sufficient conditions and the designing procedure of the estimator were introduced as well as the convergence proof. As the simulation studies show, even in the presence of the measurement noise, the presented method works and leads to an acceptable estimation error. Because of different applications of variable order dynamics, the results of this paper could be used in a number of applications especially for estimating the order which is supposed to be known is most of the previous works.
During the process of designing the estimator, some lemma and theorems were proven about the topic of non-integer order system, which, to the best knowledge of the authors, have not been reported before. These results can be used to investigate the stability properties of the variable order systems.
Future works in this topic may be done pertaining to extending the presented results in order to design an order/state estimator for nonlinear variable order system or designing an adaptive order/parameter/state estimator for variable order linear systems. Also, as a more general case, a method can be designed to estimate the order vector of an incommensurate order system.

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