Robust Observer Design for Continuous-Time and Sampled-Data Nonlinear Systems

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Problem Definition

An observer (state estimator) of a dynamic system:

\[ S : f(\dot{x}(t), x(t), u(t)) = 0, \]
\[ y(t) = h(x(t), u(t)), \]

is an auxiliary dynamical system whose inputs are both input and output vectors of the system to be observed and whose output vector is the estimated state:

\[ O : \hat{f}(\dot{z}(t), z(t), u(t), y(t)) = 0, \]
\[ \hat{x}(t) = \hat{h}(z(t), u(t), y(t)), \]

such that: \( \|e(t)\| \triangleq \|x(t) - \hat{x}(t)\| = 0, \) when \( t \to \infty. \)

- The general implicit ODEs contain state-space and descriptor representations.
Areas of Application

- Observer Based Feedback Control
- Fault Detection
- Synchronization of Chaotic Systems
- Monitoring
- Signal Processing (Filtering, Smoothing, Prediction)
The original Problem

A nominal Lipschitz system with no uncertainty or disturbance

\[
\dot{x}(t) = Ax(t) + \Phi(x, u) \quad A \in \mathbb{R}^{n \times n},
\]
\[
y(t) = Cx(t) \quad C \in \mathbb{R}^{n \times p},
\]
\[
\|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leq \gamma\|x_1 - x_2\| \quad \forall x_1(k), x_2(k) \in \mathcal{D}.
\]

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + \Phi(\hat{x}, u) + L(y - C\hat{x}). \tag{1}
\]

**Thau (1971):** If a gain matrix \( L \) can be chosen such that

\[
(A - LC)^T P + P^T (A - LC) = -Q, \quad P, Q > 0,
\]
\[
\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}
\]

then (1) yields asymptotically stable estimates.
**Methodology: LMI Optimization**

- Inequalities which are linear (affine) in a set of matrix variables.
- Many control problems can be formulated as a set of LMIs.
- Convexity of the solution space. (optimization over convex cones).
- Efficient numerical solution (interior point methods).
Previous work on Nonlinear Filtering

- **Nonlinear Observers**: Observer Design for Lipschitz Systems
  - Raghvan et. al. (1994): Riccati approach, coordinate transformation
  - Pertew and Marquez (2006): Solution to Rajamani’s formulation, Dynamic observers

- **Robust Filtering**: Robust Nonlinear $H_{\infty}$ Filtering (for conventional state space)
  - Xu (2002, 2004): LMI based approach with known constant Lipschitz constant

- **Descriptor Systems**: Observer design for nonlinear descriptor systems has been studied by
  - Shiels (1996): Unknown input observers
  - Lu et. al. (2004): Robust Observers for Lipschitz systems
We study the robust filtering problem for (nonlinear) Lipschitz systems, with the following objectives:

- **Convergence**: In the absence of disturbances, the state observer \( \hat{x} \) should asymptotically converge to that of the plant \( x \).

- **Disturbance Attenuation**: Minimization of the effect of disturbances on the estimation error.

- **Robustness**: The above two objectives remain valid, even when in the presence of model uncertainties (in a form to be described).

- **Lipschitz constant**: The maximum admissible Lipschitz constant \( \gamma \) is optimized. In other words, the Lipschitz region is maximized.

- **Multiobjective Convex Optimization**: Simultaneous optimization over the Lipschitz constant and disturbance attenuation (if possible).
Robust $H_\infty$ Observer Design

$H_\infty$ Observer Synthesis

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \Phi(x, u) + Bw(t), \\
y(t) &= Cx(t) + Dw(t),
\end{align*}
\]

where $w(t) \in \mathcal{L}_2[0, \infty)$ is an unknown exogenous disturbance. Suppose that

\[z(t) = He(t),\]

stands for the controlled output for error state where $H$ is a known matrix.

- **$H_\infty$ Design**: The observer error dynamics is asymptotically stable and the following specified $H_\infty$ norm upper bound is simultaneously guaranteed.

\[\|z\| \leq \mu \|w\|.\]
Theorem 1. The observer error dynamics is (globally) asymptotically stable with decay rate $\beta$ and simultaneously maximized admissible Lipschitz constant $\gamma^*$ and minimized $\Sigma_2(w \rightarrow e)$ gain, $\mu^*$, if there exist fixed scalers $0 \leq \lambda \leq 1$ and $\beta > 0$, scalers $\alpha > 1$, $\varepsilon > 0$, $\xi > 0$ and $\zeta > 0$ and matrices $P > 0$ and $F$ such that the following LMI optimization problem has a solution.

$$\min \left[ \lambda \cdot \xi + (1 - \lambda) \zeta \right]$$

$A^T P + PA + 2\beta P - C^T F^T - FC < -\alpha I - \varepsilon I$,

$$\begin{bmatrix}
\frac{1-\tilde{\sigma}^2(H)}{2} \cdot \xi I & P \\
P & \frac{1-\tilde{\sigma}^2(H)}{2} \cdot \xi I
\end{bmatrix} > 0,$$

$$\begin{bmatrix}
H^T H + \frac{1}{2} (\xi - 2\alpha) I & I & PB - FD \\
I & -2\xi I & 0 \\
B^T P - D^T F^T & 0 & -\zeta I
\end{bmatrix} < 0,$$

Once the problem is solved,

$L = P^{-1} F$, $\gamma^* \triangleq \max(\gamma) = \xi^{-1}$, $\mu^* \triangleq \min(\mu) = \sqrt{\zeta}$. 
LMI Optimization based observer Design

The proposed LMIs are linear in the Lipschitz constant:

- Lipschitz constant is considered unknown (it is one of the LMI variables)
- Optimization over Lipschitz constant to obtain its maximum admissible value
- Efficient numerical solution with no regularity assumptions
- Robustness against nonlinear uncertainty
- Guaranteed decay rate (exponential convergence)
- Simultaneous optimization over admissible Lipschitz constant and disturbance attenuation level via multiobjective convex optimization.
Extensions to Systems with Uncertainty

\[ (\Sigma_s) : \dot{x}(t) = (A + \Delta A(t))x(t) + \Phi(x, u) + Bw(t), \]
\[ y(t) = (C + \Delta C(t))x(t) + \Psi(x, u) + Dw(t), \]
\[ \Phi(0, u^*) = \Psi(0, u^*) = 0, \]
\[ \|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leq \gamma_1 \|x_1 - x_2\| \ \forall x_1, x_2 \in \mathcal{D}, \]
\[ \|\Psi(x_1, u^*) - \Psi(x_2, u^*)\| \leq \gamma_2 \|x_1 - x_2\| \ \forall x_1, x_2 \in \mathcal{D}. \]

- **Observer structures:**
  - Static-Gain Observer (Luenberger)
  - Dynamic Observer (Full state space representation)

- **Uncertainties:**
  - Norm-Bounded parametric uncertainty
  - Lipschitz nonlinear additive uncertainty
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Descriptor Systems - General Comments

- Descriptor systems, also referred to as
  - Singular systems, and/or
  - Differential algebraic equation systems,
  can characterize a more general class of systems than “classic” (conventional) system models.

- Descriptor systems arise naturally when modeling certain classes of
  - Power systems.
  - Large scale systems.
  - Networks and circuits.
  - Social and economic sciences.

In this work, our interest is in the filtering (observer design) problem for descriptor systems.
The General Lipschitz Class

DAE systems:
- implicit form: \( f(x(t), \dot{x}(t), y(t), u(t), w(t)) = 0 \)
- semi-explicit form:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), z(t), u(t), w(t)), \\
g(x(t), z(t), u(t), w(t)) &= 0
\end{align*}
\]

The Lipschitz system

\[
(\Sigma_s) : \begin{align*}
E \dot{x}(t) &= (A + \Delta A(t))x(t) + \Phi(x, u) + Bw(t) \\
y(t) &= (C + \Delta C(t))x(t) + \Psi(x, u) + Dw(t)
\end{align*}
\]

- \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \)
- \( \Phi(x, u) \) and \( \Psi(x, u) \) contain nonlinearities of second or higher order.
- \( A, B, C \) and \( D \) are real matrices of compatible dimensions.
- \( E \) may be singular: \( 0 < \text{rank}(E) = s < n \).
- \( x(0) = x_0 \) is a consistent (unknown) initial condition.
Assumptions

- where
  \[
  \Phi(0, u^*) = 0, \Psi(0, u^*) = 0,
  \]
  \[
  \|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| \leq \gamma_1 \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{D}
  \]
  \[
  \|\Psi(x_1, u^*) - \Psi(x_2, u^*)\| \leq \gamma_2 \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{D}
  \]

- Uncertainty

\[
\begin{bmatrix}
\Delta A(t) \\
\Delta C(t)
\end{bmatrix} = \begin{bmatrix}
M_1 \\
M_2
\end{bmatrix} F(t) N
\]

where \(M_1, M_2\) and \(N\) are known real constant matrices and \(F(t)\) is an unknown real-valued time-varying matrix satisfying

\[
F^T(t) F(t) \leq I, \quad \forall t \in [0, \infty).
\]
Problem Statement

- **General observer structure:** Define the following observer (filter)

\[
\begin{align*}
(\Sigma_o) : \dot{E}x_F(t) &= A_F x_F(t) + B_F y(t) + \Phi(x_F, u) + \mathcal{E}_1 \psi(x_F, u) \\
z_F(t) &= C_F x_F(t) + D_F y(t) + \mathcal{E}_2 \psi(x_F, u).
\end{align*}
\]

The proposed framework can capture both dynamic and static-gain observer structures by proper selection of \(\mathcal{E}_1\) and \(\mathcal{E}_2\).

We can distinguish the following special cases:
Problem Statement (cont’d)

- **Dynamical Observer**: Choosing $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = I$ structure:

  \[
  \dot{\mathbf{x}}_F(t) = A_F \mathbf{x}_F(t) + B_F y(t) + \Phi(\mathbf{x}_F, u) \\
  z_F(t) = C_F \mathbf{x}_F(t) + D_F y(t) + \Psi(\mathbf{x}_F, u).
  \]

- **Static-gain**: Letting $\mathcal{E}_1 = -L$, $\mathcal{E}_2 = 0$, and

  \[
  A_F = A - LC, \quad B_F = L, \quad C_F = I, \quad D_F = 0,
  \]

  \[
  \dot{\mathbf{x}}_F(t) = A \mathbf{x}_F(t) + \Phi(\mathbf{x}_F, u) + L[y(t) - C \mathbf{x}_F(t) - \Psi(\mathbf{x}_F, u)] \\
  z_F(t) = \mathbf{x}_F(t).
  \]
Our goal is to design an observer (i.e. find matrices $A_F$, $B_F$ $C_F$ and $D_F$) to achieve the following objectives:

- **Convergence**: When $w = 0$, $x_F \to x$ as $t \to \infty$
- **Filtering**: Minimize $\mu$ so that the effect of disturbances on the estimation error is minimized (in $L_2$ sense)

$$\|e\| \leq \mu \|w\| \quad w \in L_2$$

The effect of disturbances is minimized. $e = Hx - z_F$

- **Lipschitz constant**: The maximum admissible Lipschitz constant $\gamma$ is optimized.
- **Robustness**: The previous objectives are satisfied in the present of parametric and nonlinear uncertainties.
Main Result

**Theorem 2.** Consider the Lipschitz nonlinear system \((\Sigma_s)\) along with the general observer \((\Sigma_o)\). The observer error dynamics is (globally) asymptotically stable with maximum admissible Lipschitz constant, \(\gamma^*\), and guaranteed \(L_2(w \to e)\) gain, \(\mu\), if there exists a fixed scalar \(\mu > 0\), scalars \(\varepsilon_1 > 0\), \(\varepsilon_2 > 0\), \(\alpha_1 > 0\) and \(\alpha_2 > 0\) and matrices \(C_F\), \(D_F\), \(P_1\), \(P_2\), \(G_1\) and \(G_2\), such that the following optimization problem has a solution:

\[
\min(2\alpha_1 + \alpha_2)
\]
such that
Continuous-Time DAEs: Descriptor Systems

\[ \Xi_1 = \begin{bmatrix}
\Pi_1 & \Pi_2 & 0 & \Pi_3 & \Pi_4 \\
\ast & \Pi_5 & \Pi_6 & 0 & 0 \\
\ast & \ast & \Pi_7 & 0 & 0 \\
\ast & \ast & \ast & \Pi_8 & 0 \\
\ast & \ast & \ast & \ast & \Pi_9
\end{bmatrix} < 0 \]

\[ \Xi_2 = \begin{bmatrix}
\varepsilon_2 I & 0 & -D_F M_2 \\
\ast & I & 0 \\
\ast & \ast & I
\end{bmatrix} > 0 \]

\[ \Xi_3 = \begin{bmatrix}
\alpha_1 I & D_F^T \\
\ast & \alpha_1 I
\end{bmatrix} > 0 \]

\[ \Xi_4 = \begin{bmatrix}
I & I - P_1^T \\
\ast & I
\end{bmatrix} > 0 \]

\[ E^T P_1 = P_1^T E \geq 0 \]

\[ E^T P_2 = P_2^T E \geq 0 \]
where the elements of $\Xi_1$ are as defined in the following,

\[
\Lambda_1 = G_1^T + G_1, \quad \Lambda_2 = A^T P_2 + P_2 A + (\epsilon_1 + \epsilon_2) N^T N \quad \text{and} \quad \Lambda_3 = H^T - C^T D_F^T.
\]

\[
\Pi_1 = \begin{bmatrix}
\Lambda_1 & G_2 C & I \\
* & \Lambda_2 & 0 \\
* & * & -\alpha_2 I \\
\end{bmatrix}, \quad \Pi_4 = \begin{bmatrix}
G_2 D \\
P_2 B \\
0 \\
\end{bmatrix},
\]

\[
\Pi_2 = \begin{bmatrix}
0 & G_2 M_2 & -C_F^T \\
0 & P_2 M_1 & \Lambda_3 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \Pi_6 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -D_F M_2 \\
\end{bmatrix},
\]

\[
\Pi_3 = \begin{bmatrix}
0 & G_1 & P_1 & P_1 \delta_1 \\
P_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
Continuous-Time DAEs: Descriptor Systems

\[ \Pi_5 = \text{diag}(-\varepsilon_1 I, -\varepsilon_1 I, -\frac{1}{3} I), \Pi_7 = \text{diag}(-\frac{1}{3} \varepsilon_2 I, -\frac{1}{3} \varepsilon_2 I), \]

\[ \Pi_8 = \text{diag}(-I, -I, -I, -I), \Pi_9 = \begin{bmatrix} -\mu^2 I & -D^T D^T_F \\ \star & -\frac{1}{3} I \end{bmatrix}. \]

Once the problem is solved:

\[ A_F = P_1^{-1} G_1, \quad B_F = P_1^{-1} G_2 \]

\[ C_F \] and \[ D_F \] are directly obtained,

\[ \alpha_1^* \triangleq \min(\alpha_1), \quad \alpha_2^* \triangleq \min(\alpha_2), \]

\[ \gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha_2^*(1 + 3\|\mathcal{E}_2\|^2 + 3\alpha_1^*^2)}}. \]
The LMI conditions in Theorem 2 present two non-strict conditions and is therefore a semidefinite programming (SDP) with a quasi-convex solution space.

This SDP problem can be solved directly using free available software packages such as YALMIP, (Lofberg, 2004), which is available for free download.

More efficient numerical solutions are restricted to strict LMI problems (MATLAB LMI solver).

The SDP problem of Theorem 1 can be converted into a strict LMI optimization problem using a suitable transformations.
Corollary 1. Consider the Lipschitz nonlinear system \((\Sigma_s)\) along with the general observer \((\Sigma_o)\). The observer error dynamics is (globally) asymptotically stable with maximum admissible Lipschitz constant, \(\gamma^*\), and guaranteed \(\mathcal{L}_2(w \to e)\) gain, \(\mu\), if there exists a \(\mu > 0\), scalars \(\varepsilon_1 > 0\), \(\varepsilon_2 > 0\), \(\alpha_1 > 0\) and \(\alpha_2 > 0\) and matrices \(C_F\), \(D_F\), \(X_1 > 0\), \(X_2 > 0\), \(Y_1\), \(Y_2\), \(G_1\) and \(G_2\), such that the following LMI optimization problem has a solution.

\[
\min(2\alpha_1 + \alpha_2) \quad \Xi_1 < 0, \quad \Xi_2 > 0, \quad \Xi_3 > 0 \\
\Xi_4 = \begin{bmatrix} I & I - P^T_1 \\
* & I \\
- & I \\
\end{bmatrix} > 0,
\]
where, $\Xi_1, \Xi_2, \Xi_3$ and $\Xi_4$ are as in Theorem 1 with

\[
P_1 = X_1E + E_\perp Y_1
\]
\[
P_2 = X_2E + E_\perp Y_2.
\]

Once the problem is solved:

\[
A_F = P_1^{-1}G_1 = (X_1E + E_\perp Y_1)^{-1}G_1,
\]
\[
B_F = P_1^{-1}G_2 = (X_1E + E_\perp Y_1)^{-1}G_2,
\]
\[
C_F \text{ and } D_F \text{ are directly obtained,}
\]
\[
\alpha_1^* \triangleq \min(\alpha_1), \quad \alpha_2^* \triangleq \min(\alpha_2),
\]
\[
\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha_2^*(1 + 2\|E_2\|^2 + 2\alpha_1^2)}}.
\]
Observer Properties

- The General (dynamical) Observer Framework:
  - The use of a dynamical observer provides additional degrees of freedom in the design with respect to static-gain observers.
  - The additional freedom permits stabilizing the error dynamics in systems where static-gain observers cannot be found.
  - In cases where both dynamical and static-gain observers exist, the maximum admissible Lipschitz constant obtained in the dynamical case can be made much larger than with the static gain.

- The LMI problem(s) proposed here are linear in both the Lipschitz constant and the disturbance attenuation level. Both can be used as optimization variables.

- Maximizing of the Lipschitz constant is important. Doing so provides additional robustness against nonlinear uncertainty (as will be discussed shortly).
Robustness Against Nonlinear Uncertainty

Assume a nonlinear uncertainty as follows

\[
\begin{align*}
\Phi_\Delta(x, u) &= \Phi(x, u) + \Delta \Phi(x, u) \\
\Psi_\Delta(x, u) &= \Psi(x, u) + \Delta \Psi(x, u) \\
\dot{x}(t) &= (A + \Delta A)x(t) + \Phi_\Delta(x, u) + Bw(t) \\
y(t) &= (C + \Delta C)x(t) + \Psi_\Delta(x, u) + Dw(t),
\end{align*}
\]

where \( \Phi_\Delta \) and \( \Psi_\Delta \) are uncertain nonlinear functions and \( \Delta \Phi \) and \( \Delta \Psi \) are unknown nonlinear uncertainties. Suppose that

\[
\begin{align*}
\| \Delta \Phi(x_1, u) - \Delta \Phi(x_2, u) \| &\leq \Delta \gamma_1 \| x_1 - x_2 \|, \ \forall x_1, x_2 \in \mathcal{D}, \\
\| \Delta \Psi(x_1, u) - \Delta \Psi(x_2, u) \| &\leq \Delta \gamma_2 \| x_1 - x_2 \|, \ \forall x_1, x_2 \in \mathcal{D}.
\end{align*}
\]
Proposition 1. Suppose that the actual Lipschitz constant of the nonlinear functions Φ and Ψ are $\gamma_1$ and $\gamma_2$, respectively and the maximum admissible Lipschitz constant achieved by Theorem 2 (Corollary 1), is $\gamma^*$. Then, the observer designed based on Theorem 2 (Corollary 1), can tolerate any additive Lipschitz nonlinear uncertainties over Φ and Ψ with Lipschitz constants $\Delta \gamma_1$ and $\Delta \gamma_2$ such that

$$\sqrt{(\gamma_1 + \Delta \gamma_1)^2 + (\gamma_2 + \Delta \gamma_2)^2} \leq \gamma^*$$
A new nonlinear $H_\infty$ dynamical observer design method for a class of nonlinear descriptor systems was proposed.

The dynamical structure has additional degrees of freedom with respect to constant-gain observers.

Constant-gain observers can be obtained as a special case.

Two solutions were presented: (i) strict LMI and (ii) SDP.

The results are robust with respect to parametric uncertainties in the state space matrices.

The LMIs in the solution are linear in the admissible Lipschitz constant and the disturbance attenuation level, thus allowing both to be LMI optimization variables.

Maximization of the Lipschitz constant was exploited to provide robustness with respect to nonlinear uncertainty.
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Pervious Work

- Observer design for nonlinear sampled-data systems with approximate models:
  - Kokotovic et. al. (1999): Theoretical development of semiglobal practical convergence
  - Arcak and Nesic (2004): Observer design framework (sufficient conditions)

- Robust Filtering:
  - De Souza and Xie (1996): Riccati approach
  - Xu et. al. (2004): LMI approach
Discrete-Time Lipschitz Class

Exact Discrete-Time Models

\[ x(k + 1) = A_d x(k) + F(x(k), u(k)) \]
\[ y(k) = C_d x(k) \]
\[ \| F(x_1(k), u^*(k)) - F(x_2(k), u^*(k)) \| \leq \gamma_d \| x_1(k) - x_2(k) \| \quad \forall x_1(k), x_2(k) \in \mathcal{D} \]
\[ \hat{x}(k + 1) = A_d \hat{x}(k) + F(\hat{x}(k), u(k)) + L(y(k) - C_d \hat{x}(k)) \]

Statement of Results:

- No bound on Lipschitz Constant: Lipschitz constant is assumed to be known
  - Asymptotically convergence of the observer
  - \( H_\infty \) synthesis with optimized disturbance attenuation
- Lipschitz constant is assumed to be less that one: Lipschitz constant is unknown
  - Optimization over Lipschitz constant
  - \( H_\infty \) synthesis with optimized Lipschitz constant and/or disturbance attenuation
The Base Result: General Case

**Theorem 3.** Consider the system with given Lipschitz constant $\gamma_d$. The observer error dynamics is (globally) asymptotically stable if there exist scalar $\varepsilon > 0$, fixed matrix $Q > 0$ and matrices $P > 0$ and $G$ such that the following set of LMIs has a solution

$$
\Xi \triangleq \begin{bmatrix}
    P - Q - \varepsilon I & A_d^T P - C_d^T G^T \\
    P A_d - G C_d & P
  \end{bmatrix} > 0
$$

$$
\begin{bmatrix}
    \Psi_1 I & P \\
    P & \Psi_1 I
  \end{bmatrix} > 0
$$

where

$$
\Psi_1 = -\lambda_{\text{max}}(Q) + \sqrt{\lambda_{\text{max}}^2(Q) + \frac{1}{\gamma_d^2} \lambda_{\text{min}}^2(Q)}
$$

$$
\gamma_d + 2
$$

$P$, $G$, and $\varepsilon$ are the LMI variables and $Q$ is a design parameter to be chosen. Once the problem is solved: $L = P^{-1} G$. 
The Base Result: Special Case $\gamma_d < 1$

**Theorem 4.** The observer error dynamics is (globally) asymptotically stable with maximum admissible Lipschitz constant $\gamma^*_d$, if there exist scalars $\varepsilon > 0$, $\xi > 1$, fixed matrix $Q > 0$ and matrices $P > 0$ and $G$ such that the following LMI optimization problem has a solution

$$\min(\xi)$$

$$\Xi > 0$$

$$\begin{bmatrix} \psi_2 I & P \\ P & \psi_2 I \end{bmatrix} > 0$$

where, $\Xi$ is as defined in previous theorem and

$$\psi_2 = \frac{1}{3} [\lambda_{\min}(Q) \xi - \lambda_{\max}(Q)].$$

Once the problem is solved: $L = P^{-1}G$, $\gamma^*_d \triangleq \max(\gamma_d) = \frac{1}{\xi}$. 
Robust Observer Design

**Approximate Discrete-Time Models I**

- Locally Lipschitz with Lipschitz constant $\gamma_c$

\[
\begin{align*}
\dot{x} &= Ax + f(x, u) \\
y &= Cx
\end{align*}
\]

- Exact discretization:

\[
\begin{align*}
x(k + 1) &= A_d x(k) + F^e_T(x(k), u(k)) \\
y(k) &= C_d x(k)
\end{align*}
\]

- Approximate discretization:

\[
\begin{align*}
x^a(k + 1) &= A^a_d x(k) + F^a_T(x(k), u(k)) \\
y(k) &= C_d x^a(k)
\end{align*}
\]
Approximate Discrete-Time Models II

- Euler approximate discretization:
  \[ A_d^a = I + AT \]
  \[ F_I^a(x^a(k), u(k)) = Tf(x^a(k), u(k)) \]
  \[ \gamma_d = T\gamma_c \]

  \[ \hat{x}^a(k+1) = A_d^a\hat{x}^a(k) + F_I^a(\hat{x}^a(k), u(k)) + L(y(k) - C_d\hat{x}^a(k)) \]

  Euler approximation with Lipschitz constant less that one

- Semiglobal practical convergence of the observer
- Optimization over Lipschitz constant
- \( H_\infty \) synthesis with optimized Lipschitz constant and/or disturbance attenuation
**Definition 1.** The family $F^a_T(x, u)$ is said to be (one-step) consistent with $F^e_T(x, u)$ if for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a class-$\mathcal{K}$ function $\rho(.)$ and a constant $T_0 > 0$ such that for all $(x, u) \subset \Omega$ from and all $T \in (0, T_0]$,

$$\|F^e_T(x, u) - F^a_T(x, u)\| \leq T \rho(T).$$

**Definition 2.** We say that the observer is semiglobal practical in $T$, if there exists a class-$\mathcal{KL}$ function $\beta(., .)$ such that for any $d_1 > d_2 > 0$ and compact sets $X \in \mathbb{R}^n, U \in \mathbb{R}^m$, we can find a $T^* > 0$ with the property that for all $T \in (0, T^*]$,

$$\|\hat{x}^a(0) - x(0)\| \leq d_1, \text{ and } x_k \in X, u_k \in U, \forall k \geq 0,$$

imply

$$\|\hat{x}^a_k - x_k\| \leq \beta(\|\hat{x}^a(0) - x(0)\|, kT) + d_2.$$
Approximate Discrete-Time Models IV

**Theorem 5.** The observer designed using the Euler-approximate model is semiglobal practical in $T$ with the maximum admissible Lipschitz constant $\gamma^*_d$, if there exist scalars $\epsilon > 0$, $\xi > 1$, fixed matrix $Q > 0$ and matrices $P > 0$ and $G$ such that the LMI optimization problem in Theorem 4 has a solution where $\lambda_{\text{min}}(Q) = T$.

- For the Euler discretization proper selection of the sampling time guarantees $\gamma_d = T\gamma_c < 1$.
- Robustness against nonlinear uncertainty is similar to that of continuous-time domain; $\Delta\gamma_d \leq \gamma^*_d - \gamma_d$.
- The $H_\infty$ filter is synthesized in all cases as a trivial extension to the base results exploiting some matrix manipulation and LMI techniques.
Example: Van der Pole Oscillator I

Consider the Van der Pol oscillator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_2^2 x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w
\]

\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} x,
\]

where \( w \) is an unknown Gaussian disturbance with zero mean and standard deviation 0.05.

- This system has an unstable equilibrium point at the origin and a stable limit cycle.
- Its Lipschitz constant depends on the compact region and can be computed as \( \gamma_c = \max(|x_1|\sqrt{4x_2^2 + x_1^2}) \).
- We assume \( H = 0.25I \), and \( Q = T I \)
- \( T = 0.1 \text{ sec} \) we get \( \gamma_d^* = 0.7824, \mu^* = 0.8756 \) and \( L = \begin{bmatrix} -0.8382 & 1.0938 \end{bmatrix}^T \).
Example: Van der Pole Oscillator II

- $T = 0.05 \text{ sec}$ we get $\gamma_d^* = 0.4352$, $\mu^* = 0.9097$ and $L = [-0.9231 \quad 1.0962]^T$. 

![Graph showing the comparison of true and estimated states for Van der Pole Oscillator with Euler method at different time steps. The graphs illustrate the accuracy of the Euler method for small time steps.](image-url)
Robust Observer Design

Example: Van der Pole Oscillator III

\[ \text{abs}(x_1) (4 x_2^2 + x_1^2)^{1/2} - 978/125 = 0 \]
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Summary

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Discrete-Time Uncertain Lipschitz System

$$\begin{align*}
(\sum_1) & : \quad x(k+1) = (A + \Delta A(k))x(k) + \Phi(x,u) + Bw(k) \\
y(k) & = (C + \Delta C(k))x(k) + Dw(k)
\end{align*}$$

- Disturbance: $w(k) \in \ell_2[0,\infty)$
- Uncertainty:
  $$\begin{align*}
  \Delta A(k) & = M_1 F(k) N_1 \\
  \Delta C(k) & = M_2 F(k) N_2, \quad \forall k \quad F^T(k)F(k) \leq I
  \end{align*}$$
- $H_\infty$ performance: $z(k) = He(k), \quad \|z\| \leq \mu \|w\|$
The robust $H_\infty$ filtering problem is solved for the given class of systems. This is similar to the continuous-time time but nontrivial to cast in the LMI form.

Simultaneous Optimization over the Lipschitz constat and the disturbance attenuation level.

Nonlinear uncertainty: robustness results shown for the continuous-time are also valid in the discrete-time.

We have provided lemma for robust stability of the given class of systems which is used as a corner stone for both out filtering and SOF stabilization results (discussed in the next section).
Consider a system of the form of \((\Sigma_1)\) where

\[
A = \begin{bmatrix}
0.15 & 0.3 & 0.1 \\
0.25 & 0.15 & 0.2 \\
-0.1 & 0.25 & 0.1
\end{bmatrix}, \quad \Phi(x) = \begin{bmatrix}
0.2x_2x_3 \\
0.3x_2\sin(x_2) \\
0.3x_3\sin(x_3)
\end{bmatrix}, \quad C = \begin{bmatrix}
0.5 & 0.2 & 0 \\
0 & 0.2 & 0.1
\end{bmatrix}
\]

\[
M_1 = \begin{bmatrix}
0.1 & 0 \\
0.1 & 0.1 \\
0.1 & 0.1
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
-0.1 & 0.2 \\
0.05 & 0.1
\end{bmatrix}, \quad N = \begin{bmatrix}
0.3 & 0.15 & 0.1 \\
0.1 & 0.2 & 0.1
\end{bmatrix},
\]

Assuming

\[
\mu = 2, \quad \beta = 0.05, \quad B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \quad D = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^T, \quad H = 0.5I_3.
\]
Systems with Uncertainty

Example II

With $\lambda = 0.99$ we get

$$\gamma^* = 0.4762$$
$$\mu^* = 2.6200$$

$$L = \begin{bmatrix}
0.2934 & 1.2408 \\
0.4519 & 0.0877 \\
-0.1903 & 1.4930
\end{bmatrix}.$$  

Now, with $\lambda = 0.8$ we get

$$\gamma^* = 0.4559$$
$$\mu^* = 1.9750$$

$$L = \begin{bmatrix}
0.1845 & 1.4658 \\
0.3497 & 0.7353 \\
-0.3168 & 1.7043
\end{bmatrix},$$
Systems with Uncertainty

Example III

Figure: The true and estimated states of the Example
Figure: $\gamma^*$, $\mu^*$ and $\bar{\sigma}(L)$, and the optimal trade-off curve
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4. Summary
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Most of the control literature deals with either

1. State feedback control, or.
2. Dynamic output feedback.

1. Requires measuring every state and is therefore, impractical.
2. Results in higher order controller and might be undesirable in industrial applications.

As a consequence, many researchers have tried to characterize the problem of finding a stabilizing state output feedback (SOF) control law. This problem is, however, non-convex, and therefore difficult to solve both analytically and numerically.
Previous work

- Structural pole assignment (Yang et. al, Automatica, 2006, Franze et. al, IEE, 2006).
- Riccati-based approach (Trofino-Neto et. al IEEE TAC, 1993, Kucera et. al, 1995)
- Min-max optimization (Geromel et. al, IEEE TAC, 1998).
- Iterative LMI approach (Garcia et. al, IEEE TAC, 2001, Bara et. al, IEEE TAC, 2005)

The original (non-convex) problem can be directly formulated using bilinear matrix inequalities (BMIs) - whose solution is NP-hard.

- Semi-definite “cone” programming approach to solve the BMI problem (Mesbahi et. al, SIAM, 2000)
In this work

- Consider Lipschitz (nonlinear) systems with parametric uncertainties.

- Present a non-iterative method that provides a non restrictive solution (compared to previous work).

- Our solution is robust with respect to:
  - parametric uncertainties in the state space matrices, and
  - nonlinear uncertainties (with explicit bounds)

- Has optimal disturbance attenuation, in the $H_\infty$ sense.
The system: We consider the system “Σ” defined as follows

\[
\begin{align*}
x(k+1) &= (A + \Delta A(k))x(k) + \Phi(x, u) + B_1 u(k) + B_2 w(k) \\
y(k) &= (C + \Delta C(k))x(k) + Dw(k) \\
z(k) &= Hx(k) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p,
\end{align*}
\]

where we assume that

\[
\text{rank}(B_1) = m < n, \quad \text{and} \quad \text{rank}(C) = p < n
\]

The nonlinearity \( \Phi \) satisfies a local Lipschitz continuity condition of the form

\[
\begin{align*}
\|\Phi(0, u^*)\| &= 0 \\
\|\Phi(x_1, u^*) - \Phi(x_2, u^*)\| &\leq \gamma \|x_1 - x_2\| \quad \forall x(k) \in D
\end{align*}
\]
$w(k)$ represents the effect of exogenous disturbance and is assumed to satisfy

$$w(k) \in \ell_2[0, \infty)$$

i.e. $w(k)$ is an unknown function with bounded energy. Finally, the parametric uncertainty is assumed to have the following structure:

$$\begin{bmatrix} \Delta A(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(k)N$$

where $M_1$, $M_2$ and $N$ are known real constant matrices and $F(k)$ is an unknown real-valued time-varying matrix satisfying (Khargonekar et al, 1990)

$$\forall k, \quad F^T(k)F(k) \leq I.$$
The objective of this work is to find a static output feedback control law

\[ u = Ky \]

such that

- Asymptotically stabilizes the system \( \Sigma \) when \( w = 0 \).
- The solution minimizes the effect of disturbances:

\[ \|z\| \leq \mu \|w\|. \]

- The solution is robust with respect to parametric uncertainties in the state space matrices.
- The maximum admissible Lipschitz constant is optimized.
- The solution is robust with respect to nonlinear uncertainties of Lipschitz form.
Our first solution is summarized in the following Theorem:

Consider the nonlinear uncertain system \( \Sigma \)

**Theorem 6.** The output feedback \( u = Ky \) robustly asymptotically stabilizes this system with \( \|z\| \leq \mu \|w\| \) and maximum admissible Lipschitz constant \( \gamma^* \), if there exist scalars \( \epsilon_1 > 0, \alpha > 0 \), and matrices \( P > 0 \) and \( G \) such that the following LMI optimization problem has a solution:
Strict LMI Solution (cont’d)

\[
\begin{align*}
\min (\alpha + \varepsilon_1)
\end{align*}
\]

\[
\begin{align*}
s.t. \\
\begin{bmatrix}
\Lambda_1 & I & \Lambda_2 & 0 & 0 \\
* & -\alpha I & 0 & 0 & 0 \\
* & * & -\frac{1}{2}P & P & PM_1 + GM_2 \\
* & * & * & P - 2\varepsilon_1 I & 0 \\
* & * & * & * & -\varepsilon_2 I \\
\end{bmatrix}
< 0
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
-\mu^2 I & B_2^T P + D^T G^T & B_2^T P + D^T G^T \\
* & -\frac{1}{2}P & 0 \\
* & * & -I \\
\end{bmatrix}
< 0
\end{align*}
\]
where $\Lambda_1 = H^T H - P + \varepsilon_2 N^T N$ and $\Lambda_2 = A^T P + C^T G^T$. Once the problem is solved

\[
\alpha^* \triangleq \min(\alpha), \quad \varepsilon_1^* \triangleq \min(\varepsilon_1)
\]

\[
\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1 + \varepsilon_1^*)}}, \quad K = B_1^T \overline{K}
\]

where $\overline{K}$ is an unknown matrix that can be found from:

\[
P B_1 B_1^T \overline{K} = G \Rightarrow (I_p \otimes P B_1 B_1^T) \text{vec}(\overline{K}) = \text{vec}(G)
\]

if and only if

\[
\text{rank}(I_p \otimes P B_1 B_1^T) = \text{rank}([I_p \otimes P B_1 B_1^T \text{vec}(G)]). \quad (\ast)
\]
Remarks

Positive elements of the solution:

- Theorem 6 solves the objective of this work:
  - If $w = 0$, then the origin is an asymptotically stable equilibrium point.
  - The LMIs in Theorem 6 are linear in $\alpha, \varepsilon_1,$ and $\zeta = \mu^2$, thus they can be used as constants, or as optimization variables.
  - This property permits minimization of the disturbance attenuation (minimizing $\zeta = \mu^2$ thus reducing $\|z\|$ via $\|z\| \leq \mu \|w\|$).
  - $\alpha + \varepsilon_1$ and $\zeta = \mu^2$ can be simultaneously optimized via multiobjective optimization.
  - Maximizing the Lipschitz constant provides robustness against nonlinear uncertainty (as will be shown shortly).
Remarks (cont’d)

Negative elements of the solution:

- The main shortcoming of Theorem 6 is that an exact solution for $K$ can only be found if the condition (*) is satisfied.
- This condition cannot be verified beforehand since we need to know $P$ and $G$.

Alternative solution

- An alternative solution can be found converting the original (non-convex) BMI into a convex Semi-definite programming problem.
- The price paid is an additional equality constraint in the optimization.
Corollary 2. Consider a nonlinear uncertain system of class \((\Sigma_1)\). The output feedback \(u = Ky\) robustly asymptotically stabilizes this system with \(\|z\| \leq \mu \|w\|\) and maximum admissible Lipschitz constant \(\gamma^*\), if there exist scalars \(\varepsilon_1 > 0, \varepsilon_1 > 0\) and \(\alpha > 0\) and matrices \(P > 0, Q\) and \(G\) such that the following SDP problem has a solution:
min(\(\alpha + \varepsilon_1\))

s.t.

\[ PB_1 = B_1 Q \]  \hspace{1cm} (2)

\[ \Pi_1 \triangleq \begin{bmatrix} I & I - Q \\ * & I \end{bmatrix} > 0 \]  \hspace{1cm} (3)

\[ \Pi_2 \triangleq \begin{bmatrix} -\mu^2 I & \Lambda_4 & \Lambda_4 \\ * & -\frac{1}{2}P & 0 \\ * & * & -I \end{bmatrix} < 0 \]  \hspace{1cm} (4)

\[ \begin{bmatrix} \Lambda_1 & I & \Lambda_3 & 0 & 0 \\ * & -\alpha I & 0 & 0 & 0 \\ * & * & -\frac{1}{2}P & P & PM_1 + B_1 GM_2 \\ * & * & * & P - 2\varepsilon_1 I & 0 \\ * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \]  \hspace{1cm} (5)
where $\Lambda_1$ is as in Theorem 6, $\Lambda_3 = A^T P + C^T G^T B_1^T$ and $\Lambda_4 = B_2^T P + D^T G^T B_1^T$.

Once the problem is solved

\[
\begin{align*}
\alpha^* & \triangleq \min(\alpha), \\
\epsilon_1^* & \triangleq \min(\epsilon_1), \\
\gamma^* & \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1 + \epsilon_1^*)}}, \\
K & = Q^{-1} G.
\end{align*}
\]
Assume a nonlinear uncertainty as follows

\[ \Phi_{\Delta}(x, u) = \Phi(x, u) + \Delta \Phi(x, u) \]
\[ x(k+1) = (A + \Delta A)x(k) + \Phi_{\Delta}(x, u) \]
\[ \|\Delta \Phi(x_1, u) - \Delta \Phi(x_2, u)\| \leq \Delta \gamma \|x_1 - x_2\|. \]

**Proposition 2.** If the actual Lipschitz constant in \( \Sigma \) is \( \gamma \) and the maximum admissible Lipschitz constant achieved by Corollary 2 (Theorem 6), is \( \gamma^* \), then, the controller obtained by Corollary 2 (Theorem 6), can tolerate additive Lipschitz nonlinear uncertainty with Lipschitz constant satisfying \( \Delta \gamma \leq (\gamma^* - \gamma) \).
Consider a system of class $\Sigma$ where,

$$A = \begin{bmatrix}
0.5000 & -0.5975 & 0.3735 & 0.0457 & 0.3575 \\
0.2500 & 0.3000 & 0.4017 & 0.1114 & 0.0227 \\
0.4880 & 0.1384 & 0.2500 & 0.7500 & 0.7500 \\
0.3838 & 0.0974 & 0.5000 & 0.2500 & 0.5000 \\
0.0347 & 0.1865 & -0.2500 & 0.5000 & 0.2500
\end{bmatrix}, \quad \Phi(x,u) = \begin{bmatrix}
0.1 \sin(x_3) \\
0.2 \sin(x_4) \\
0.3 \sin(x_1) \\
0 \\
0.1 \sin(x_2)
\end{bmatrix},$$

$$B_1 = \begin{bmatrix}
0.7 & 0.8 & 0 \\
0.4 & 0.9 & 0.9 \\
0.9 & 0.9 & 0.2 \\
0.9 & 0.6 & 0.7 \\
0 & 0.5 & 0.3
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
0.1 & 0 \\
0.1 & 0.1 \\
0.1 & 0.1 \\
0 & 0.1 \\
0 & 0.2
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
0 & 0.1 \\
0.1 & 0.2
\end{bmatrix},$$

$$D = \begin{bmatrix}
0.2 \\
0.2
\end{bmatrix}, \quad C = \begin{bmatrix}
0.5 & 0.2 & 0 & 0 & 0.3 \\
0 & 0.2 & 0.1 & 0.3 & 0
\end{bmatrix}, \quad N = \begin{bmatrix}
0.3 & 0.15 & 0.1 & 0 & 0.2 \\
0.1 & 0.2 & 0.1 & 0.2 & 0
\end{bmatrix}.$$

- The system is globally Lipschitz with Lipschitz constant 0.3.
Example II

The nonlinear system is unstable. The matrix $A$ is unstable, which means the linear part of the systems is itself unstable. The matrix $A$ is singular, so the approach of Garcia is not applicable even to the nominal linear part.

The submatrix of $A$ obtained by omitting the first $p(=2)$ rows and columns in singular, too. Therefore, we can not use the results of Bara, neither.

Using YALMIP with SeDuMi engine, we solve the proposed SDP problem and we get:

$$
\varepsilon_1^* = 0.2076 \\
\alpha^* = 0.3013 \\
\gamma^* = 1.6584 \\
K = \begin{bmatrix} -0.7026 & -0.8334 \\ 0.4825 & -1.8664 \\ 0.3292 & -1.1758 \end{bmatrix},
$$
Robust Static Output Feedback Stabilization

Example III

Figure: The stabilized states of the Example
A new LMI optimization approach to the robust static output feedback stabilization for nonlinear discrete-time uncertain is systems with $H_\infty$ performance was proposed.

The class of nonlinear systems contains norm-bounded time-varying model uncertainties as well as additive Lipschitz nonlinear model uncertainties.

Explicit bounds on the tolerable uncertainty were derived.
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Conclusions

References
In summary, the following contributions are made:

- Robust observer design for nonlinear systems in the presence of unknown exogenous disturbances with a maximized Lipschitz constant and disturbance attenuation level through multiobjective convex optimization; both in continuous-time and discrete-time domains.

- Robust observer design for nonlinear systems with unknown parametric and nonlinear uncertainties.

- Robust observer design for nonlinear sampled data systems with approximate discrete-time models.
Conclusions II

- Casting the nonlinear state estimation problem of the above cases into a computationally efficient LMI optimization framework, introducing new combined optimality criterions and proposing a new design method and a more general observer structure that obtains more accurate results and can tolerate more uncertainty in the system model than the traditional designs.

- New results towards the robust static output feedback stabilization of nonlinear uncertain systems, a well-known open problem in systems control.
References I

**Continuous-Time**


**Discrete-Time**

Masoud Abbaszadeh, Horacio J. Marquez, **LMI Optimization Approach to Robust $H_\infty$ Observer Design and Static Output Feedback Stabilization for Discrete-Time Nonlinear Uncertain Systems**, Accepted for publication in the International Journal of Robust and Nonlinear Control, 2008, DOI: 10.1002/rnc.1310, (Preprint available online)


References III