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Non-integer Variable Order Dynamic Modeling and Identification of Soft Tissue Deformation

S. Sepehr Tabatabaei [1], Heidar Ali Talebi [2] and Mahdi Tavakoli [3]

Abstract— **This paper presents the concept of non-integer analytical dynamic modeling of soft tissue deformation. The main idea of the paper is to introduce a variable order equation for the single-dimensional viscoelastic stress-strain relation. A set of experimental data and an identification method are used to validate such idea. Afterwards, we extend the stress-strain relationship to the multi-dimensional case. Combining this equation with the equations of motion for a soft continuum body leads to the set of force-displacement equations. The model is rearranged to form a standard state space dynamical system. Then, these equations are analyzed and compared with a previously developed integer-order model.**

I. INTRODUCTION

Soft tissue deformation modeling mainly refers to deriving a model to describe the force – displacement relation of the tissue. Tissue models are used in surgery simulators [1], needle – tissue interaction in needle insertion procedure [2], brachytherapy [3], etc. A precise and accurate model of tissue facilitates the insertion procedure and results in less surgical injuries. It is reported that neglecting this important issue in brachytherapy leads to inaccurate seed implantation [4].

Among all possible approaches, here we focus on deriving an analytical model, i.e., we aim to derive a set of Partial Differential Equations (PDE) describing the tissue deformation. There are several approaches to find an analytical model for soft tissue. The tissue models can be categorized in two main classes of static and dynamic models [5], [6]. The former studies the tissue in relaxation mode and the time is neglected [7], [8]. This view point is not sufficient for needle insertion applications where the tissue deformation is being changed during the procedure [9], [10]. Hence, the tissue should be considered as a dynamic environment in such cases. The tissue is considered to be a viscoelastic environment [11]. A viscoelastic material is something "between" the elastic and viscous ones. The most prevalent approach for modeling such behavior is to use the Kelvin-Voigt model [12], [13] where the viscoelastic material is considered to be a combination of a Hookean spring and a Newtonian damper, [12]. The stress-strain relationship is the main characteristic used to categorize different materials. The stress-strain relation of a Hookean spring is given by $\sigma = k\varepsilon$ where stress and strain vectors are indicated by σ and ε, respectively, and k is the elasticity tensor. For the Newtonian dampers, the stress-strain relation is considered to be $\sigma = b\dot{\varepsilon}$ where, $\dot{\varepsilon} = \frac{d\varepsilon}{dt}$ and b is the viscosity tensor. The main idea of the Kelvin-Voigt model is to consider a viscoelastic material as a parallel-series combination of an ideal Hookean spring and a Newtonian damper [12], i.e.,

$$\sigma = k\varepsilon + b\dot{\varepsilon} \tag{1}$$

There can be a different interpretation for the word "between". In this point of view, the stress-strain relationship of the elastic and viscous materials can be respectively written as $\sigma = kD_t^0\varepsilon$ and $\sigma = bD_t^1\varepsilon$, where, D_t^n is defined as the n^{th} time derivation operator (the 0^{th} derivative of a function is considered to be itself.) Now, if a material behaves between these, its stress-strain relation can be considered as:

$$\sigma = \eta D_t^q\varepsilon, \qquad 0 < q < 1 \tag{2}$$

Equation (2) is the main idea of many recently published papers, based on the fractional (or, to be more precise, non-integer) order modeling of viscoelasticity, some examples of which can be found in [14], [15]. In (2), D_t^q (the non-integer order derivation operator) can be defined in various ways with the most common ones being the definition in the sense of Caputo and Riemann-Liouville [16]. The following equations show the definitions of left-sided non-integer variable order derivation of order $0 < q(t) < 1$ in the sense of Caputo and Riemann-Liouville, respectively, [16]:

$$^{C}D_t^{q(t)}f = \frac{1}{\Gamma(1-q(t))}\int_0^t (t-\tau)^{-q(t)}\frac{d}{d\tau}f(\tau)\mathrm{d}\tau \tag{3}$$

$$^{RL}D_t^{q(t)}f = \frac{1}{\Gamma(1-q(t))}\frac{d}{dt}\int_0^t (t-\tau)^{-q(t)}f(\tau)\mathrm{d}\tau \tag{4}$$

where $\Gamma(.)$ i the extension of the factorial function to non-integer arguments, $\Gamma(w) = \int_0^\infty r^{w-1}e^{-r}dr$.

In (2), when $q \to 0$, the behavior tends to elasticity and as $q \to 1$, it tends to viscosity. As shown in (3) and 4, the order can vary in time. Variable order dynamics are widely used in modeling various phenomena. Such operators are interpreted as the human's ability to forget and remember [17], also used to develop mechanical laws in [18]. In [19], the theory of viscoelasticity and the abilities of variable order calculus build a framework for modeling viscoelastic behavior. In [20], variable order differential equations are used to describe anomalous diffusion modeling. The effect of tension on cable deformation is modeled in [21] using variable order

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[1]S. Sepehr Tabatabaei is a PhD Candidate at the Department of Electrical Engineering, Amirkabair University of Technology (Tehran Polytechnic), Tehran, Iran s.tabatabaei@aut.ac.ir

[2] Heidar Ali Talebi is with the Department of Electrical Engineering, Amirkabair University of Technology (Tehran Polytechnic), Tehran, Iran. alit@aut.ac.ir

[2] Mahdi Tavakoli is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada. mahdi.tavakoli@ualberta.ca

dynamics. In-line with such papers, this manuscript aims to propose a novel variable order model for describing soft tissue difformation and validate it using experimental data. In this regard, the rest of the paper is organized as follows:

In Section II, the definitions of stress and strain and the equations of motion for a continuum environment are introduced. The experimental study will be presented in Section III. In Section IV, the linear non-integer model of viscoelasticity in two dimensions is introduced and the model of the soft tissue displacement is derived. Then, the model is discretized and rearranged to the state-space form, the proposed model is compared to the model introduced in Section II, and its advantages are explained. Finally, Section IV concludes the paper.

II. Dynamics of a Soft Continuum Body

This section studies the effect of the force applied on a soft continuum body on its formation as a function of time and space. Due to its non-rigidity, each single point of the body should be considered, thus, the body is assumed to consist of a combination of infinitesimal parts. Hence, the mechanical behavior of the material is modeled as a continuous mass rather than an aggregation of discrete particles. Modeling an object as a continuum assumes that the substance of the object completely fills the space it occupies. Modeling objects in this way makes it possible to utilize mathematical tools (e.g., PDEs) to describe them.

For analyzing the kinematics, mechanics and dynamics of a continuum material, some quantities such as stress and strain have to be defined. In fact, these quantities help us relate the external applied force to the deflection of each point of the body. Moreover, physical laws such as Newton's second law should be rewritten for such materials. Our aim in this section is to derive such equations.

A. Strain and Stress

Strain is the normalized measure of deformation representing the displacement between particles in a body relative to a reference length. More precisely, in a single-dimensional environment, it can be defined as $\varepsilon = \lim_{\Delta l \to 0} \frac{\Delta l}{l_0}$ [22], where, ε denotes the strain, l_0 is the initial length and Δl is the displacement. In the three-dimensional space, the strain can be defined for each direction. In a continuum body, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ are the strain along x, y, z axes, respectively. The definitions of these quantities are [22]

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{5}$$

where u_x, u_y, u_z are the displacements in x, y, z axes, respectively. In addition, the engineering shear strains can be defined according to the following equations [22]:

$$\begin{aligned} \varepsilon_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \varepsilon_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \varepsilon_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{aligned} \tag{6}$$

Stress is defined as the average force per unit area that some particle of a body exerts on an adjacent particle, across an imaginary surface that separates them [22]. The stress tensor can be represented in any chosen Cartesian coordinate system by a 3×3 matrix of real numbers. Depending on whether the coordinates are numbered (x_1, x_2, x_3) or named (x, y, z), the matrix may be written as [22]:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

For each particle of the body, each entry of the stress tensor is defined as $F_i = \sum_{j=1}^{3} \sigma_{ij} A_j$ [22], where F_i is the force applied on the particle along the i^{th} axis [22]. The relation between stress and strain in a material is a basis of categorizing material to the classes such as elastic, viscous, viscoelastic, hyper-elastic, etc. [22].

B. Two-Dimensional Motion of a Continuum Body

The motion of a continuum body mainly refers to the displacement of each particle caused by an external force or an initial displacement of a sub-region of it. The equations of motion are derived as follows, using the Newton's second law [22]:

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + F_x = \rho \frac{\partial^2 u_x}{\partial t^2} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = \rho \frac{\partial^2 u_y}{\partial t^2} \end{cases} \tag{7}$$

In the above equations, F_x and F_y are the components of the external force in x and y axes, respectively and ρ is the density, considered here to be a constant parameter. It should be noted that the arguments of the functions are omitted for brevity. In fact,

$$\begin{cases} \sigma_{xx} = \sigma_{xx}(x, y, t) & , \quad \sigma_{yx} = \sigma_{yx}(x, y, t) \\ \sigma_{xy} = \sigma_{xy}(x, y, t) & , \quad \sigma_{yy} = \sigma_{yy}(x, y, t) \\ u_x = u_x(x, y, t) & , \quad u_y = u_y(x, y, t) \\ F_x = F_x(x, y, t) & , \quad F_y = F_y(x, y, t) \end{cases} \tag{8}$$

In addition to the equations of motion and the definition of strain, the characteristic property of the material (i.e., the stress-strain relation) is needed for deriving the force-displacement equations. Fig. 1 shows such relations as a block diagram. The input of the main model is the applied force and the output is the displacement. Furthermore, there are some internal blocks, the first with force and stress as inputs and displacement as output, the second for relating strain and displacement, and the third one for relating stress and strain. Blocks 1 and 2 are fixed and determined based on the physical laws. Different modeling approaches affect, the various ways in which block 2 is chosen. Our novelty in this paper is to use a variable order relation for Block 2. To provide a comparative study, the traditional Kelvin-Voigt model is used in the next subsection for the stress-strain relation. Using this relation, a set of PDEs will be derived as a model for soft tissue. Considering this model makes it possible to compare and evaluate the suggested model proposed in this paper.

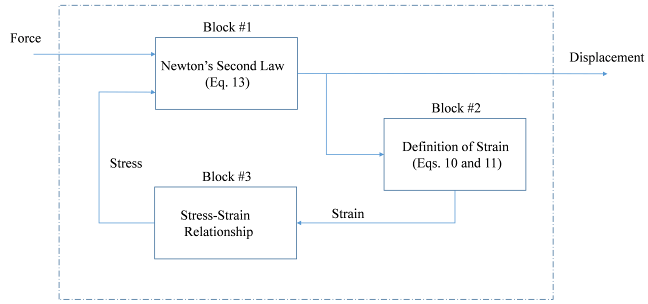

Fig. 1. Force-Displacement relation block diagram

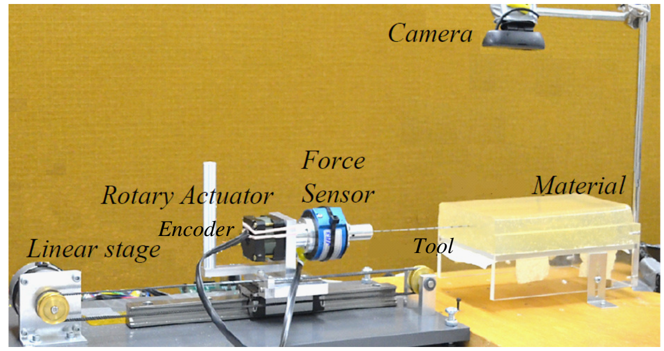

Fig. 2. Data gathering apparatus

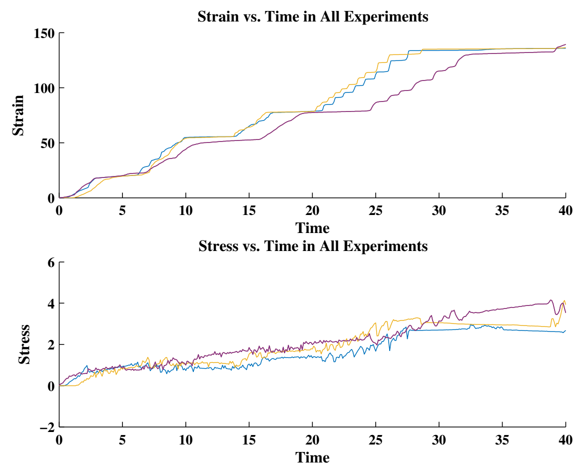

Fig. 3. Top: Strain profiles in different experiments. Bottom: Stress in Different Experiments

C. Force-Displacement Equations Using a Simple Kelvin-Voigt Model

The generalization of (1) in two-dimensional environment leads to the following equation for stress-strain relation:

$$\begin{cases} \sigma_{xx} = (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + (b+2\mu)\dot{\varepsilon}_{xx} + b\dot{\varepsilon}_{yy} \\ \sigma_{yx} = \sigma_{xy} = \mu\varepsilon_{xy} + b\dot{\varepsilon}_{xy} \\ \sigma_{yy} = (\lambda+2\mu)\varepsilon_{yy} + \lambda\varepsilon_{xx} + (b+2\mu)\dot{\varepsilon}_{yy} + b\dot{\varepsilon}_{xx} \end{cases} \tag{9}$$

here, λ, b, μ are three independent parameters related to the structure of the body. Substituting (9) into (7) and using the definition of strain given in (5,6), it is straightforward to derive the force-displacement equations as

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial^2 u_x}{\partial x^2} + \lambda\dfrac{\partial^2 u_y}{\partial x\partial y} + (b+2\mu)\dfrac{\partial^2 \dot{u}_x}{\partial x^2} + b\dfrac{\partial^2 \dot{u}_y}{\partial x\partial y} \\ \quad + \mu(\dfrac{\partial^2 u_x}{\partial y^2} + \dfrac{\partial^2 u_y}{\partial y\partial x}) + b(\dfrac{\partial^2 \dot{u}_x}{\partial y^2} + \dfrac{\partial^2 \dot{u}_y}{\partial y\partial x}) + F_x = \rho\ddot{u_x} \\ (\lambda+2\mu)\dfrac{\partial^2 u_y}{\partial y^2} + \lambda\dfrac{\partial^2 u_x}{\partial y\partial x} + (b+2\mu)\dfrac{\partial^2 \dot{u}_y}{\partial y^2} + b\dfrac{\partial^2 \dot{u}_x}{\partial y\partial x} \\ \quad + \mu(\dfrac{\partial^2 u_y}{\partial x^2} + \dfrac{\partial^2 u_x}{\partial x\partial y}) + b(\dfrac{\partial^2 \dot{u}_y}{\partial x^2} + \dfrac{\partial^2 \dot{u}_x}{\partial x\partial y}) + F_y = \rho\ddot{u_y} \end{cases} \tag{10}$$

The existence of the derivatives of $\dot{u}_x$ and $\dot{u}_y$ with respect to displacement makes the model complicated. It is not easy to express (10) as a dynamical system in time domain. As will be shown, using non-integer order modeling leads to a simpler more flexible model. The next section presents a variable order stress-strain model verified using a single-dimensional set of data.

III. Single-Dimensional Variable Order Modeling and Identification of Soft Tisuue Diformation

Before deriving the soft tissue model using variable order model, we should make sure that such modeling concept agrees with real data. To this aim consider (2), a single-dimensional stress strain relationship with two parameters, the constant coefficient η and the variable order $q(t)$. The material chosen for the experimental study is a symmetrical slice of real beef tissue. It is indented in one dimension using a needle insertion robot shown in Fig. 2. The setup consists of a robotic system with two degrees of freedom (DOF) for translational and rotational motions of the needle. Forces and torques are measured at the needle base using a 6-DOF force sensor. After we replace the needle with a blunt indenter, the setup can be used to apply controlled force or displacement to the tissue for indentation tests. We have used the encoder data to record the displacement and calculate the strain. Since the tissue is symmetric, we will just consider the indentation direction to get one dimensional stress-strain data. The data gathering process is done through obtaining force and displacement data from the force sensor and the encoder and converting them to stress and strain, respectively, using the one dimensional equations $\sigma = \frac{F}{a}, \varepsilon = \frac{d}{l}$, where F is force, d is displacement, a is area, l is tissue length, σ is stress and ϵ is strain. We have used controlled tissue displacement with various similar profiles. The results are shown in Fig. 3.

The identification process is converted to a minimization process by defining the cost function as the integrated squared error between measured and estimated stress. To deal with the variable order q, we consider it as a piecewise constant function, i.e., the interval $[0 \quad T]$ is split into N subintervals $[t_{i-1} \quad t_i), i = 1, ...N, t_0 = 0, t_N = T$ and $q = q_0 + \sum_{i=1}^{N} q_i\big(\Theta(t-t_i) - \Theta(t-t_{i-1})\big)$ where $\Theta(.)$ is the Heaviside step function. Minimization process is done using genetic algorithm with minimizing parameters

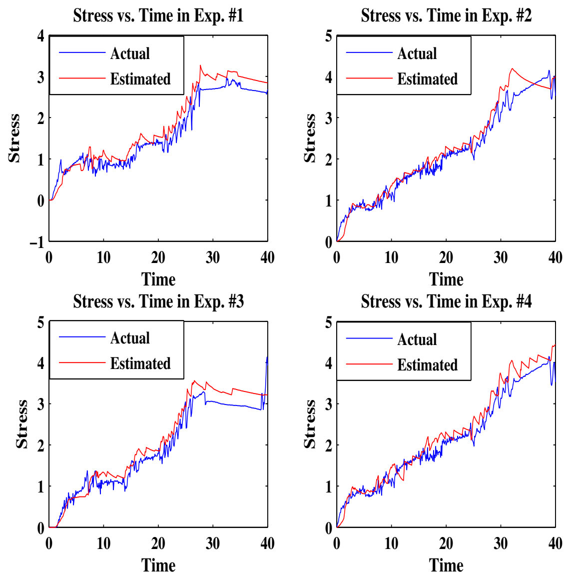

Fig. 4. Measured and estimated stress in four different experiment

η and q_is. Fig. 4 depicts the results. As shown, in all experiments, the proposed model and identification approach are able to identify the tissue deformation with an acceptable level of precision. The presented approach is enough to validate the idea of using a variable order. Hence, choosing a variable order model for describing soft tissue stress-strain relationship also makes sense from the experimental point of view. Accordingly, since variable order modeling worked in the single-dimensional case, in the next section we will extend it to the two-dimensional case and derive the force-displacement equations.

IV. NON-INTEGER ANALYTICAL MODEL OF SOFT TISSUE DEFORMATION

A. Deriving the Dynamic Model

The non-integer order dynamic of soft tissue is the result of considering a non-integer stress-strain relation for the tissue. Assuming the symmetry, the generalization of (2) to the two-dimensional case is obtained in a manner similar to (9):

$$\sigma_{xx} = \alpha \, {}^{RL}D_t^q \, \varepsilon_{xx} + \beta \, {}^{RL}D_t^q \, \varepsilon_{yy} \tag{11}$$

$$\sigma_{yx} = \sigma_{xy} = \gamma {}^{RL}D_t^q \varepsilon_{xy} \tag{12}$$

$$\sigma_{yy} = \beta \, {}^{RL}D_t^q \, \varepsilon_{xx} + \alpha \, {}^{RL}D_t^q \varepsilon_{yy} \tag{13}$$

To the best knowledge of the authors, this set of equations are first introduced and used in this paper as an extension to one-dimensional non-integer order viscoelastic relation. As will be shown, this extension is well defined in the sense that it leads to a set of PDEs which is consistent with the former model of soft tissue in the relaxation mode. It should be noted thatt here are four independent parameters in the set of equations (11)-(13) which makes the model parametrically richer than (9).

Using the equations of motion and a procedure similar to the one used in part C of Section II, one may easily derive the force - displacement equations as

$$\begin{cases} {}^{RL}D_t^q\big(\alpha\frac{\partial^2 u_x}{\partial x^2} + \beta\frac{\partial^2 u_y}{\partial x \partial y} + \gamma\frac{\partial^2 u_y}{\partial y \partial x} + \gamma\frac{\partial^2 u_x}{\partial y^2}\big) + F_x = \rho \ddot{u}_x \\ {}^{RL}D_t^q\big(\alpha\frac{\partial^2 u_y}{\partial y^2} + \beta\frac{\partial^2 u_x}{\partial y \partial x} + \gamma\frac{\partial^2 u_x}{\partial x \partial y} + \gamma\frac{\partial^2 u_y}{\partial x^2}\big) + F_y = \rho \ddot{u}_y \end{cases} \tag{14}$$

It can be seen that the spatial derivatives of $\dot{u}_x$ and $\dot{u}_y$ are eliminated and the model is simplified as compared with (10). Furthermore, it can be shown that the model can be presented in a more suitable way for simulation, estimation and control goals. The following theorems help to simplify the above equations to a standard state space format.

Theorem 1: When w is nonsingular, ${}_0I_t^p w|_{t=0} = 0, \forall 0 < p < 1$; where ${}_0I_t^p w$ is the non-integer order integral of w of order p, defined as [23]

$${}_0I_t^p w = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} w(\tau) \mathrm{d}\tau, \qquad 0 < p < 1 \tag{15}$$

Proof: Define $H(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} w(\tau) \mathrm{d}\tau$. It is worth mentioning that merely concluding $H(0) = 0$ due to the fact that the lower and upper bounds of the integral are equal is not true since the argument is not valid for singular integrands [24]. In fact, $H(0) = \lim_{\epsilon \to 0} H(\epsilon) = \frac{1}{\Gamma(p)} \int_0^\epsilon (\epsilon-\tau)^{p-1} w(\tau) \mathrm{d}\tau$
However, the change of variable $\tau = t - r^{\frac{1}{p}}$ leads to a nonsingular integrand:

$$\tau = 0 \Rightarrow r = t^p, \quad \tau = t \Rightarrow r = 0, \quad d\tau = -\frac{1}{p} r^{\frac{1-p}{p}}$$

$$\begin{aligned} H(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} w(\tau) \mathrm{d}\tau \\ &= \frac{1}{p\Gamma(p)} \int_0^{t^p} w(t - r^{\frac{1}{p}}) \mathrm{d}r \end{aligned}$$

Now, since w is nonsingular, it can be concluded that $H(0) = 0$ and this proves the Theorem. ■

Theorem 2: If w is nonsingular, the equation ${}^{RL}D_t^q w + F = \rho \ddot{u}, 0 < q < 1$ can be rewritten in the following state-space format:

$$\begin{aligned} \dot{u} &= \frac{1}{\rho} z + \frac{1}{\rho} g \quad &, u(0) = 0 \\ \dot{g} &= F &, g(0) = \rho \dot{u}(0) \\ {}^{RL}D_t^p z &= w &, z(0) = 0 \end{aligned} \tag{16}$$

where $p = 1 - q$ and z and g are the internal states.

Proof: According to definition (3), we have

$$\frac{d}{dt} {}_0I_t^{1-q} w + F = \rho \frac{d}{dt} \dot{u} \tag{17}$$

Now, integrating both sides of (17) leads to

$${}_0I_t^{1-q} w - {}_0I_t^{1-q} w|_{t=0} + \int_0^t F d\tau = \rho \dot{u} - \rho \dot{u}(0) \tag{18}$$

According to Theorem 3.1, ${}_0I_t^p w|_{t=0} = 0$. Now, define

$$g = \int_0^t F d\tau + \rho \dot{u}(0) \Rightarrow g(0) = \rho \dot{u}(0)$$

$$z = {}_0I_t^{1-q} w \Rightarrow z(0) = 0$$

The main property of the non-integer order operators is that $\left[{}^{RL}D_{t\,0}^{p}I_{t}^{p}\right]w = w$ [23]. Hence, ${}^{RL}D_{t}^{1-q}z = w$, and

$$\begin{aligned}\dot{u} &= \frac{1}{\rho}z + \frac{1}{\rho}g, u(0) = 0\\ \dot{g} &= F, g(0) = \rho\dot{u}(0)\\ {}^{RL}D_{t}^{p}z &= w, z(0) = 0\end{aligned}$$

and the Theorem is proved. ∎

According to Theorem 3.2, (14) can be reformed as the following form which is more suitable for many applications, as it can be discretized using finite difference approximation.

$$\begin{aligned}
\dot{u}_x(x,y,t) &= \frac{1}{\rho}z_x(x,y,t) + \frac{1}{\rho}g_x(x,y,t)\\
&\quad , u_x(x,y,0) = u_{x0}\\
\dot{g}_x(x,y,t) &= F_x(x,y,t)\\
&\quad , g_x(x,y,0) = \rho\dot{u}_x(x,y,0)\\
{}^{RL}D_t^p z_x &= \alpha\frac{\partial^2 u_x}{\partial x^2} + \beta\frac{\partial^2 u_y}{\partial x\partial y} + \gamma\frac{\partial^2 u_y}{\partial y\partial x} + \gamma\frac{\partial^2 u_x}{\partial y^2}\\
&\quad , z_x(x,y,0) = 0\\
\dot{u}_y(x,y,t) &= \frac{1}{\rho}z_y(x,y,t) + \frac{1}{\rho}g_y(x,y,t)\\
&\quad , u_y(x,y,0) = u_{y0}\\
\dot{g}_y(x,y,t) &= F_y(x,y,t)\\
&\quad , g_y(x,y,0) = \rho\dot{u}_y(x,y,0)\\
{}^{RL}D_t^p z_y &= \alpha\frac{\partial^2 u_y}{\partial y^2} + \beta\frac{\partial^2 u_x}{\partial y\partial x} + \gamma\frac{\partial^2 u_x}{\partial x\partial y} + \gamma\frac{\partial^2 u_y}{\partial x^2}\\
&\quad , z_y(x,y,0) = 0
\end{aligned} \tag{19}$$

Keeping the time derivations, the spatial derivatives are approximated. Consider a two-dimensional layer of the tissue gridded into $(N_x+1)\times(N_y+1)$ points with equal step length h. On such grid, we define $x_i = ih, i = 0,1,...,N_x$ and $y_j = jh, j = 0,1,...,N_y$ and for a smooth enough function V defined on the surface, $V(i,j,t) = V(x_i,y_j,t)$ The spatial derivatives of V can be approximated using the following formulas [25]:

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &\approx \frac{\left[V(i+1,j) - 2V(i,j) + V(i-1,j)\right]}{h^2}\\
\frac{\partial^2 V}{\partial x\partial y} = \frac{\partial^2 V}{\partial y\partial x} &\approx \frac{V(i+1,j+1) + V(i-1,j-1)]}{4h^2}\\
&\quad - \frac{V(i+1,j-1) + V(i-1,j+1)}{4h^2}\\
\frac{\partial^2 V}{\partial y^2} &\approx \frac{\left[V(i,j+1) - 2V(i,j) + V(i,j-1)\right]}{h^2}\\
i &= 1,...,N_x - 1\\
j &= 1,...,N_y - 1
\end{aligned} \tag{20}$$

Substituting these approximations in (19) leads to the following equations, in which if we consider the state variables $u_x(i,j,t)$, $u_y(i,j,t)$, $g_x(i,j,t)$, $g_y(i,j,t)$, $z_x(i,j,t)$, $z_y(i,j,t)$, $i = 1,...,N_x - 1, j = 1,...,N_y - 1$, it can be considered as a pure state space system with $6\times(N_x-1)\times(N_y-1)$ state variables.

$$\begin{cases}
\dot{u}_x(i,j,t) = \frac{1}{\rho}z_x(i,j,t) + \frac{1}{\rho}g_x(i,j,t)\\
\qquad , u_x(i,j,0) = u_{x0}\\
\dot{g}_x(i,j,t) = F_x(i,j,t), g_x(i,j,0) = \rho\dot{u}_x(i,j,0)\\
{}^{RL}D_t^p z_x(i,j,t) = \frac{\alpha}{h^2}\big(u_x(i+1,j) - 2u_x(i,j)\\
\qquad + u_x(i-1,j)\big)\\
\qquad + \frac{\beta+\gamma}{4h^2}\big(u_y(i+1,j+1) - u_y(i+1,j-1)\\
\qquad\quad - u_y(i-1,j+1) + u_y(i-1,j-1)\big)\\
\qquad + \frac{\gamma}{h^2}\big(u_x(i,j+1) - 2u_x(i,j) + u_x(i,j-1)\big)\\
\qquad , z_x(i,j,0) = 0\\
\dot{u}_y(i,j,t) = \frac{1}{\rho}z_y(i,j,t) + \frac{1}{\rho}g_y(i,j,t)\\
\qquad , u_y(i,j,0) = u_{y0}\\
\dot{g}_y(i,j,t) = F_y(i,j,t), g_y(i,j,0) = \rho\dot{u}_y(i,j,0)\\
{}^{RL}D_t^p z_y(i,j,t) = \frac{\gamma}{h^2}\big(u_y(i,j+1) - 2u_y(i,j)\\
\qquad + u_y(i,j-1)\big)\\
\qquad + \frac{\beta+\gamma}{4h^2}\big(u_x(i+1,j+1) - u_x(i+1,j-1)\\
\qquad\quad - u_x(i-1,j+1) + u_x(i-1,j-1)\big)\\
\qquad + \frac{\alpha}{h^2}\big(u_y(i+1,j) - 2u_y(i,j) + u_y(i-1,j)\big)\\
\qquad , z_y(i,j,0) = 0
\end{cases} \tag{21}$$

B. Relaxation Mode Model

The relaxation mode is considered as the state where the tissue rests in a static manner, without being exposed to an external force. Hence, the tissues transient time domain dynamic has died out. Since the tissue deformation will not then change with respect to time, the time derivations can be omitted and the model can be interpreted using a spatial partial differential equation. Kelvin-Voigt is a well-known model for describing soft tissue in relaxation mode [26]. Hence, the model proposed in this paper should agree with Kelvin-Voigt in the relaxation mode. Consider (10) where the Kelvin-Voigt based model is described. Using this model, the relaxation mode is described by the following equation:

$$\begin{cases}(\lambda+2\mu)\frac{\partial^2 u_x}{\partial x^2} + \lambda\frac{\partial^2 u_y}{\partial x\partial y} + \mu(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial y\partial x}) = 0\\ (\lambda+2\mu)\frac{\partial^2 u_y}{\partial y^2} + \lambda\frac{\partial^2 u_x}{\partial y\partial x} + \mu(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_x}{\partial x\partial y}) = 0\end{cases} \tag{22}$$

In fact, the relaxation mode formation of the tissue is the response of (22). Let us calculate the relaxation mode model of our proposed model, which is the equilibrium state of the state-space equations, given in (19) by setting the time derivatives and external forces to zero. The third and sixth equations of (19) lead to

$$\begin{cases}\alpha\frac{\partial^2 u_x}{\partial x^2} + \beta\frac{\partial^2 u_y}{\partial x\partial y} + \gamma\frac{\partial^2 u_y}{\partial y\partial x} + \gamma\frac{\partial^2 u_x}{\partial y^2} = 0\\ \alpha\frac{\partial^2 u_y}{\partial y^2} + \beta\frac{\partial^2 u_x}{\partial y\partial x} + \gamma\frac{\partial^2 u_x}{\partial x\partial y} + \gamma\frac{\partial^2 u_y}{\partial x^2} = 0\end{cases} \tag{23}$$

Comparing (22) and (23), it can be easily seen that the relaxation mode formations are structurally the same, using integer or non-integer models. In fact, there is no difference between our proposed model and the Kelvin-Voigt one in the steady state.

V. CONCLUSION

The goal of this paper is to derive a dynamic model for soft tissue displacement, based on a non-integer variable order model for stress-strain relation. Such a model can be used for enhancing the accuracy of tissue behavior represented in surgery simulators.

As shown, the concept of the non-integer modeling of the tissue is validated using an experimental study. Also, it leads to the general state-space equations of the soft tissue displacement. The proposed model has been compared to the Kelvin-Voigt model based analytical tissue dynamic model. Bringing all the results together, one may conclude that:

1. The proposed model is the same as the Kelvin-Voigt model in the steady state.
2. In the transient mode, the proposed model is simpler than the previous ones and it can be described in a state space form.
3. Due to the parameter order, the model is more flexible in comparison with the Kelvin-Voigt model.

The identification approach used in this paper is an offline one based on Genetic Algorithm. Future researches may concentrate on developing real-time or adaptive identification methods to estimate the order and parameters. Also, the proposed model can be combined with the needle dynamical model to compose a model for describing needle-tissue interactions.

REFERENCES

[1] X. Liu, C. Wang, J. Fan, C. Yu, and Y. Fan, "A innovative model for feedback force in interaction with soft tissue during simulation of surgery," in *in: World Congress on Medical Physics and Biomedical Engineering May 26-31, 2012, Beijing, China*. Springer, 2013, pp. 2162–2165.

[2] T. Yang, H. Yin, X. Zhao, J. Han, and W. Xu, "Interaction modeling and simulation of a flexible needle insertion into soft tissues," in *ISR/Robotik 2014; 41st International Symposium on Robotics; Proceedings of*. VDE, 2014, pp. 1–6.

[3] D. Liang, S. Jiang, Z. Yang, and X. Wang, "Simulation and experiment of soft-tissue deformation in prostate brachytherapy," *Proceedings of the Institution of Mechanical Engineers, Part H: Journal of Engineering in Medicine*, vol. 230, no. 6, pp. 532–544, 2016.

[4] H.-W. Nienhuys and A. F. van der Stappen, "A computational technique for interactive needle insertions in 3d nonlinear material," in *in: Robotics and Automation, 2004. Proceedings. ICRA'04. 2004 IEEE International Conference on*, vol. 2. IEEE, 2004, pp. 2061–2067.

[5] H. Delingette and N. Ayache, "Soft tissue modeling for surgery simulation," *Computational models for the human body*, pp. 453–550, 2004.

[6] H. Delingette, "Toward realistic soft-tissue modeling in medical simulation," *in: Proceedings of the IEEE*, vol. 86, no. 3, pp. 512–523, 1998.

[7] S. H. Pun, Y. M. Gao, P. Mak, M. I. Vai, and M. Du, "Quasi-static modeling of human limb for intra-body communications with experiments," *IEEE Trans. Inf. Technol. Biomed.*, vol. 15, no. 6, pp. 870–876, 2011.

[8] A. L. Robert, G. Chagnon, I. Bricault, P. Cinquin, and A. Moreau-Gaudry, "A generic three-dimensional static force distribution basis for a medical needle inserted into soft tissue," *J. Mech. Behav. Biomed. Mater.*, vol. 28, pp. 156–170, 2013.

[9] M. Khadem, C. Rossa, R. S. Sloboda, N. Usmani, and M. Tavakoli, "Mechanics of tissue cutting during needle insertion in biological tissue," *IEEE Robotics and Automation Letters*, vol. 1, no. 2, pp. 800–807, 2016.

[10] C. Rossa, R. Sloboda, N. Usmani, and M. Tavakoli, "Estimating needle tip deflection in biological tissue from a single transverse ultrasound image: application to brachytherapy," *International journal of computer assisted radiology and surgery*, pp. 1–13, 2016.

[11] C.-J. Chuong, M. Ma, R. C. Eberhart, and P. Zimmern, "Viscoelastic properties measurement of the prolapsed anterior vaginal wall: a patient-directed methodology," *Eur. J. Obstet. Gyn. R. B.*, vol. 173, pp. 106–112, 2014.

[12] C. Jamison, R. Marangoni, and A. Glaser, "Viscoelastic properties of soft tissue by discrete model characterization," *J. Manuf. Sci. Eng.*, vol. 90, no. 2, pp. 239–247, 1968.

[13] M.-A. Abellan, J.-M. Bergheau, and H. Zahouani, "Comparison of different viscoelastic models for the characterisation of mechanical properties of human skin in vivo by indentation test," *Comput. Meth. Biomech. Biomed. Eng.*, vol. 17, no. sup1, pp. 22–23, 2014.

[14] J. J. Shen, C. G. Li, H. T. Wu, and M. Kalantari, "Fractional order viscoelasticity in characterization for atrial tissue," *Korea-Aust. Rheol. J.*, vol. 25, no. 2, pp. 87–93, 2013.

[15] F. Mainardi, "An historical perspective on fractional calculus in linear viscoelasticity," *Fract. Calc. Appl. Anal.*, vol. 15, no. 4, pp. 712–717, 2012.

[16] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic press, 1998, vol. 198.

[17] S. S. Tabatabaei, M. J. Yazdanpanah, S. Jafari, and J. C. Sprott, "Extensions in dynamic models of happiness: effect of memory," *International Journal of Happiness and Development*, vol. 1, no. 4, pp. 344–356, 2014.

[18] C. F. Coimbra, "Mechanics with variable-order differential operators," *Annalen der Physik*, vol. 12, no. 11-12, pp. 692–703, 2003.

[19] L. E. Ramirez and C. F. Coimbra, "A variable order constitutive relation for viscoelasticity," *Annalen der Physik*, vol. 16, no. 7-8, pp. 543–552, 2007.

[20] H. Sun, W. Chen, and Y. Chen, "Variable-order fractional differential operators in anomalous diffusion modeling," *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 21, pp. 4586–4592, 2009.

[21] A. Bhrawy and M. Zaky, "Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation," *Nonlinear Dynamics*, vol. 80, no. 1-2, pp. 101–116, 2015.

[22] P. Chadwick, *Continuum mechanics: concise theory and problems*. Courier Corporation, 2012.

[23] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier Science Limited, 2006, vol. 204.

[24] V. Sladek, J. Sladek, S. Atluri, and R. Van Keer, "Numerical integration of singularities in meshless implementation of local boundary integral equations," *Comput. Mech.*, vol. 25, no. 4, pp. 394–403, 2000.

[25] W.-M. Ni, *The mathematics of diffusion*. siam, 2011, vol. 82.

[26] W. Maurel, Y. Wu, D. Thalmann, and N. M. Thalmann, *Biomechanical models for soft tissue simulation*. Springer, 1998.