Stabilization of remote control systems with unknown time varying delays by LMI techniques

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In this paper, the stabilization of one class of remote control systems with unknown time varying delays is analysed and discussed using LMI techniques. A discrete time state space model under a static control law for remote control systems is first introduced based on some assumptions on the uncertain term. The time delay is unknown, time varying, and can be decomposed into two parts: one fixed part which is unknown and is an integer multiple of the sampling time; the other part which is randomly varying but bounded by one sampling time. Static controller designs based on delay dependent stability conditions are presented. This system is then extended to a more general case when the randomly varying part of the time delay is not limited to one sampling time. The derivative of the time delay is not limited to be bounded. Hence, the contributions are as follow: (i) for a given controller, we can use these stability criteria to test stability of the resulted system; (ii) we can design a remote controller to stabilize an unstable system. Finally, simulation examples are presented to show the effectiveness of the proposed method and to demonstrate remote stabilization of open loop unstable systems.

1. Introduction

Over the past few years, increasing attention has been drawn to the control of time delay systems (Dugard and Verriest 1998, Nilsson 1998, Mahmoud 2000). The stability problem is one of the most important issues in time delay systems since delays caused by transmission may result in instability, especially when there are uncertainties (Zhang et al. 2001, Pan et al. 2004b).

Many delay independent and delay dependent stability criteria, which are mainly concerning continuous systems, have been presented (Xie and de Souza 1997, Cao et al. 1998). However, most practical applications of remote controllers through network transmission are implemented digitally. Up until now, little attention has been paid to discrete time systems with delays and controlled in a remote way. One possible reason is that, for a known delay, the delay difference equation can be rewritten to be a high order system without delay by augmentation (Lozano et al. 2004). However, in a case with large known delay, this augmentation approach can lead to high dimensional systems; and moreover, it is not applicable for systems with unknown delays.

In several references (Song et al. 1999, Lee and Kwon 2002, Xu et al. 2004), no input delay was considered in the problem formulation. This simplifies the design of complex controllers for robust stabilization. In Lee and Lee (1999), two delay dependent conditions were developed for discrete time systems but no control design and uncertainties were considered in the system parameters. The stability conditions in Lee and Lee (1999) are the analogues corresponding to the results for continuous time delay systems as in Xie and de Souza (1997), Cao et al. (1998) and Park (1999). In Nilsson (1998), stochastic control of time delay systems in discrete time is considered while the limitation is that delays in both channels should be shorter than one sampling time.

Xiao et al. (2000) model the problem via a discrete-time jump linear system. However the assumption of this approach is that the transition probability matrix for the time delay should be known as a priori. It is usually
difficult to get this matrix in reality. In Stipanovic and Siljak (2001), discrete time delay systems with non-linear perturbations are considered and it is proposed for the case with constant time delay. Most recently, Yue et al. (2005) propose a robust $H_{\infty}$ controller for continuous time delay systems while the lower bound of the network induced delay can be used in the design. However, Assumption 3 on the time delay distribution in this paper is a strong condition. In Yu et al. (2005), only one channel time delay is considered. In these two recent approaches, sampled-data approach is incorporated in analysing the communication channel(s) which reflects the physical point of real network transmission.

Based on the above analysis and from a novel aspect, here we look further into the stability problem of remote control systems with unknown time varying delays in the control input in the discrete time domain. In this case, the existing theory using augmentation cannot be applied because of the unknown time varying delay. One approach is to decompose the time delay into two parts: one fixed part which is unknown and is an integer multiple of the sampling time; the other part which is randomly varying but bounded by one sampling time. Then the stability condition of the discrete time state space model under a static control law for the remote control system is first introduced based on a certain assumption on the uncertain term. Note that one motivation to study the problem in discrete time is that most systems with a communication channel time delay is considered. In these two recent papers is a strong condition. In Yu et al. (2005), only one delay is not necessary to be bounded. Another advantage is that for a given controller, we can use these stability criteria to test stability of the resulted closed loop systems.

The paper is organized as follows. In §2, the problem formulation is first presented with the analysis on the discrete time system when the delay variation part is less than one sampling time. In §3, static controllers based on delay dependent conditions are analysed in detail. In §4, the system description of the general case and the controller design are analysed when the delay variation part is longer than one sampling time. Section 6 draws the conclusions. Notations: $\mathbb{R}^n$ denotes an $n$-dimensional real vector space; $\| \cdot \|$ is the Euclidean norm or induced matrix norm.

2. Problem formulation

The following system with uncertain delay is considered

$$\dot{x}(t) = A_px(t) + B_pu(t - \tau(t)), \quad (1)$$

where $x \in \mathbb{R}^n$ is the measurable state vector, $u \in \mathbb{R}^m$ denotes the remote control signal, $A_p \in \mathbb{R}^{n \times n}$ and $B_p \in \mathbb{R}^{n \times m}$ are known constant matrices. The system in (1) can be illustrated as in figure 1, where $\tau(t)$ is the time delay from the sensor to the remote controller and $u_c \in \mathbb{R}^m$ is the control input to the physical system. In this paper, the sensor is assumed to be time driven; the controller and the actuator are event driven.

Assumption 1: The time delays in the two channels can be represented as $\tau_i(t) = h_i T_s + \varepsilon_i(t), i = 1, 2$, where $T_s$ is the sampling time and $\varepsilon_i(t)$ is unknown but is bounded as $0 \leq \varepsilon_i(t) < L_i T_s$, with $L_i \ll h_i$, where $L_i$ is a known integer. $h_i$ is an unknown integer constant and is assumed to be: $h_i \in [0, h_i]$, where $h_i$ is known.

Remark 1: Note that the controller and the actuator are event driven; thus there is no holding-up on the controller (pure gain) and actuator side. The total delay from the sensor to the actuator can be represented as $\tau(t) = \tau_2(t) + \tau_1(t) - \tau_3(t)$. According to Assumption 1, similarly $\tau(t)$ can be rewritten as $\tau(t) = h T_s + \varepsilon(t)$, with $h \in [h_1 + h_2, h_1 + h_2 + 1]$ being an unknown integer and $\varepsilon(t)$ being unknown but bounded with $0 \leq \varepsilon(t) < L T_s$ and $L < h \leq h_1$, where $L$ and $h$ are known integers.

![Figure 1. Block diagram of the remote control systems.](image-url)
Here we first consider the case when $L = 1$ which means that the randomly varying part of the time delay is bounded by one sampling time. The case when $h > L > 1$ is extended in §4. The timing flow of the control system is shown in figure 2. Suppose the measured state $x_{k-h}$ is sent at time $(k-h)T_s$ from the sensor, it arrives at the controller in the sampling interval $[(k-h)T_s, (k-h+1)T_s)$ according to the delay structure in Assumption 1. At the controller side, the control signal is then denoted as $u_{k-h} = Kx_{k-h}$ if a static gain is designed and the sampled state arrives at the controller at time $t \in [(k-h)T_s, (k-h+1)T_s]$. This control signal arrives at the actuator at time $t = kT_s + \varepsilon(t)$. Figure 3 shows an example of a time delay profile satisfying $\tau(t) = 3T_s + \varepsilon(t)$ with $0 < \varepsilon(t) < T_s$, measured in the teleoperation experiment (Pan et al. 2004a); this can be used to validate the delay assumptions introduced earlier.

Integrating the open loop system in the interval $[kT_s, (k+1)T_s]$, and using figure 2, we have

$$
x_{k+1} = A x_k + \int_{kT_s}^{(k+1)T_s} e^{A\tau(s)} ds B P u_k(s - \tau_2(s)) ds
$$

$$
= A x_k + \int_{kT_s}^{(k+1)T_s} e^{A\tau(s)} ds B P u_{k-h} + \int_{kT_s}^{(k+1)T_s} e^{A\tau(s)} ds B P u_{k-h-1}
$$

$$
= A x_k + (B + \Delta B_0) u_{k-h} + (B + \Delta B_1) u_{k-h-1},
$$

where

$$
A = e^{AhT_s}, \quad B = \int_{kT_s}^{(k+1)T_s} e^{Ah\tau(s)} ds B P = e^{AhT_s} \int_{0}^{T_s} e^{Ah\tau(s)} ds B P,
$$

$$
\Delta B_0 = - \int_{kT_s}^{(k+1)T_s} e^{Ah\tau(s)} ds B P = - \int_{0}^{T_s} e^{Ah\tau(s)} ds B P,
$$

$$
\Delta B_1 = - \int_{kT_s}^{(k+1)T_s} e^{Ah\tau(s)} ds B P = - \int_{0}^{T_s} e^{Ah\tau(s)} ds B P,
$$

and $\Delta B_0 + \Delta B_1 = -B$.

Because there are uncertainties in the time delay, we cannot use the prediction method to stabilize the system which requires accurate information of the delay in order to predict the future states. Hence a static controller is designed as $u(t) = Kx(t - \tau_1(t))$, where $\tau_1(t)$ is the delay from the sensor to the controller in the continuous time domain. The main task here is how to design a proper $K$ such that the closed loop control system is stable. Furthermore, as previously discussed in figure 2, in discrete time domain we have $u_{k-h} = Kx_{k-h}$. Hence it is straightforward that (2) becomes

$$
x_{k+1} = A x_k + (B + \Delta B_0) u_{k-h} + (B + \Delta B_1) u_{k-h-1},
$$

where $A$ and $B$ can be derived directly from $A_P$ and $B_P$ in (1). Based on the above analysis, though $\Delta B_i, i = 0, 1,$ are uncertain, embedding them into standard uncertain terms would allow us to treat the problem as a robust
stabilization problem. Hence in this paper, we assume that $\Delta B_i$ satisfies the following assumption.

**Assumption:** $\Delta B_i$ can be expressed as $\Delta B_i = (k)F_i$, $i = 0, 1$, where $E$, $F_i$ are known with appropriate dimensions and $\Gamma_i(k)^T \Gamma_i(k) < I$.

Note that Assumption 2 on uncertainties is a commonly used condition for many existing approaches for uncertain time delay systems (Mahmoud 2000, Xu et al. 2004, Yue et al. 2005). There always exist $E$ and $F_i$ such that $\Gamma_i(k)^T \Gamma_i(k) < I$ for $\Delta B_i$, $i = 0, 1$. However, in this paper, for a specific system pair $(A_p, B_p)$ and sampling time $T_s$, we can use numerical methods to get the matrices $E$ and $F_i$ which are required to be known for the controller design.

**Lemma 1** (Mahmoud 2000): Let $\Sigma_1$ and $\Sigma_2$ be real constant matrices of compatible dimensions, and $H(t)$ be a real matrix function satisfying $H(t)^T H(t) \leq I$. Then the following inequality holds

$$
\Sigma_1^T H(t) \Sigma_2 + \Sigma_2^T H(t)^T \Sigma_1 \leq \varepsilon \Sigma_1^T \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_2,
$$

where $\varepsilon$ is a positive constant.

### 3. Controller design

Rewrite the system in (3) as

$$
x_{k+1} = Ax_k + A_0 \Delta x_{k-h} + A_1 \Delta x_{k-h-1},
$$

where $A_0 \Delta \triangleq BK + \Delta B_0 K = A_0 + \Delta_0$ with $A_0 = BK$ and $\Delta_0 = \Delta B_0 K$, and $A_1 \Delta \triangleq BK + \Delta B_1 K = A_1 + \Delta_1$ with $A_1 = BK$ and $\Delta_1 = \Delta B_1 K$.

In this part, we focus on the controller design based on the delay department condition by introducing some slack matrix variables which result in a less conservative condition. The similar approach for the continuous time system is introduced in Wu et al. (2004). At first, we extend the method in the continuous time domain to the discrete time domain. For any matrices $N_i$, $S_j$, and $M_i$ ($i = 1, 2, 3, 4$) of appropriate dimensions, the following equations hold

$$
\Psi_1 = 2 \left[ x_k^T N_1 + x_{k-h}^T N_2 + x_{k-h-1}^T N_3 + x_{k+1}^T N_4 \right] 
\times \left[ x_k - x_{k-h} - \sum_{j=k-h+1}^k (x_j - x_{j-1}) \right] = 0, \quad (6)
$$

$$
\Psi_2 = 2 \left[ x_k^T S_1 + x_{k-h}^T S_2 + x_{k-h-1}^T S_3 + x_{k+1}^T S_4 \right] 
\times \left[ x_k - x_{k-h-1} - \sum_{j=k-h}^{k-h-1} (x_j - x_{j-1}) \right] = 0, \quad (7)
$$

$$
\Psi_3 = 2 \left[ x_k^T M_1 + x_{k-h}^T M_2 + x_{k-h-1}^T M_3 + x_{k+1}^T M_4 \right] 
\times \left[ x_{k+1} - Ax_k - A_0 \Delta x_{k-h} - A_1 \Delta x_{k-h-1} \right] = 0. \quad (8)
$$

Based on the Lyapunov functional analysis while combining the equations (6)–(8) in the derivation, first we have the following proposition.

---

**Figure 3.** The time delay when $h = 3$, $L = 1$ and $T_s = 0.05$ sec.
Proposition 1: For a given control gain $K$ and a given upperbound, $\bar{h}$ of $h$ the system in $(3)$ is asymptotically stable if there exist symmetric positive definite matrices $P$, $Q_1, Q_2, Z_1, Z_2 \in \mathbb{R}^{n \times n}$, and matrices $N_i, S_i, M_i, i = 1, 2, 3, 4$ with appropriate dimensions such that the following inequality holds,

$$
\begin{bmatrix}
D_{11} & \ast & \ast & \ast & \ast & \ast \\
D_{21} & D_{22} & \ast & \ast & \ast & \ast \\
D_{31} & D_{32} & D_{33} & \ast & \ast & \ast \\
D_{41} & D_{42} & D_{43} & D_{44} & \ast & \ast \\
\hat{h}N_1^T & \hat{h}N_2^T & \hat{h}N_3^T & \hat{h}N_4^T & -\hat{h}Z_1 & \ast \\
(\bar{h}+1)S_1^T & (\bar{h}+1)S_2^T & (\bar{h}+1)S_3^T & (\bar{h}+1)S_4^T & 0 & -(\bar{h}+1)Z_1
\end{bmatrix} < 0,
$$

(9)

where

$$
\begin{align*}
D_{11} &= -P + \bar{h}Z_1 + Q_1 + (\bar{h}+1)Z_2 + Q_2 + N_1 + N_1^T + S_1 + S_1^T - M_1A - AT_1^TM_1^T, \\
D_{21} &= -N_2 - N_2^T + S_2 - M_2A - AT_2^TM_2^T, \\
D_{31} &= -N_3 - S_3^T + S_3 - M_3A - AT_3^TM_3^T, \\
D_{32} &= -N_3 - S_3^T - M_3A_0 + AT_3^TM_3^T, \\
D_{22} &= -Q_1 - N_2 - N_2^T - M_2A_0 - AT_2^TM_2^T, \\
D_{33} &= -Q_2 - S_3 - S_3^T - M_3A_1 + AT_3^TM_3^T, \\
D_{41} &= -\hat{h}Z_1 - (\bar{h}+1)Z_2 + N_4 + S_4 - M_4A + M_4^T, \\
D_{42} &= -N_4 - M_4A_0 + M_4^T, \\
D_{43} &= -S_4 - M_4A_1 + M_4^T, \\
D_{44} &= P + \bar{h}Z_1 + (\bar{h}+1)Z_2 + M_4 + M_4^T.
\end{align*}
$$

Proof: Consider the following Lyapunov function for the discrete system in $(3),$

$$
V_k = V_{1,k} + V_{2,k} + V_{3,k} + V_{4,k} + V_{5,k}
\triangleq x_k^T P x_k + \sum_{i=-h}^{-1} \sum_{j=k+i+1}^{k} (x_j - x_{j-1})^T Z_1(x_j - x_{j-1})
+ \sum_{j=-h}^{k} x_j^T Q_1 x_j
+ \sum_{i=-h}^{-1} \sum_{j=k+i+1}^{k} (x_j - x_{j-1})^T Z_2(x_j - x_{j-1})
+ \sum_{j=-h}^{k} x_j^T Q_2 x_j.
$$

Then we have

$$
\begin{align*}
\Delta V_{1,k} &= V_{1,k+1} - V_{1,k} = x_{k+1}^T P x_{k+1} - x_k^T P x_k, \\
\Delta V_{2,k} &= \sum_{i=-h}^{k-1} \sum_{j=k+i+1}^k (x_j - x_{j-1})^T Z_1(x_j - x_{j-1})
- \sum_{i=-h}^{k-1} \sum_{j=k+i+1}^k (x_j - x_{j-1})^T Z_1(x_j - x_{j-1})
\leq \bar{h}(x_{k+1} - x_k)^T Z_1(x_{k+1} - x_k)
- \sum_{j=-h+k-1}^k (x_j - x_{j-1})^T Z_1(x_j - x_{j-1}), \\
\Delta V_{3,k} &= x_k^T Q_1 x_k - x_{k-h}^T Q_1 x_{k-h}, \\
\Delta V_{4,k} &= \sum_{i=-h}^{k-1} \sum_{j=k+i+1}^k (x_j - x_{j-1})^T Z_2(x_j - x_{j-1})
- \sum_{i=-h}^{k-1} \sum_{j=k+i+1}^k (x_j - x_{j-1})^T Z_2(x_j - x_{j-1})
\leq \bar{h}(x_{k+1} - x_k)^T Z_2(x_{k+1} - x_k)
- \sum_{j=-h+k-1}^k (x_j - x_{j-1})^T Z_2(x_j - x_{j-1}), \\
\Delta V_{5,k} &= x_k^T Q_2 x_k - x_{k-h}^T Q_2 x_{k-h}.
\end{align*}
$$

Defining the following new variables

$$
\begin{align*}
z &\triangleq [x_k, x_{k-h}, x_{k-h-1}, x_{k+1}]^T, \\
N &\triangleq [N_1, N_2, N_3, N_4]^T, \\
S &\triangleq [S_1, S_2, S_3, S_4]^T, \\
M &\triangleq [M_1, M_2, M_3, M_4]^T,
\end{align*}
$$

and combining $(6)$–$(8)$, we have that $\Delta V_k$ becomes

$$
\begin{align*}
\Delta V_k &= \Delta V_{1,k} + \Delta V_{2,k} + \Delta V_{3,k} + \Delta V_{4,k} + \Delta V_{5,k}
+ \Psi_1 + \Psi_2 + \Psi_3
\leq x_{k+1}^T P x_{k+1} - x_k^T P x_k \\
+ \bar{h}(x_{k+1} - x_k)^T Z_1(x_{k+1} - x_k)
- \sum_{j=-h+k-1}^{k} (x_j - x_{j-1})^T Z_1(x_j - x_{j-1})
+ x_k^T Q_1 x_k - x_{k-h}^T Q_1 x_{k-h} + (\bar{h}+1)
\times (x_{k+1} - x_k)^T Z_2(x_{k+1} - x_k)
+ x_k^T Q_2 x_k - x_{k-h}^T Q_2 x_{k-h} + \bar{h}(x_{k+1} - x_k)^T Z_1(x_{k+1} - x_k)
+ 2z^T N \left[ x_k - x_{k-h} - \sum_{j=-h+k-1}^{k} (x_j - x_{j-1}) \right]
+ 2z^T S \left[ x_k - x_{k-h} - \sum_{j=-h+k-1}^{k} (x_j - x_{j-1}) \right]
+ 2z^T M \left[ x_{k+1} - A x_k - A_0 x_k - A_1 x_{k-h} - A_{1\Delta} x_{k-h-1} \right].
\end{align*}
$$

(10)
Moreover,
\[-2z^T N \sum_{j=k-h+1}^{k} (x_j - x_{j-1}) \]
\[= -2 \sum_{j=k-h+1}^{k} z^T N (x_j - x_{j-1}) \leq \overline{h} z^T N z + \sum_{j=k-h+1}^{k} (x_j - x_{j-1})^T Z_1 (x_j - x_{j-1}). \tag{11} \]

Similarly we have
\[-2z^T S \sum_{j=k-h}^{k} (x_j - x_{j-1}) \leq (\overline{h} + 1) z^T S z + \sum_{j=k-h}^{k} (x_j - x_{j-1})^T Z_2 (x_j - x_{j-1}). \tag{12} \]

Using (11) and (12), (10) can be written as
\[
\Delta V_k \leq x_{k+1}^T P x_{k+1} - x_k^T P x_k + \overline{h} (x_{k+1} - x_k)^T \times Z_1 (x_{k+1} - x_k) + x_k^T Q x_k - x_{k-h}^T Q x_{k-h} + (\overline{h} + 1) (x_{k+1} - x_k)^T Z_2 (x_{k+1} - x_k) + x_k^T Q x_k - x_{k-h}^T Q x_{k-h} + 2z^T N (x_k - x_{k-h}) + 2z^T S (x_k - x_{k-h}) + \overline{h} z^T N z + (\overline{h} + 1) z^T S z + \overline{h} z^T M [x_{k+1} - A x_k - A_0 x_k - A_1 x_{k-h}] + (\overline{h} + 1) z^T S z + \overline{h} z^T M [x_{k+1} - A x_k - A_0 x_k - A_1 x_{k-h} - A_2 x_{k-h} - A_3 x_{k-h}]. \tag{13} \]

where \( \overline{h} \) is the upper bound of \( h \). By using the Schur complement, then \( \Delta V_k < 0 \) holds if the following condition is satisfied
\[
\begin{bmatrix}
D_{11} & * & * & * & * & * \\
D_{21} & D_{22} & * & * & * & * \\
D_{31} & D_{32} & D_{33} & * & * & * \\
D_{41} & D_{42} & D_{43} & D_{44} & * & * \\
\overline{h}_N^T & \overline{h}_N^T & \overline{h}_N^T & \overline{h}_N^T & -\overline{h}_Z \\
(\overline{h} + 1)S_1^T & (\overline{h} + 1)S_2^T & (\overline{h} + 1)S_3^T & (\overline{h} + 1)S_4^T & 0 & -\overline{h} \overline{Z}_2 & * & * & * \\
E^T M_1^T & E^T M_2^T & E^T M_3^T & E^T M_4^T & 0 & 0 & 0 & -\rho I & * & * \\
0 & 0 & \rho F_0 K & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \\
0 & 0 & \rho F_1 K & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \\
\end{bmatrix} < 0, \tag{14} \]

Furthermore by using the Lyapunov-Krasovskii stability theorem (Hale 1993), \( \Delta V_k < 0 \) means that the system in (3) is asymptotically stable.

Equation (9) contains the uncertainties \( \Delta_0 \) and \( \Delta_1 \). Thus we cannot use it directly to check the system stability. The following proposition is given as a sufficient condition for the feasibility of (9) by dealing with the uncertainties in the inequality. First we can represent the uncertainties as
\[
[\Delta_0 \Delta_1] = E [\Gamma_0 \Gamma_1] [F_0 K \quad 0 \quad 0] \triangleq E \chi G \tag{15} \]

With \( \Gamma = [\Gamma_0 \Gamma_1] \) and \( \Gamma^T \Gamma < 1 \), and
\[
G = [G_0 \quad G_1] = [F_0 K \quad 0 \quad 0]. \]

**Proposition 2:** For a given control gain \( K \) and a given upperbound \( \overline{h} \) of \( h \), the system (3) is asymptotically stable if there exist symmetric positive definite matrices \( P, Q_1, Q_2, Z_1, Z_2 \in \mathbb{R}^{n \times n} \) matrices \( N_i, M_i, i = 1, 2, 3, 4 \) with appropriate dimensions and constant \( \rho > 0 \) such that the following inequality holds.

\[
\begin{bmatrix}
D_{11} & * & * & * & * & * & * & * & * & * \\
D_{21} & D_{22} & * & * & * & * & * & * & * & * \\
D_{31} & D_{32} & D_{33} & * & * & * & * & * & * & * \\
D_{41} & D_{42} & D_{43} & D_{44} & * & * & * & * & * & * \\
\overline{h}_N^T & \overline{h}_N^T & \overline{h}_N^T & \overline{h}_N^T & -\overline{h}_Z \\
(\overline{h} + 1)S_1^T & (\overline{h} + 1)S_2^T & (\overline{h} + 1)S_3^T & (\overline{h} + 1)S_4^T & 0 & -\overline{h} \overline{Z}_2 & * & * & * \\
E^T M_1^T & E^T M_2^T & E^T M_3^T & E^T M_4^T & 0 & 0 & 0 & -\rho I & * & * \\
0 & 0 & \rho F_0 K & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \\
0 & 0 & \rho F_1 K & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \\
\end{bmatrix} < 0, \tag{16} \]
where

\[
\begin{align*}
D'_{11} &= -P + \tilde{h}Z_1 + Q_1 + (\tilde{h} + 1)Z_2 + Q_2 + N_1 + N_1^T + S_1 + S_1^T - M_1A - A^TM_1^T, \\
D'_{21} &= N_2 - N_2^T + S_2 - M_2A = K^TB^TM_2^T, \\
D'_{31} &= N_3 - S_3^T + S_3 - M_3A = K^TB^TM_3^T, \\
D'_{32} &= -N_3 - S_3^T - M_3BK - K^TB^TM_3^T, \\
D'_{22} &= -Q_1 - N_2 - N_2^T - M_2BK - K^TB^TM_2^T, \\
D'_{33} &= -Q_2 - S_3 - S_3^T - M_3BK - K^TB^TM_3^T, \\
D'_{41} &= D_{41} = \tilde{h}Z_1 - (\tilde{h} + 1)Z_2 + N_4 + S_4 - M_4A + A^T \\
D'_{42} &= -N_4 + S_4^T - M_4BK, \\
D'_{43} &= S_4 + S_4^T - M_4BK, \\
D'_{44} &= D_{44} = P + \tilde{h}Z_1 + (\tilde{h} + 1)Z_2 + M_4 + M_4^T.
\end{align*}
\]

**Proof:** The uncertainty part in (9) can be dealt with by using the property in Lemma 1. The left side of the inequality in (9) can be further represented as

\[
(\tilde{h} + 1)S_1^T (\tilde{h} + 1)S_2^T (\tilde{h} + 1)S_3^T (\tilde{h} + 1)S_4^T 0 - (\tilde{h} + 1)Z_2 + \Psi_4,
\]

where

\[
\Psi_4 = -\begin{bmatrix}
A_{0\Delta}^T & M_1^T & M_2^T & M_3^T \\
A_{1\Delta}^T & M_2^T & M_3^T & M_4^T \\
M_1 & 0 & A_{0\Delta} & A_{1\Delta} \\
M_2 & 0 & A_{0\Delta} & A_{1\Delta} \\
M_3 & 0 & A_{0\Delta} & A_{1\Delta} \\
M_4 & 0 & A_{0\Delta} & A_{1\Delta}
\end{bmatrix}
\]

\[
\leq \rho^{-1}ME(M) + \rho \begin{bmatrix}
G_0^T & G_1 \\
G_1 & G_0 \\
0 & 0
\end{bmatrix}
\]

Using (18) and Schur complement, it is obvious to obtain that the following condition is a sufficient condition of inequality (9),

\[
\begin{pmatrix}
D'_{11} & * & * & * & * & * & * & * \\
D'_{21} & D'_{22} & * & * & * & * & * & * \\
D'_{31} & D'_{32} & D'_{33} & * & * & * & * & * \\
D'_{41} & D'_{42} & D'_{43} & D'_{44} & * & * & * & * \\
\tilde{h}N_1^T & \tilde{h}N_2^T & \tilde{h}N_3^T & \tilde{h}N_4^T & -\tilde{h}Z_1 & * & * & * \\
(\tilde{h} + 1)S_1^T & (\tilde{h} + 1)S_2^T & (\tilde{h} + 1)S_3^T & (\tilde{h} + 1)S_4^T & 0 & - (\tilde{h} + 1)Z_2 & * & * \\
E^TM_1^T & E^TM_2^T & E^TM_3^T & E^TM_4^T & 0 & 0 & -\rho I & *
\end{pmatrix}
> 0.
\]
Based on the analysis in Propositions 1 and 2, we are now able to design the feedback gain $K$ which can ensure the asymptotical stability of the networked control system in (3).

**Theorem 1:** For given scalars $\theta_i, \ i = 1, 2, 3, 4$, and a given upperbound $h$ of $h$, if there exist symmetric positive definite matrices $\hat{P}, \hat{Q}_1, \hat{Q}_2, \hat{Z}_1, \hat{Z}_2 \in \mathbb{R}^{n \times n}$, matrices $Y, \hat{N}_i, \hat{S}_i, i = 1, 2, 3, 4$, non-singular matrix $X$ with appropriate dimensions and constant $\sigma > 0$ such that the following inequality holds,

$$
\begin{bmatrix}
\phi_{11} & * & * & * & * & * & * & * & * \\
\phi_{21} & \phi_{22} & * & * & * & * & * & * & * \\
\phi_{31} & \phi_{32} & \phi_{33} & * & * & * & * & * & * \\
\phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} & * & * & * & * & * \\
\hat{N}_1 \hat{N}_1^T & \hat{N}_2 \hat{N}_2^T & \hat{N}_3 \hat{N}_3^T & \hat{N}_4 \hat{N}_4^T & \sigma \theta_1 E^T & \sigma \theta_2 E^T & \sigma \theta_3 E^T & \sigma \theta_4 E^T & 0 \\
(\hat{h} + 1) \hat{S}_1 \hat{S}_1^T & (\hat{h} + 1) \hat{S}_2 \hat{S}_2^T & (\hat{h} + 1) \hat{S}_3 \hat{S}_3^T & (\hat{h} + 1) \hat{S}_4 \hat{S}_4^T & 0 & 0 & -\sigma I & * & 0 \\
0 & F_0 Y & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma I \\
0 & 0 & F_1 Y & 0 & 0 & 0 & 0 & 0 & -\sigma I
\end{bmatrix} < 0, 
$$

(20)

where

$$
\phi_{11} = -\hat{P} + \hat{h} \hat{Z}_1 + \hat{Q}_1 + (\hat{h} + 1) \hat{Z}_2 + \hat{Q}_2 + \hat{N}_1 + \hat{N}_2^T + \hat{S}_1 + \hat{S}_2^T - \theta_1 A X^T - \theta_1 X A^T,
$$

$$
\phi_{21} = \hat{N}_2 - \hat{N}_1^T + \hat{S}_2 - \theta_2 A X^T - \theta_1 Y^T B^T,
$$

$$
\phi_{31} = \hat{N}_3 - \hat{N}_1^T + \hat{S}_3 - \theta_3 A X^T - \theta_1 Y^T B^T,
$$

$$
\phi_{32} = -\hat{N}_3 - \hat{S}_3^T - \theta_3 B Y + \theta_2 Y^T B^T,
$$

$$
\phi_{22} = -\hat{Q}_1 + \hat{N}_2 - \hat{N}_2^T + \theta_2 B Y - \theta_2 Y^T B^T,
$$

$$
\phi_{33} = -\hat{Q}_3 - \hat{S}_3 - \hat{S}_2^T - \theta_3 B Y + \theta_1 Y^T B^T,
$$

$$
\phi_{41} = -\hat{h} \hat{Z}_1 - (\hat{h} + 1) \hat{Z}_2 + \hat{N}_4 + \hat{S}_4 - \theta_4 A X^T + \theta_1 X,
$$

$$
\phi_{42} = -\hat{N}_4 + \theta_4 X - \theta_4 B Y,
$$

$$
\phi_{43} = -\hat{S}_4 + \theta_4 X - \theta_4 B Y,
$$

$$
\phi_{44} = \hat{P} + \hat{h} \hat{Z}_1 + (\hat{h} + 1) \hat{Z}_2 + \theta_4 X^T + \theta_4 X,
$$

then with the control law

$$
\begin{align*}
\mathbf{u} &= \mathbf{Kx}(\cdot), \\
\mathbf{K} &= \mathbf{YX}^{-T},
\end{align*}
$$

(21)

the system in (3) is asymptotically stable for all admissible network-induced delays.

**Proof:** In order to transform the non-convex LMI in (16) into a solvable LMI, at first we assume that we have some relations in $M_i, i = 1, 2, 3, 4$. One possibility is that $M_i = \theta_i M_0$ where $M_0$ is non-singular and $\theta_i$ is known and given. Define $X = M_0^{-1}, W = \text{diag}(X, X, X, X, X, X, X, I), \sigma = \rho^{-1}$ and $Y = K X^{-T}$. Then pre-multiplying the inequality in (16) by $W$ and post-multiplying by $W^T$, the inequality in (20) can be obtained. Note that the inequality in (20) is only a sufficient condition for the solvability of (16) based on these derivations.

**Remark 2:** Note that the diagonal term $\Phi_{44}$ in (20) contains three positive definite matrices $\hat{P}, \hat{Z}_1, \hat{Z}_2$.

Hence the negativeness of $\Phi_{44}$ depends on the term $\theta_4 X^T + \theta_4 X$. In practice, in order to solve the LMI in (20), one strategy is that $\theta_4$ should be given as a relatively large value compared with $\theta_i, i = 1, 2, 3$. Since the LMI in (20) is only a sufficient condition, there are many possibilities in getting a solution $K$ by assigning different values of $\theta_i$’s and $h$.

### 4. Extension to the general case

The system in (3) can be extended to a more general case based on a different assumption of the network induced delay. In this section, we further extend the case in §2 to the case when $h > L > 1$, i.e., the varying part of the time delay has a bound which is more than one sampling time. When $L > 1$, the discretized system becomes more complicated. It can be derived by the same procedure as in §2 and is represented as follows:

$$
\begin{align*}
x_{k+1} &= A x_k + (B + \Delta B_0) x_{k-h} + (B + \Delta B_1) x_{k-h-1} + \cdots + (B + \Delta B_L) x_{k-h-L},
\end{align*}
$$

(22)

where $A$ and $B$ are the same as in (3). The $\Delta B_i$ term is similar as in Assumption 2 but with $i = 0, 1, \ldots, L$. Then the corresponding results on the control gain design can be extended from Theorem 1. The remarks and proofs
are similar and hence they are omitted here. Rewrite the system in (22) as
\[
x_{k+1} = AX_k + A_{0\Delta}x_{k-h} + A_{1\Delta}x_{k-h-1} + \cdots + A_{L\Delta}x_{k-h-L}, \tag{23}
\]
where \(A_{i\Delta} = BK + \Delta B_i K = A_i + \Delta_i\) with \(A_i = BK\) and \(\Delta_i = \Delta B_i K, i = 0, 1, \ldots, L\).

From Theorem 1, the controller design can be concluded as follows.

**Theorem 2:** For given scalars \(\theta_i, i = 1, \ldots, L + 3,\) and a given upperbound \(\bar{h}\) of \(h\), if there exit symmetric positive definite matrices \(\hat{P}, \tilde{Q}_j, \tilde{Z}_j \in \mathbb{R}^{n \times n}, j = 1, \ldots, L + 1,\) matrices \(Y, \tilde{N}_{ij}\), non-singular matrix \(X\) with appropriate dimensions and constants \(\sigma > 0\) such that the following inequality holds,

\[
\begin{bmatrix}
\Phi_{1,1} & * & * & * & * & * \\
\Phi_{2,1} & \Phi_{2,2} & * & * & * & * \\
\Phi_{3,1} & \Phi_{3,2} & \Phi_{3,3} & * & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_{L+3,1} & \Phi_{L+3,2} & \Phi_{L+3,3} & * & * & * \\
\tilde{h}_L\tilde{N}_{i,1}^T & \tilde{h}_L\tilde{N}_{i,2}^T & \tilde{h}_L\tilde{N}_{i,3}^T & \tilde{h}_L\tilde{N}_{i,L+1}^T & \tilde{h}_L\tilde{N}_{i,L+2}^T & \tilde{h}_L\tilde{N}_{i,L+3}^T \\
\sigma \theta_1 E^T & \sigma \theta_2 E^T & \sigma \theta_3 E^T & \sigma \theta_4 E^T & \sigma \theta_5 E^T & \sigma \theta_6 E^T \\
0 & 0 & \cdots & F_0Y & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & F_L Y & 0 & 0 \\
\end{bmatrix}
< 0, \tag{24}
\]

where \(\tilde{h}_L = \bar{h} + L\) and

\[
\Phi_{1,1} = -\hat{P} + \sum_{i=0}^L (\bar{h} + i)\tilde{Z}_{i+1}
+ \sum_{i=1}^{L+1} \left[ \hat{Q}_i + \tilde{N}_{i,1} + \tilde{N}_{i,1}^T \right] - \theta_1 AX^T - \theta_1 XA^T,
\]

\[
\Phi_{2,1} = \sum_{i=1}^{L+1} \tilde{N}_{i,2} - \tilde{N}_{i,1}^T - \theta_2 AX^T - \theta_1 YB^T,
\]

\[
\Phi_{3,1} = \sum_{i=1}^{L+1} \tilde{N}_{i,3} - \tilde{N}_{i,2}^T - \theta_3 AX^T - \theta_1 YB^T,
\]

\[
\Phi_{3,2} = -\tilde{N}_{i,3} - \tilde{N}_{i,2}^T - \theta_3 BY - \theta_2 YB^T,
\]

\[
\Phi_{3,3} = -\hat{Q}_1 - \tilde{N}_{i,1} - \tilde{N}_{i,1}^T - \theta_1 BY - \theta_2 YB^T - \theta_3 YB^T,
\]

\[
\Phi_{L+3,1} = -\sum_{i=0}^L (\bar{h} + i)\tilde{Z}_{i+1} + \tilde{N}_{L+3,L+3}^T
+ \tilde{N}_{L+3,L+3} + \tilde{N}_{L+3,L+3} - \theta_2 Y - \theta_3 BY,
\]

\[
\Phi_{L+3,2} = -\tilde{N}_{L+3,L+3} - \theta_2 Y - \theta_3 BY,
\]

\[
\Phi_{L+3,3} = \hat{P} + \sum_{i=0}^L (\bar{h} + i)\tilde{Z}_{i+1} + \theta_{L+3} XT + \theta_{L+3} X,
\]

then with the control law \(u = Kx\), \(K = X^T\), the system in (3) is asymptotically stable for all admissible network-induced delays.

**5. Illustrative examples**

Consider the following open loop unstable continuous system with

\[
A_p = \begin{bmatrix}
0.01999 \\
0 & 3.048 \\
\end{bmatrix}, \quad B_p = \begin{bmatrix}
-1.698 \\
0 & 27.73 \\
\end{bmatrix}. \tag{25}
\]

The time delay profiles of the two channels: from sensor to controller and from controller to actuator, are set to be same as shown in figure 3. Hence the time delay parameters are \(h = 3\), \(L = 1\) with the sampling time...
By zero-order holder method, the corresponding unstable discrete system in (3) is
\[
A = \begin{bmatrix} 1.001 & 0.11 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]
where the initial condition is \(x(0) = [0.5, 0.5]^T\). Choosing
\[
E = \begin{bmatrix} 0.1 & 0.02 \\ 0.01 & 0.1 \end{bmatrix}
\]
and \(F_0 = F_1 = [1, 1]^T, \Gamma_k \Gamma_k^T < I\) is proven to be satisfied by calculating the terms \(\Delta B_0, \Delta B_1\) according to their expressions, \(\forall \epsilon(k) \in [0, T_s]\), in Matlab. The initial condition is \(x(0) = [0.5, 0.5]^T\).

At first, by pole placement without considering the time delays, a gain \(K = [-0.2, 0.45]\) gives the poles of the \((A + BK)\) within the unit circle as: \(0.9755 \pm 0.1461i\). However, this is not sufficient to stabilize the system. As shown in figure 4, the system response is still unstable.

Secondly, through solving the corresponding LMI (20) in Theorem 1 on the delay dependent analysis, and with given values \(\theta_1 = \theta_2 = \theta_3 = 1, \theta_4 = 45\) and \(h = 3\), we have
\[
X = \begin{bmatrix} -1.7198 & 0.2112 \\ 0.3270 & -0.8741 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.069 \\ -0.0144 \end{bmatrix}.
\]
Hence \(K = YX^{-T} = [-0.0399, 0.0016]\). The resulted system response is stable as in figure 5. Thus the unstable system is stabilized by the remote controller designed according to Theorem 1 which is based on discrete time domain analysis.

Furthermore, as discussed in the introduction, most existing works consider designs in continuous time domain or for sample-data systems. In discrete time domain, less attention has been paid to the problem than in this paper, e.g., Xiao et al. (2000) model the problem via a discrete-time jump linear system and Lee and Kwon (2002) is for a system with state delay only. Nilsson (1998) considers the networked control problem with delays shorter than one sampling time. Hence, here we compare the proposed scheme with the stability criterion in Lee and Lee (1999). In Lee and Lee (1999), the discrete time delay system is of form
\[
x(k + 1) = Ax(k) + A_dx(k - h)
\]
and the delay dependent criterion in Theorem 2.1 is developed for discrete systems with analogues corresponding to the results obtained for continuous time delay systems in Xie and de Souza (1997) and in Cao et al. (1998). Let
\[
A_d = BK = \begin{bmatrix} 0 & 0 \\ -0.0399 & 0.0016 \end{bmatrix}
\]
for (26) which is identical to the proposed case when \(\Delta B_0 = 0\) and \(\Delta B_1 = -B\) since \(\Delta B_0 + \Delta B_1 = -B\) in this paper. The maximum time delay \(h = 68\) sampling steps for (26) in Lee and Lee (1999). However, in the proposed
scheme, after checking the delay-dependent condition in Proposition 2, the LMI in 16 is still feasible when we have times delay steps as $h \leq 136$. When checking the response of the closed-loop system as shown in figure 6, it is still stable though the settling time is longer. Hence, the delay-dependent criterion in Proposition 2 is less conservative than the one in Theorem 2.1 in Lee and Lee (1999).
6. Conclusions

The controller designs based on delay dependent stability conditions for discrete time remote control systems have been proposed in this paper by using LMI technique. Two cases classified according to the knowledge of the time delay bound have been discussed. Future work may be in the following areas: (i) on the robust stabilization of the remote control systems with uncertain external disturbances and (ii) on the case with data dropouts.

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References


