Conditions for the existence of continuous storage functions for nonlinear dissipative systems

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Abstract

The problem of existence of continuous storage functions for dissipative nonlinear systems is considered. It is shown that, if a nonlinear system is dissipative in the state $x_*$, then, under certain assumptions, a continuous storage function can be constructed on a set of points accessible from $x_*$ by concatenation of a finite number of forward and backward motions of the system. Most of these assumptions are weaker than certain controllability-type properties and can be checked using similar tests.

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1. Introduction

The theory of dissipative systems was established in the pioneering work of Willems [14] and subsequently developed in the series of papers by Hill and Moylan [3–5]. Since then this theory has played an increasingly important role in nonlinear control and, in particular, has found applications in nonlinear $H_\infty$ control [7,13], control of mechanical and port-controlled Hamiltonian systems [8], ISS systems and related notions [12], and other areas. See [8,13] for general references on the subject. Roughly speaking, a system is said to be dissipative if a certain integral functional is nonnegative along the system trajectories. One of the most important results in this theory states that the dissipativity property is equivalent to the existence of a nonnegative function, called storage function, which is defined on the state space of the system and satisfies a so-called dissipation inequality. This result provides a connection between the theory of dissipative systems and a variety of nonlinear control problems, since the storage function may serve as (or can be used for construction of) a Lyapunov function corresponding to a given control problem.
In general, however, a storage function is discontinuous, while the Lyapunov function is usually required to satisfy some regularity assumptions. For example, essentially stronger results in nonlinear $H_\infty$ control theory can be obtained under the assumption that the corresponding storage function is continuously differentiable [7]; recursive design procedures like backstepping require the corresponding Lyapunov function to be sufficiently smooth; the possibility of stabilization by feedback depends on existence of a continuous control Lyapunov function [1], etc. Thus the problem of finding conditions for the existence of more regular (continuous, smooth, etc) storage functions is of essential interest. However, this problem has received scant attention in the literature. For dissipative systems with supply rate of general form such a problem was addressed in [5,6,10]. In particular, in [6] it is proved that any dissipative system has a lower semicontinuous storage function. Hill and Moylan [5] stated that any storage function is continuous, if the dissipative system has the local $w$-uniform reachability property in all points of the state space. A more refined version of this result was presented in [10]; namely, if the system is dissipative in one point $x_*$ and satisfies the local $w$-uniform reachability assumption in the same point $x_*$, then there exists a continuous storage function defined on the set $\mathcal{R}(x_*)$ of points reachable from $x_*$. The main purpose of this paper is to provide conditions for the existence of a continuous storage function defined on a set larger than $\mathcal{R}(x_*)$. In fact, we show that, under some assumptions, a continuous storage function can be constructed on a set of points accessible from $x_*$ by concatenation of a finite number of forward and backward motions of the system. It is also shown that most of our assumptions are weaker than some well-studied controllability-type properties of the system and can be checked in a similar manner.

The structure of the paper is as follows. In Section 2, we introduce several definitions and preliminary needed throughout the rest of the paper. In particular, we introduce the relationship between our assumptions and the corresponding controllability-type properties of the system. The main results together with the proofs are presented in Section 3. Finally, Section 4 contains some concluding remarks.

2. Preliminaries

Consider a nonlinear control system

$$\dot{x} = F(x, u).$$

Here $x \in X \subset \mathbb{R}^n$ is state, $u \in U \subset \mathbb{R}^m$ is the input function, $X$ and $U$ are open and connected sets, $U$ contains 0, $F(x, u)$ is assumed to have following properties: for every fixed $u$ the function $F(\cdot, u)$ is of class $C^1$ (continuously differentiable), and both $F(x, u)$ and $\partial F/\partial x(x, u)$ are continuous on $x, u$. Under these conditions for every measurable essentially bounded control $u : [0, s) \rightarrow U$, where $s > 0$, and every initial condition $x_0 \in X$ the corresponding solution $x(t) = \phi(t, x_0, u)$ of system (1) exists at least for all $t \in [0, t_0]$, where $t_0 \in (0, s]$. A measurable essentially bounded control $u : [0, s) \rightarrow U$ is said to be admissible for an initial condition $x_0 \in X \subset \mathcal{U}^{[0,s)}$ if the corresponding solution $x(t) = \phi(t, x_0, u)$ is defined for all $t \in [0, s]$.

Let $\mathcal{E}$ be a subset of $X$. We will say that the set $\mathcal{E}$ is forward invariant with respect to trajectories of system (1) if for any $x_0 \in \mathcal{E}$, $u \in \mathcal{U}^{[0,t]}$, $t \geq 0$ the corresponding solution satisfies $\phi(s, x_0, u) \in \mathcal{E}$ for all $s \in [0, t]$. The set $\mathcal{E}$ is said to be backward invariant with respect to trajectories of (1) if it is forward invariant with respect to trajectories of the time-reversed system

$$\dot{x} = -F(x, u).$$

Finally, the set is said to be invariant if it is both forward and backward invariant.

Again, let $\mathcal{E}$ be a subset of $X$ ($\mathcal{E} \subset X$). The state $\xi \in \mathcal{E}$ is said to be reachable from $\eta \in \mathcal{E}$ (equivalently, $\eta \in \mathcal{E}$ is controllable to $\xi \in \mathcal{E}$) without leaving $\mathcal{E}$ if there exists $t \geq 0$ and $u \in \mathcal{U}^{[0,t]}$ such that $\phi(s, \eta, u) \in \mathcal{E}$ for all $s \in [0, t]$, and $\phi(t, \eta) = \xi$. If $\mathcal{E} = X$, we will simply say the state $\xi$ is reachable from $\eta$ ($\eta$ is controllable to $\xi$). The set of all points reachable from $\xi$ (controllable to $\eta$) without leaving $\mathcal{E}$ is denoted by $\mathcal{R}_\mathcal{E}(\xi)$ ($\mathcal{C}_{\mathcal{E}}(\eta)$), or, if $\mathcal{E} = X$, simply by $\mathcal{R}(\xi)$ ($\mathcal{C}(\eta)$).

Now let $\mathcal{N} \subset X$. The set $\mathcal{R}(\mathcal{N})$ of points reachable from $\mathcal{N}$ is defined as a union of sets reachable from each point of $\mathcal{N}$:

$$\mathcal{R}(\mathcal{N}) := \bigcup_{\eta \in \mathcal{N}} \mathcal{R}(\eta).$$
On the other hand, the set $\mathcal{C}(\mathcal{N})$ of points controllable to $\mathcal{N}$ is defined by the formula
\[
\mathcal{C}(\mathcal{N}) := \{x \in X : \mathcal{R}(x) \cap \mathcal{N} \neq \emptyset\}.
\]

Finally, given a set $A$, then its interior will be denoted by $\text{int} A$. Its closure by $\bar{A}$, and its boundary by $\partial A$.

2.1. Dissipativity, virtual storage and storage functions

Let $w(x,u)$ be a scalar function continuous on $X \times U$ in both arguments. Following [5,14] we will say that system (1) in the state $x_0 \in X$ is dissipative with respect to supply rate $w(x,u)$ if there exists $\beta \geq 0$ such that for each $t \geq 0$ and each $u \in \mathcal{U}_{x_0}^{[0,t)}$,
\[
\int_0^t w(\phi(s,x_0,u),u(s)) \, ds + \beta \geq 0.
\]
The following notion is closely related with the dissipativity property of the system.

Definition 1. The function $V : X \to R$ is said to be a virtual storage function for system (1) with supply rate $w(x,u)$ if for every $x_0 \in X$, every $t \geq 0$ and every $u \in \mathcal{U}_{x_0}^{[0,t)}$,
\[
V(\phi(t,x_0,u)) - V(x_0) \leq \int_0^t w(\phi(s,x_0,u),u(s)) \, ds.
\]

If, in addition, $V(x) \geq 0$ for all $x \in X$, then $V$ is called a storage function.

It is a well-known fact (see [5]) that the dissipativity property of the system is equivalent to the existence of a storage function, while so-called cyclo-dissipativity (i.e. dissipativity along the pieces of trajectories where the initial and the final states coincide) is equivalent to the existence of a virtual storage function. In general, however, the storage function is discontinuous. To formulate conditions for the existence of a continuous storage function, we need the following notion.

2.2. w-uniform reachability

Definition 2 (Hill and Moylan [5], Polushin and Marquez [10]). Let $w : X \times U \to R$ be a measurable function. The system (1) is said to be locally w-uniformly reachable in the state $x_*$ if there exists a neighborhood $\Omega$ of $x_*$ and a class-$\mathcal{K}$ function $\rho$ such that for each $x \in \Omega$ there exist $t > 0$ and $u \in \mathcal{U}_{x_*}^{[0,t)}$ such that $x = \phi(t,x_*,u)$ and
\[
\left| \int_0^t w(\phi(s,x_*,u),u(s)) \, ds \right| \leq \rho(|x-x_*|).
\]

It is shown in [5] that under the assumptions of dissipativity in every state and local uniform reachability in every state, every storage function is continuous. On the other hand, we have the following result.

Theorem 1 (Polushin and Marquez [10]). Suppose system (1) is dissipative in the state $x_* \in X$ with supply rate $w(x,u)$ and uniformly w-reachable in the same state $x_*$. Then there exists a continuous storage function defined on the set $\mathcal{R}(x_*)$.

The purpose of this paper is to present conditions under which a continuous storage function can be defined on a set larger than $\mathcal{R}(x_*)$.

Remark 1. In some cases the local w-uniform reachability property follows from the local controllability and, therefore, can be checked using controllability-type tests [10]. Below we formulate a simple statement of this type which generalizes the one given in [10]. To this end, denote
\[
\mathcal{R}_{T,L}(x_*) := \left\{ \phi(t,x_*,u) \middle| t \in [0,T], \begin{array}{c} u \in \mathcal{U}_{x_*}^{[0,t)}, \\
\sup_{s \in [0,t]} |u(s)| \leq L \end{array} \right\}.
\]

In words, $\mathcal{R}_{T,L}(x_*)$ is the set of states reachable from $x_*$ in time less than or equal to $T$ by an admissible control essentially bounded by $L$. System (1) in the state $x_* \in X$ is said to be small-time locally controllable by uniformly bounded control (STLC-UBC) if there exists $L < \infty$ such that for any $T > 0$,
\[
x_* \in \text{int} \mathcal{R}_{T,L}(x_*).
\]

In other words, the system is STLC-UBC if $\mathcal{R}_{T,L}(x_*)$ contains an open neighborhood of $x_*$. The notion of STLC-UBC is a version of small-time local controllability notion extensively studied in the literature (see [9,11] and the bibliography therein).
**Proposition 1.** Let \( w : X \times U \rightarrow R \) be a measurable locally bounded function. System (1) is locally \( w \)-uniformly reachable in the state \( x_0 \) if it is STLC-UBD in \( x_0 \).

**Proof.** Take an arbitrary \( \delta > 0 \) and consider the closed ball \( B_\delta(x_0) := \{ x \in X : |x - x_0| \leq \delta \} \). By assumption, the right-hand side \( F(x, u) \) as well as supply rate \( w(x, u) \) are continuous in both arguments, therefore we have

\[
\sup_{x \in B_\delta(x_0)} |F(x, u)| = F^* < \infty
\]

and

\[
\sup_{x \in B_\delta(x_0)} |w(x, u)| = D^* < \infty.
\]

Put \( T^* = \delta/F^* \). From (4) we see that

\[
\mathcal{R}_{T,L}(x_0) \subset B_\delta(x_0)
\]

for all \( T \in [0, T^*] \).

Now for each \( T \in [0, T^*] \) define \( \beta^*(T) \) as the supremum of all possible \( \beta \geq 0 \) such that

\[
\{ x \in X : |x - x_0| < \beta \} \subset \mathcal{R}_{T,L}(x_0).
\]

Clearly, \( \beta^*(0) = 0 \), \( \beta^*(\cdot) \) is nondecreasing, and, since the system is STLC-UBD, we have \( \beta^*(T) > 0 \) for all \( T \in (0, T^*) \). Further, take any continuous function \( \beta : [0, T^*] \rightarrow \mathbb{R}^+ \) strictly increasing and satisfying \( \beta(t) \leq \beta^*(t) \) for all \( t \in [0, T^*] \). Such a function \( \beta \) always exists (for the proof of this fact see Appendix A). Due to the properties of \( \beta \), the inverse function \( \beta^{-1}(s) \) is well defined for \( s \in [0, \beta(T^*)] \), satisfies \( \beta^{-1}(0) = 0 \), and is strictly increasing. By construction of \( \beta^{-1}(\cdot) \), we see that \( |x - x_0| \leq \beta(T^*) \) implies \( x \in \mathcal{R}_{\beta^{-1}(|x-x_0|),L}(x_0) \), i.e. there exists \( t \in [0, \beta^{-1}(|x-x_0|)] \), and a control \( u \in \mathcal{U}_{\beta^{-1}(|x-x_0|)} \) such that \( \phi(t, x, u) = x \). Using (5), (6), we see that along the corresponding trajectory

\[
\left| \int_0^T w(\phi(s, x, u), u(s)) \, ds \right| \leq D^* \beta^{-1}(|x - x_0|).
\]

This completes the proof. \( \square \)

Proposition 1 shows that a number of existing tests for local small-time controllability can also be applied to determine the \( w \)-uniform reachability property.

A nonlocal version of \( w \)-uniform reachability notion can be defined as follows.

**Definition 3.** Let \( \mathcal{M}, \mathcal{N} \) be subsets of \( X \). We will say that \( \mathcal{M} \) is \( w \)-uniformly reachable from \( \mathcal{N} \) if \( \mathcal{M} \subset \mathcal{R}(\mathcal{N}) \), and there exists \( K < +\infty \) such that for each \( \mu \in \mathcal{M} \) there exists \( \eta \in \mathcal{N} \) and \( u \in \mathcal{U}_{\eta}^{0,1}, t \geq 0 \) such that \( \phi(t, \eta, u) = \mu \) and

\[
\int_0^T w(\phi(s, \eta, u), u(s)) \, ds \leq K.
\]

Analogously, one can define the following “dual” notion of \( w \)-uniform controllability.

**Definition 4.** Let \( \mathcal{M}, \mathcal{N} \) be subsets of \( X \). We will say that \( \mathcal{N} \) is \( w \)-uniformly controllable to \( \mathcal{M} \), if \( \mathcal{N} \subset \mathcal{C}(\mathcal{M}) \), and there exists \( K < +\infty \) such that for each \( \eta \in \mathcal{N} \) there exists \( \mu \in \mathcal{M} \) and \( u \in \mathcal{U}_{\eta}^{0,1}, t \geq 0 \) such that \( \phi(t, \eta, u) = \mu \) and (7) holds.

2.3. Weak accessibility

**Definition 5** (Hermann and Krener [2]). The state \( \xi \in X \) is said to be weakly accessible from the state \( \eta \in X \) if there exists a finite number of states \( x_0, x_1, \ldots, x_n \in X \) such that \( x_0 = \xi \), \( x_n = \eta \), and \( x_{i+1} \in \mathcal{R}(x_i) \cup \mathcal{C}(x_i) \) for any integer \( 0 \leq i \leq n - 1 \).

It is worth noting that, according to this definition, weak accessibility is an equivalence relation, while the reachability and the controllability are not. More precisely; both controllability and reachability are reflexive and transitive relations, but in general they are not symmetric. Weak accessibility, on the other hand, is clearly symmetric. To see this note that if \( x \in \mathcal{R}(y) \) then \( y \in \mathcal{C}(x) \), and vice versa. Since weak accessibility is an equivalence relation, one can consider the following partition of the state space

\[
X = \bigcup_{i \in \Xi} W_i,
\]

where the \( W_i \)'s are equivalence classes with respect to the weak accessibility relation. Note that each set \( W_i \) is invariant with respect to trajectories of the controlled system.
We now provide a construction that will be used in the sequel. For a given $\xi \in X$ define the sequence of sets $\Omega_i$, $i = 1, 2, \ldots$ as follows:

$$\Omega_1(\xi) = \mathcal{R}(\xi)$$

(8)

and

$$\Omega_i(\xi) = \mathcal{R}^i(\xi)$$

(9)

for $i = 2, 3, \ldots$. Thus, each set $\Omega_i(\xi)$, $i = 1, 2, \ldots$, consists of points accessible from $\xi$ by concatenation of $k$ possible forward and backward motions of system (1), where $k \in \{0, 1, \ldots, 2i - 1\}$. Clearly,

$$\ldots \subset \Omega_{i-1}(\xi) \subset \Omega_i(\xi) \subset \Omega_{i+1}(\xi) \subset \ldots$$

It is also easy to see that each set $\Omega_i(\xi)$, $i = 1, 2, \ldots$, is forward invariant with respect to trajectories of the controlled system (1), and

$$\mathcal{W}(\xi) = \bigcup_{i=1,2,\ldots} \Omega_i(\xi),$$

(10)

where $\mathcal{W}(\xi)$ is the class of states equivalent to $\xi$ with respect to the weak accessibility relation.

2.4. Local $w$-uniform accessibility

**Definition 6.** Let $w: X \times U \to \mathbb{R}$ be a measurable function. The system is called locally $w$-uniformly accessible at the state $\xi \in X$, if for any $\varepsilon > 0$ and for any open neighbourhood $Q$ of $\xi$ the set

$$\mathcal{A}_{Q, \varepsilon}(\xi) = \left\{ x \in Q: \exists t \geq 0, u \in \mathcal{W}^{[0, t]}(\xi), \text{s.t.} \phi(t, \xi, u) = x, \int_0^t |w(\phi(s, \xi, u), u(s))| \, ds < \varepsilon \right\}$$

has a nonempty interior.

The concept of local $w$-uniform accessibility, introduced in Definition 6, is also related to the more traditional accessibility-type property. Indeed, using notation (3), the following statement can be formulated.

**Proposition 2.** Let $w: X \times U \to \mathbb{R}$ be a measurable locally bounded function. System (1) is locally $w$-uniformly accessible in the state $x_* \in X$ if there exists $L < \infty$ such that for any $T > 0$

$$\text{int} \mathcal{R}_{T, L}(x_*) \neq \emptyset.$$  

**Proof.** Given $\varepsilon > 0$, and let $Q$ be an open neighbourhood of $x_*$. Take an arbitrary $\delta > 0$ such that $B_{\delta}(x_*) := \{ x \in X: |x - x_*| \leq \delta \} \subset Q$. We have (see the proof of Proposition 1)

$$\sup_{x \in B_{\delta}(x_*)} |F(x, u)| = F^* < \infty,$$

$$\sup_{x \in B_{\delta}(x_*)} |w(x, u)| = D^* < \infty.$$  

Put

$$T^* = \min \left\{ \frac{\delta}{F^*}, \frac{\varepsilon}{D^*} \right\}.$$  

For any $T \in [0, T^*)$ we have

$$\mathcal{R}_{T, L}(x_*) \subset \mathcal{A}_{Q, \varepsilon}(x_*),$$

therefore

$$\text{int} \mathcal{R}_{T, L}(x_*) \subset \text{int} \mathcal{A}_{Q, \varepsilon}(x_*).$$

By assumption, $\text{int} \mathcal{R}_{T, L}(x_*) \neq \emptyset$, therefore $\text{int} \mathcal{A}_{Q, \varepsilon}(x_*) \neq \emptyset$. This completes the proof. \[\square\]

Proposition 2 shows that the local $w$-uniform accessibility property follows from a version of the local accessibility property widely studied in the literature [2,9,11]. In particular, the local $w$-uniform accessibility can be checked by calculating the rank of the corresponding Lie algebra. Let $\mathcal{L}$ be the Lie algebra generated by the set of vector fields $\mathcal{F}_u := \{ F(\cdot, u), u \in U \}$. Using standard results in nonlinear control theory [2,9,11], one can easily prove the following consequence of Proposition 2:

**Corollary 1.** Let $w: X \times U \to \mathbb{R}$ be a measurable locally bounded function. System (1) is locally $w$-uniform accessible in the state $x_* \in X$ if

$$\text{rank} \mathcal{L}(x_*) = n.$$  

3. Main results

In this section we present the main results of the paper. More explicitly; we obtain conditions
Theorem 2. Suppose system (1) is dissipative at some state $x_\star \in X$ with supply rate $w(x, u)$ continuous in both arguments, and the following properties are satisfied:

(i) system (1) is locally $w$-uniformly reachable at $x_\star$;
(ii) for each $i = 1, 2, \ldots$ the set $\mathcal{C}(\Omega_i(x_\star))$ of states controllable to $\Omega_i(x_\star)$ is $w$-uniformly controllable to $\Omega_i(x_\star)$, and for each $i = 2, 3, \ldots$ the set $\Omega_i(x_\star)$ of points reachable from $\mathcal{C}(\Omega_{i-1}(x_\star))$ is $w$-uniformly reachable from $\mathcal{C}(\Omega_{i-1}(x_\star))$.

(iii) system (1) is locally $w$-uniformly accessible on the set

$$\Omega^*(x_\star) := \bigcup_{i=1, 2, \ldots} \left( (\hat{c}\mathcal{C}\Omega_i(x_\star) \cap \mathcal{C}(\Omega_i(x_\star))) \right.$$

$$\left. \cup (\hat{c}\mathcal{C}\mathcal{C}\mathcal{C}\Omega_i(x_\star) \cap \Omega_{i+1}(x_\star))) \right).$$

Then there exists a continuous virtual storage function defined on the set $\mathcal{H}(x_\star)$.

Remark 2. Note that properties (i) and (iii) can be checked constructively using controllability-type tests, as discussed in Section 2. On the other hand, assumption (ii) guarantees that the virtual storage function is bounded uniformly on each set $\Omega_i$, $i = 1, 2, \ldots$.

Remark 3. Recall that the set $\mathcal{H}(x_\star)$ is invariant with respect to both forward and backward motions of system (1). Moreover, $\mathcal{H}(x_\star)$ is the minimal invariant set that contains $x_\star$.

Proof. For each state $x \in \mathcal{H}(x_\star) = \Omega_1(x_\star)$ define a function $V$ as follows:

$$V(x) = \inf_{\substack{\tau \geq 0, u \in \mathcal{U}(0, \tau), \\ \phi(\tau, x, u) = x}} \int_0^\tau w(\phi(s, x, u), u(s)) \, ds.$$  (13)

Thus defined function $V$ is called the required supply [5, 14]. Taking into account the dissipativity property and assumption (ii), we see that

$$-\beta(x_\star) \leq V(x) \leq K_1 - \beta(x_\star)$$

on the set $\Omega_1(x_\star)$ for some $K_1 < \infty$. By construction, the set $\Omega_1(x_\star)$ is connected. On the other hand, as shown in [10], assumption (i) implies that $\Omega_1(x_\star)$ is an open set, and $V$ is continuous on $\Omega_1(x_\star)$.

Consider now the set $\mathcal{C}(\Omega_1(x_\star))$. This set is clearly open and connected. For each $x \in \mathcal{C}(\Omega_1(x_\star)) \Omega_1(x_\star)$ define $V(x)$ by the formula

$$V(x) = \sup_{\substack{\tau \geq 0, u \in \mathcal{U}(0, \tau), \phi(\tau, x, u) = x}} \left( V(\tilde{x}) - \int_0^\tau w(\phi(s, x, u), u(s)) \, ds \right).$$  (14)
Due to (ii), we see that
\[-\beta(x_\ast) - K_2 \leq V(x) \leq K_1 + K_2 - \beta(x_\ast)\]
on the set $\mathcal{C}((\Omega_1(x_\ast)))$ for some $K_2 < +\infty$. To show that defined in this way the function $V$ satisfies the dissipation inequality (2), take any point $x_0 \in \mathcal{C}(\Omega_1(x_\ast))$. Take an arbitrary $\hat{u} \in \mathcal{U}^{[0,t]}_{x_0}$, $t > 0$ and suppose $\phi(t_1, x_0, \hat{u}) = x_1 \in \mathcal{C}(\Omega_1(x_\ast)) \setminus \Omega_1(x_\ast)$. Then
\[
V(x_0) = \sup_{t \geq 0, u \in \mathcal{U}^{[0,t]}_{x_0} \setminus \Omega_1(x_\ast)} \left( V(\phi(t, x_0, u)) - \int_0^t w(\phi(s, x_0, u), u(s)) \, ds \right) \geq \sup_{t \geq 0, u \in \mathcal{U}^{[0,t]}_{x_0} \setminus \Omega_1(x_\ast)} \left( V(\phi(t, x_1, u)) - \int_0^t w(\phi(s, x_1, u), u(s)) \, ds \right) = -\int_0^t w(\phi(s, x_0, \hat{u}), \hat{u}(s)) \, ds + V(x_1).
\]
Otherwise, suppose $\phi(t_1, x_0, \hat{u}) = x_1 \in \Omega_1(x_\ast)$, then
\[
V(x_0) = \sup_{t \geq 0, u \in \mathcal{U}^{[0,t]}_{x_0} \setminus \Omega_1(x_\ast)} \left( V(\phi(t, x_0, u)) - \int_0^t w(\phi(s, x_0, u), u(s)) \, ds \right) \geq V(x_1) - \int_0^t w(\phi(s, x_0, \hat{u}), \hat{u}(s)) \, ds.
\]
Now we claim that under the conditions of the theorem the function $V$ is continuous on the set $\mathcal{C}(\Omega_1(x_\ast))$. Indeed, continuity of $V$ on the set $\operatorname{int}(\mathcal{C}(\Omega_1(x_\ast)) \setminus \Omega_1(x_\ast))$ clearly follows from the continuity of $V$ on the set $\Omega_1(x_\ast)$ and continuity of $w(x, u)$. To prove continuity of $V$ on the set $\partial \Omega_1(x_\ast) \cap \mathcal{C}(\Omega_1(x_\ast))$, take any point $x_0 \in \partial \Omega_1(x_\ast) \cap \mathcal{C}(\Omega_1(x_\ast))$ and fix $\varepsilon > 0$. It is easy to see using standard continuous dependence arguments that there exists $\delta > 0$ such that for any $x \in \mathcal{C}(\Omega_1(x_\ast)) \setminus \Omega_1(x_\ast)$, $|x - x_0| < \delta$ we have $|V(x) - V(x_0)| < \varepsilon$. On the other hand, take an arbitrary $x \in \Omega_1(x_\ast)$ sufficiently close to $x_0$. Suppose $\hat{u} \in \mathcal{U}^{[0,t]}_{x_0}$ is an arbitrary control such that $\phi(t, x_0, \hat{u}) \in \Omega_1(x_\ast)$ and
\[
V(\phi(t, x_0, \hat{u})) - \int_0^t w(\phi(s, x_0, \hat{u}), \hat{u}(s)) \, ds \geq V(x_0) - \frac{\varepsilon}{3}.
\]
If $x$ is sufficiently close to $x_0$, then we have $\phi(t, x, \hat{u}) \in \Omega_1(x_\ast)$,
\[
|V(\phi(t, x, \hat{u})) - V(\phi(t, x_0, \hat{u}))| \leq \frac{\varepsilon}{3}
\]
and
\[
\left| \int_0^t w(\phi(s, x, \hat{u}), \hat{u}(s)) \, ds \right| \leq \frac{\varepsilon}{3}.
\]
Combining the above formulas with the dissipation inequality
\[
V(\phi(t, x, \hat{u})) \leq V(x) + \int_0^t w(\phi(s, x, \hat{u}), \hat{u}(s)),
\]
we see that
\[
V(x) \geq V(x_0) - \varepsilon. \tag{15}
\]
To prove the inequality opposite to (15), take a sufficiently small open neighborhood $\mathcal{U}(x_0)$ of the point $x_0$ such that for any $x_1, x_2 \in \mathcal{U}(x_0) \cap \Omega_1(x_\ast)$ we have $|V(x_1) - V(x_2)| < \varepsilon/2$. By the assumption (iii), there exists a nonempty open subset $\mathcal{U}_0 \subset \mathcal{U}(x_0)$ with the following property: for any state $\zeta \in \mathcal{U}_0$ there exists a control $u \in \mathcal{U}^{[0,t]}_{x_0}$, $t \geq 0$ such that $\phi(t, x_0, u) = \zeta$, and
\[
\left| \int_0^t w(\phi(s, x_0, u), u(s)) \, ds \right| < \frac{\varepsilon}{2}.
\]
First, we claim that
\[
\mathcal{U}_0 \cap \Omega_1(x_\ast) \neq \emptyset. \tag{16}
\]
Indeed, if $\mathcal{U}_0 \cap \Omega_1(x_\ast) = \emptyset$, then $\mathcal{U}_0 \subset \operatorname{int}(X \setminus \Omega_1(x_\ast))$. Then for any state $x_0 \in \Omega_1(x_\ast)$ sufficiently close to $x_0$, we have $\phi(t, x_0, u) \in \mathcal{U}_0 \subset \operatorname{int}(X \setminus \Omega_1(x_\ast))$, which contradicts the fact that $\Omega_1(x_\ast)$ is invariant with respect to trajectories of the controlled system. Now, take any point $x_1 \in \mathcal{U}_0 \cap \Omega_1(x_\ast)$. By definition of $\mathcal{U}_0$
we have \( V(x_1) \leq V(x_0) + \varepsilon/2 \). Therefore, by definition of \( \Upsilon(x_0) \) for any \( x \in \Upsilon(x_0) \cap \Omega_1(x_1) \) we have
\[
V(x_0) \geq V(x) - \varepsilon. \tag{17}
\]

Combining (15) and (17), we get that the function \( V \) is continuous on the set \( \partial \Omega_1(x_1) \cap \mathcal{C}(\Omega_1(x_1)) \). Therefore, it is continuous on \( \mathcal{C}(\Omega_1(x_1)) \).

Consider now the set \( \Omega_2(x_1) := \mathcal{R}(\mathcal{C}(\Omega_1(x_1))) \).

Define an extension of the function \( V \) on the set \( \Omega_2(x_1) \backslash \mathcal{C}(\Omega_1(x_1)) \) as follows:
\[
V(x) := \inf_{\tilde{x} \in \Omega_1(x_1), t \geq 0, u \in \mathcal{W}[0,t]} \left( V(\tilde{x}) + \int_0^t u(\phi(s, x, u), u(s)) \, ds \right). \tag{18}
\]

By (ii) we see that thus defined function \( V \) is uniformly bounded on the set \( \Omega_2(x_1) \). Using the same line of reasoning as above, one can prove that \( V \) is continuous on the set \( \Omega_2(x_1) \) and satisfies the dissipation inequality (2) along the trajectories of the system.

Thus we have proven that the function \( V \) is a continuous virtual storage function defined on the set \( \Omega_2(x_1) \). Continuing this line of reasoning, one can extend \( V \) to the sets \( \Omega_3, \Omega_4, \ldots \) etc. Taking into account formula (10), we get the result of Theorem 2.

Under assumptions of Theorem 2, the existence of a continuous storage function is guaranteed by the following corollary.

**Corollary 2.** Under the assumptions of Theorem 2, for any set \( \Omega_i(x_1), i = 1, 2, \ldots \), there exists a continuous storage function defined on \( \Omega_i(x_1) \).

**Proof.** By the construction of Theorem 2, the continuous virtual storage function \( V \) is uniformly bounded on each set \( \Omega_i(x_1), i = 1, 2, \ldots \). Therefore, for any \( i = 1, 2, \ldots \) the function \( V \) can be made nonnegative on \( \Omega_i(x_1) \) simply by adding an appropriate constant.

4. Concluding remarks

In this paper, we have shown that if a nonlinear system of the general form (1) is dissipative in the state \( x_s \) and locally \( w \)-uniformly reachable from the same state \( x_s \), then under additional assumptions (ii), (iii) of Theorem 2 there exists a continuous virtual storage function defined on the set of points weakly accessible from \( x_s \). By construction, this function is bounded from below on each set \( \Omega_i(x_1), i = 1, 2, \ldots \), therefore it can be made nonnegative on each such a set simply by adding an appropriate positive constant. We also show that most of our assumptions follow from the well-studied controllability-type properties and can be checked by the similar tests.

As mentioned in the introduction, one of the main interests of studying dissipative systems is that a storage function of a dissipative system can be used as a Lyapunov (control Lyapunov) function of a certain stability (stabilization) problem. Note that continuity of a storage function may already be a sufficient regularity assumption to state (using a suitably generalized notion of derivative) certain stability results. For example, as shown in [1], for a general nonlinear system the existence of a continuous control-Lyapunov function is both necessary and sufficient for the existence of a stabilizing feedback, while, for example, a smooth control-Lyapunov function may fail to exist in this case. In general, however, much stronger stability results can be obtained if the corresponding storage function is more regular (locally Lipschitz, smooth, etc.). Thus, the development of conditions for existence of more regular storage functions should be a topic for future research.

**Appendix A. Proof of a technical fact**

The following technical fact has been used in the proof of Proposition 1. Although it is almost obvious, for the sake of completeness we give a proof here.

**Lemma 1.** Suppose there exists a function \( \beta^* : [0, T^*] \to \mathbb{R}^+, T^* > 0 \), with the following properties:

- it is nondecreasing and satisfies \( \beta^*(0) = 0, \beta^*(t) > 0 \) for all \( t \in (0, T^*) \).
- Then there exists a continuous function \( \beta : [0, T^*] \to \mathbb{R}^+ \) such that \( \beta(0) = 0, \beta(\cdot) \) is strictly increasing and satisfies \( \beta(t) \leq \beta^*(t) \) for all \( t \in [0, T^*] \).

**Proof.** Given \( \beta^*(\cdot) \), consider a function \( \phi_1 : [0, T] \to \mathbb{R}^+ \), \( \phi_1(0) = 0 \), defined for \( s \in (0, T^*] \) as follows:
\[
\phi_1(s) := \lim_{t \to s^-} \beta^*(t).
\]
Since $\beta^*(\cdot)$ is nondecreasing, we see that $\phi_1(\cdot)$ is well-defined and satisfies $\phi_1(t) \leq \beta^*(t)$ for all $t \in [0, T^*]$. By construction, $\phi_1(\cdot)$ is left semicontinuous and therefore integrable. Take an arbitrary $s_0 \in (0, 1)$, and define the function $\phi_2(\cdot)$ as follows:

$$\phi_2(s) := \frac{1}{R(s)} \int_0^s \phi_1(t) \, dt,$$

where

$$R(s) = \begin{cases} s & \text{if } s \geq s_0, \\ s_0 & \text{if } s \in [0, s_0). \end{cases}$$

Function $\phi_2(\cdot)$ is clearly continuous and satisfies $\phi_2(0) = 0$. Since $\phi_1(\cdot)$ is nondecreasing, it is easy to see that $\phi_2(\cdot)$ is also nondecreasing, and $\phi_2(t) \leq \phi_1(t)$ for all $t \in [0, T^*]$. Finally, put

$$\beta(t) = \frac{t + T^*}{2T^*} \phi_2(t).$$

We see that $\beta(0) = 0$, $\beta(\cdot)$ is strictly increasing and satisfies $\beta(t) \leq \phi_2(t)$ for all $t \in [0, T^*]$. This completes the proof. □

References


