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Preface

I began writing this textbook several years ago. At that time my intention was to write a research monograph with focus on the input–output theory of systems and its connection with robust control, including a thorough discussion of passivity and dissipativity of systems. In the middle of that venture I began teaching a first-year graduate-level course in nonlinear control, and my interests quickly shifted into writing something more useful to my students. The result of this effort is the present book, which doesn’t even resemble the original plan. I have tried to write the kind of textbook that I would have enjoyed myself as a student. My goal was to write something that is thorough, yet readable.

The first chapter discusses linear and nonlinear systems and introduces phase plane analysis. Chapter 2 introduces the notation used throughout the book and briefly summarizes the basic mathematical notions needed to understand the rest of the book. This material is intended as a reference source and not a full coverage of these topics. Chapters 3 and 4 contain the essentials of the Lyapunov stability theory. Autonomous systems are discussed in Chapter 3 and nonautonomous systems in Chapter 4. I have chosen this separation because I am convinced that the subject is better understood by developing the main ideas and theorems for the simpler case of autonomous systems, leaving the more subtle technicalities for later. Chapter 5 briefly discusses feedback stabilization based on backstepping. I find that introducing this technique right after the main stability concepts greatly increases students’ interest in the subject. Chapter 6 considers input–output systems. The chapter begins with the basic notions of extended spaces, causality, and system gains and introduces the concept of input-output stability. The same chapter also discusses the stability of feedback interconnections via the celebrated small gain theorem. The approach in this chapter is classical; input–output systems are considered without assuming the existence of an internal (i.e. state space) description. As such, Chapters 3-5 and 6 present two complementary views of the notion of stability: Lyapunov, where the focus is on the stability of equilibrium points of unforced systems (i.e. without external excitations); and the input–output theory, where systems are assumed to be relaxed (i.e. with zero initial conditions) and subject to an external input. Chapter 7 focuses on the important concept of input-to-state stability and thus starts to bridge across the two alternative views of stability. In Chapters 8 and 9 we pursue a rather complete discussion of dissipative systems, an active area of research, including its importance in the so-called nonlinear $L_2$ gain control problem. Passive systems are studied first in Chapter
8, along with some of the most important results that derive from this concept. Chapter 9 generalizes these ideas and introduces the notion of dissipative system. I have chosen this presentation for historical reasons and also because it makes the presentation easier and enhances the student's understanding of the subject. Finally, Chapters 10 and 11 provide a brief introduction to feedback linearization and nonlinear observers, respectively.

Although some aspects of control design are covered in Chapters 5, 9, and 10, the emphasis of the book is on analysis and covers the fundamentals of the theory of nonlinear control. I have restrained myself from falling into the temptation of writing an encyclopedia of everything ever written on nonlinear control, and focused on those parts of the theory that seem more fundamental. In fact, I would argue that most of the material in this book is essential enough that it should be taught to every graduate student majoring in control systems.

There are many examples scattered throughout the book. Most of them are not meant to be real-life applications, but have been designed to be pedagogical. My philosophy is that real physical examples tend to be complex, require elaboration, and often distract the reader's attention from the main point of the book, which is the explanation of a particular technique or a discussion of its limitations.

I have tried my best to clean up all the typographical errors as well as the more embarrassing mistakes that I found in my early writing. However, like many before me, and the many that will come after, I am sure that I have failed! I would very much appreciate to hear of any error found by the readers. Please email your comments to

marquez@ee.ualberta.ca

I will keep an up-to-date errata list on my website:

http://www.ee.ualberta.ca/~marquez

Like most authors, I owe much to many people who directly or indirectly had an influence in the writing of this textbook. I will not provide a list because I do not want to forget anyone, but I would like to acknowledge four people to whom I feel specially indebted: Panajotis Agathoklis (University of Victoria), Chris Damaren (University of Toronto), Chris Diduch, and Rajamani Doraiswami (both of the University of New Brunswick). Each one of them had a profound impact in my career, and without their example this book would have never been written. I would also like to thank the many researchers in the field, most of whom I never had the pleasure to meet in person, for the beautiful things that they have published. It was through their writings that I became interested in the subject. I have not attempted to list every article by every author who has made a contribution to nonlinear control, simply because this would be impossible. I have tried to acknowledge those references that have drawn my attention during the preparation of my lectures and later during the several stages of the writing of this book. I sincerely apologize to every
author who may feel that his or her work has not been properly acknowledged here and encourage them to write to me.

I am deeply grateful to the University of Alberta for providing me with an excellent working environment and to the Natural Sciences and Engineering Research Council of Canada (NSERC) for supporting my research. I am also thankful to John Wiley and Son’s representatives: John Telecki, Kristin Cooke Fasano, Kirsten Rohstedt and Brendan Cody, for their professionalism and assistance.

I would like to thank my wife Goody for her encouragement during the writing of this book, as well as my son Francisco and my daughter Madison. To all three of them I owe many hours of quality time. Guess what guys? It’s over (until the next project). Tonight I’ll be home early.

Horacio J. Marquez

Edmonton, Alberta
October 2002
Nonlinear Control Systems:
Analysis and Design

Horacio J. Marquez