Abstract—This paper generates improved stability conditions for a class of feedback systems consisting of two elements, one of which is a linear time invariant plant with parametric uncertainty and the other, an odd, monotone nondecreasing nonlinearity confined to the sector \([0, k]\). The improvement results from the use of noncausal multipliers which reduces the requirement of passivity on the feedback elements.

I. INTRODUCTION

In this paper, we are interested in the absolute stability of systems that consist of a linear time invariant-subsystem with parametric uncertainties in the forward path and a nonlinear subsystem in the feedback path. In the case where the feedback elements are both nonlinear, there has been well-known contributions to the field of absolute stability of feedback systems by Sandberg, Zames, Desoer, and Vidyasagar, and Willems to name but a few. One of the most important results is the Passivity Theorem, which states that the closed loop system is stable if both feedback elements are passive, and one of them is strongly passive with finite gain [4], [11]. The Popov stability condition is another well-known result that may be derived using the Passivity Theorem when one of the feedback elements is a linear system with no uncertainty, and the other is a memoryless nonlinearity that satisfies a sector condition. This class of systems was extensively studied in the 1960's and 1970's, and many stability results were obtained [4], [11]–[13], all of them based on the well established technique of introducing an artificial multiplier into the loop and then applying the Passivity Theorem on the transformed system. We will cast our problem in a similar framework and then derive improved stability conditions for a class of nonlinear feedback systems that contain a linear time invariant plant with parametric uncertainty.

If the nonlinearity belongs to the sector \([0, k]\), then the classical approach does not directly allow determination of the maximum value, say \(k = k_{\text{max}}\), that guarantees closed loop stability. This is usually an important stability issue. However, under the mild assumption that the nonlinearity is slope bounded, \(k_{\text{max}}\) can be determined by inserting the multiplier after a loop transformation, [12]. If this procedure is applied to a plant that contains parametric uncertainties, then the loop transformation will propagate the uncertainty around the interval plant to the extent that the results may be more conservative. Our development permits the determination of \(k_{\text{max}}\) without using loop transformations and does not require the nonlinearity to be slope-bounded.

A substantial amount of work has been done in recent years on strict positive realness (SPR) of interval plants. See [2], [14], [15] and the references therein. These results are extensions of Kharitonov’s work [7] on the stability of a family of plants. Passivity of the family of plants is determined indirectly using the well-known relationship between passivity and strict positive realness (or more precisely, passivity and positive realness). Thus, stability results can be obtained in this framework using the Passivity Theorem. One of the shortcomings of using SPR conditions for ensuring passivity is the requirement for the linear system under consideration to be causal. Although physical systems are always causal, it is desirable to remove this requirement because an artificial noncausal multiplier may be introduced into the analysis as a mechanism for reducing the conservativeness of the resulting absolute stability conditions.

The main result of the paper is the derivation of sufficient conditions for stability of the feedback interconnection of an interval plant and an odd, monotone non-decreasing nonlinearity in the sector \([0, k]\), a case that has practical application. Our stability condition involves checking for the strong positivity of a shifted family of plants cascaded with a, not necessarily rational, noncausal multiplier, a concept that differs from the more restrictive notion of SPR studied in [1], [3], [8], and [10] among others.

II. PROBLEM FORMULATION

Let \(\mathbb{R}\) and \(\mathbb{R}^+\) denote the real and nonnegative real numbers. A function \(z: \mathbb{R}^+ \to \mathbb{R}\) belongs to \(L_2\) if and only if \(\int_0^\infty |z(t)|^2 dt < \infty\). Define the usual inner product in \(L_2\) and the \(L_2\) norm by \((x, y) = \int_0^\infty x(t)y(t) dt\), and \(\|x\|^2 = (x, x)\). Denote \(x_T(.)\) a truncation of the function \(x(.)\) to the interval \([0, T]\). Then \(x_T(t) = x(t)\) for \(t < T\), and \(x_T = 0\) elsewhere. The extension of the space \(L_2\), denoted by \(L_{2e}\), is defined as follows: \(L_{2e} = \{x(.): x_T(.) \in L_2, \forall T \in \mathbb{R}\}\).

We now introduce the following definitions and refer the reader to [4] or [13] for further details.

Definition 1: An operator \(H: L_{2e} \to L_{2e}\) is said to be strongly passive if there exists a \(\delta > 0\) such that \(\delta(x_T, x_T) \geq \delta(x_T, x_T), \forall x(.) \in L_{2e}\), and \(H\) is passive if the inequality is satisfied with \(\delta = 0\).
A. Factorization of Positive Operators

Let \( M \) be a convolution operator defined by \( Mx(t) = h(t) \otimes x(t) \), where the symbol, \( \otimes \), denotes convolution. Suppose in addition, \( M \) satisfies the following conditions:

1. \( M \in L_2 \),
2. \( M \) is invertible in \( L_2 \),
3. \( \gamma(M) = \sup \| (Mx) / \|x\| < \infty, x(\cdot) \in L_2 \).

The operators, \( M \), satisfying conditions 1)-3) form a commutative Banach algebra \( B = B_c \cup B_ac \), where \( B_c \) and \( B_ac \) are the Banach algebras of causal and anticausal operators, respectively. Let \( P_a \) and \( P_c \) be the projection on \( B \) with ranges \( B_c \) and \( B_c \), respectively. Thus, \( B_c \) and \( B_ac \) are the subspaces spanned by \( B_c \) and \( B_c \), respectively. It can be shown [9] that if \( M \) is an arbitrary strongly positive operator in \( B \), then there exist elements \( M_+ = \exp[P_a \log M] \) and \( M_- = \exp[P_c \log M] \) such that

1. \( M_+ \in B_ac \), and \( M_- \in B_c \),
2. \( M = M_+ M_- \),
3. \( M_+ \) and \( M_- \) are invertible with \( M_+^{-1} \in B_c \) and \( M_-^{-1} \in B_ac \).

B. Interval Plants

Let \( \Delta \) be a real polynomial of \( n \)th degree, \( \Delta(s) = \delta_n + \delta_{n-1}s + \cdots + \delta_2s^2 + \delta_1s + \delta_0 \). We say that \( \Delta(s) \) belong to the family \( F_\Delta \) of interval polynomials if each coefficient \( \delta_i \) can take values between a lower and an upper bound, i.e., \( \delta_i \in [x_i, y_i] \). We will denote by \( K_{1,\Delta}^l(s), K_{1,\Delta}^u(s), K_{2,\Delta}^l(s), K_{2,\Delta}^u(s) \), and \( K_{3,\Delta}^l(s), K_{3,\Delta}^u(s) \) the four Kharitonov polynomials associated with \( F_\Delta \) [15], and \( S_{1,\Delta}^l, S_{1,\Delta}^u, S_{2,\Delta}^l, S_{2,\Delta}^u, S_{3,\Delta}^l, S_{3,\Delta}^u \) the associated line segments, defined by \( S_{i,\Delta}^l = (1 - \lambda)K_{i,\Delta}^l + \lambda K_{i,\Delta}^u \), \( \lambda \in [0, 1] \). Notice that there are six possible combinations of the four Kharitonov polynomials \( K_{i,\Delta} \). The indices \( i \), \( j \) of \( K_{i,\Delta}^l \) and \( K_{i,\Delta}^u \) should be chosen such that \( K_{1,\Delta}^l \) and \( K_{1,\Delta}^u \) have either the same \( \text{even} \) order coefficients, or the same \( \text{odd} \) order coefficients.

Given a linear plant with transfer function \( H(s) = n(s)/d(s) \) where \( n(s) \) and \( d(s) \) belong to the families \( F_n \) and \( F_d \) of interval polynomials, we define the Kharitonov Systems \( K_{sys} \) and the Kharitonov Segments \( K_{seg} \)

\[
K_{sys} = K_{1,n}^u / K_{1,d}^l, \quad i, j = 1, 2, 3, 4
\]

\[
K_{seg} = K_{1,n}^u / S_{1,d}^l, \quad k = 1, 2, 3, 4
\]

\[
(i, j) = (1, 2), (1, 4), (2, 3), (3, 4).
\]

In the sequel, we will make use of the geometric properties of interval plants. Consider a family \( F_\Delta \) of interval polynomials, and let \( F(\omega) \triangleright \delta \{ \delta(j\omega) : \delta \in F_\Delta \} \). Then at each frequency \( \omega_0 \), \( F(\omega_0) \) represents a rectangle in the complex plane known as the value set, with vertices \( K_{1,\Delta}^l(j\omega_0), K_{2,\Delta}^l(j\omega_0), K_{3,\Delta}^l(j\omega_0), \) and \( K_{3,\Delta}^u(j\omega_0) \). See for example, [2].
(-\infty, \infty), b) G is strongly positive iff \text{Re}[G(j\omega)] \geq \delta > 0, \forall \omega \in (-\infty, \infty). Notice that g need not be causal.

Consider now the function, \( R = MP + k \), where \( P \) represents a linear time invariant plant with transfer function \( P(s) = n(s)/d(s) \), \( n(s) \) and \( d(s) \) belong to the families \( \mathcal{F}_n \) and \( \mathcal{F}_d \) of interval polynomials, and \( k \) is a positive real number. \( M \) is a noncausal convolution operator that represents a multiplier. Denote the Fourier transform of \( M(j\omega) \), \( \mathcal{F}(\omega) \) and \( \mathcal{F}_n(\omega) \) as \( M(j\omega) \), \( \mathcal{F}(\omega) \) and \( \mathcal{F}_n(\omega) \) respectively.

In other words, \( R(j\omega, \omega, \mu) = (M(j\omega)N(\omega, \lambda) + kD_{\omega}(\omega, \mu))/D_{\omega}(\omega, \mu) \), \( \lambda, \mu \in [0, 1], \), \( i, j, h, q = 1, 2, 3, 4. \) (3.1)

Then, at each frequency, the function \( r(\omega, x, y) \) defines a set, \( Q \), that is bounded, and \( \partial Q \subseteq R(j\omega, \lambda, \mu). \)

Proof: \( Q \) is bounded since \( N \) and \( D \) are bounded, and \( y \in D \neq 0. \) To prove that \( \partial Q \subseteq R(j\omega, \lambda, \mu) \), we first recognize that since both \( N \) and \( D \) are closed, then \( N = N^o \cup \partial N \) and \( D = D^o \cup \partial D \). Consider a point \( z_1 \in \partial Q \), and assume that \( z_1 \notin R(j\omega, \lambda, \mu). \) If \( z_1 = [(M(j\omega)N_1(\omega, \lambda) + kD_{\omega}(\omega, \mu))/D_{\omega}(\omega, \mu) \), \( \lambda, \mu \in [0, 1], \), then one of the following conditions must be true: i) \( z_1 \in \partial N \) and \( y_1 \in D^o \), ii) \( z_1 \in N^o \) and \( y_1 \in \partial D \), or iii) \( x_1 \in N^o \) and \( y_1 \in D^o \).

Thus, the points \( z_1 \in \partial N \) and \( y_1 \in \partial D \) define a neighborhood \( N(z_1, \delta) = \{ z \in N : d(z, z_1) < \delta \} \), such that for all \( z_2 \in N(z_1, \delta), \) \( r(z_2, y_1) \in Q \). In this case, we can write \( z_2 = [M(j\omega)x_2 + k\gamma_1]/y_1, \) \( z_2 \in Q \), and since \( y_1 \) is constant and different from zero, then \( z_2 = ax_2 + k \).

Thus, the points \( z_1 \in \partial N \) and \( y_1 \in \partial D \) define a neighborhood \( N(z_1, \delta) = \{ z \in N : d(z, z_1) < \delta \} \), such that \( z_1 \in \partial N \), which contradicts the assumption that \( z_1 \in \partial Q \).

Lemma 3.1 is important in subsequent developments as a result of the following observation. Since the image of \( R(j\omega) = M(j\omega)P(j\omega) + k \) is bordered by \( R(j\omega, \lambda, \mu) \), then to find the infimum of \( \text{Re}[R(j\omega)] \), it is enough to consider points, \( z_1 \in R(j\omega, \lambda, \mu), \) \( \mu \). It is possible to compute the infimum of \( \text{Re}[R(j\omega)] \) at each frequency, \( \omega_0 \), by fixing one of the two parameters, \( \lambda \) or \( \mu \) (\( \mu \) say), and finding the infimum when \( \lambda \) varies between 0 and 1, then change \( \mu \) and iterate the process. Even though this procedure is theoretically correct, it is impractical due to the computations involved. Fortunately, the following theorem proves that this search is actually unnecessary.

Theorem 3.1: Consider the function \( R = MP + k \), where \( P \rightarrow L_2 \) represents a linear time invariant plant with transfer function \( P(s) = n(s)/d(s) \), where \( n(s) \) and \( d(s) \) belong to the families \( \mathcal{F}_n \) and \( \mathcal{F}_d \) of interval polynomials, respectively.

Thus, the union of the convex combinations of the elements of \( R(j\omega,h) \) for \( h = 1, 2, 3, 4 \) coincides with the function \( R(j\omega,\lambda,\mu) \) defined in (3.1), and the result follows by Lemma 3.1.

IV. STABILITY RESULTS

The main problem addressed in this section is the following. Given a feedback interconnection of a linear time invariant plant containing parametric uncertainty, i.e., a family of interval plants, and a nonlinearity in the sector \([0, k]\), find sufficient conditions for the stability of the closed loop system. We first derive a new stability condition for systems without uncertainty, which generalizes the results of [12]. The proof of the following theorem and its corollary appear in the Appendix.
**Theorem 4.1:** Consider the feedback connection of the subsystems, \( H_1 \) and \( H_2 \), defined by (2.3a)–(2.3b). Let \( H_1 \) and \( H_2: L_2 \rightarrow L_2 \). Then, a sufficient condition for closed-loop stability is the existence of a \( \beta \) and \( \delta > 0 \) and a (possibly noncausal) strongly positive multiplier, \( M \in B \), which satisfies

\[
\begin{align*}
\langle f, NM^{-1} f \rangle &\geq \beta^{-1} ||NM^{-1} f||^2, \quad \forall f \in L_2 \\
\langle f, MH_1 f \rangle &\geq (\delta - \beta^{-1})||f||^2, \quad \forall f \in L_2 \\
\gamma(MH_1) &< \infty
\end{align*}
\]  

(4.1)(4.2)(4.3)

**Corollary 4.1:** Under the same assumptions as Theorem 4.1, if \( H_1 \) is a linear convolution operator with transfer function \( P(s) \), where \( P(s) \) has no poles in the open right half-plane, and \( H_2 \) is an odd, monotone nondecreasing nonlinearity in the sector \([0, \beta]\), then a sufficient condition for closed-loop stability is that \( M \) satisfy the following conditions:

\[
\begin{align*}
\text{(a)} \quad M &= I - T, \quad ||T|| < 1, \\
\text{(b)} \quad \text{Re}[M(j\omega)P(j\omega) + \beta^{-1}] &\geq 0, \\
\text{(c)} \quad \gamma(MH_1) &< \infty.
\end{align*}
\]  

(4.4)(4.5)(4.6)

The following theorem and its corollary generalize the results of Theorem 4.1 and Corollary 4.1 by incorporating parametric uncertainties in the plant.

**Theorem 4.2:** Consider the feedback connection of the subsystems \( H_1 \) and \( H_2 \) defined by (2.3a) and (2.3b). Let \( H_1 \) and \( H_2: L_2 \rightarrow L_2 \) and assume \( H_1 \) is a linear time invariant plant defined by its transfer function \( H_1(s) = \mathcal{F}(n(s))/d(s) \) where \( n(s) \) and \( d(s) \) belong to the families \( \mathcal{F}_n \) and \( \mathcal{F}_d \) of interval polynomials, respectively, and assume \( d(s) \) has no roots in the open right half-plane. Then, a sufficient condition for closed-loop stability is the existence of a \( \beta \) and \( \delta > 0 \) and a noncausal multiplier, \( M \in B \), which satisfies

\[
\begin{align*}
\langle f, NM^{-1} f \rangle &\geq \beta^{-1} ||NM^{-1} f||^2, \quad \forall f \in L_2 \\
\langle f, MP^* f \rangle &\geq (\delta - \beta^{-1})||f||^2, \quad \forall f \in L_2, \quad P^* \text{ is defined as in (3.3)} \\
\gamma(MH_1) &< \infty
\end{align*}
\]  

(4.7)(4.8)(4.9)

**Proof:** The proof is a straightforward application of Theorem 4.1. Notice that the only difference between Theorem 4.2 and Theorem 4.1 (condition 4.7), which implies that \((MP + \beta^{-1})\) is strongly positive, where \( P \) is the inverse Laplace transform of \( K_S = k_S^j/s_j^j \), for \( k = 1, 2, 3, 4 \), \((i, j) = (1, 2), (1, 4), (2, 3), \) or \((3, 4)\), and \( \lambda \in \{0, 1\} \). Therefore, the result follows by application of Theorem 3.1.

**Corollary 4.2:** Under the same assumptions as Theorem 4.2, if \( H_2 \) is an odd, monotone nondecreasing nonlinearity in the sector \([0, \beta]\), and \( M \) is such that conditions (a) and (b) of Corollary 4.1 are satisfied with \( P(j\omega) \) replaced by \( P^*(j\omega) \), where \( P^*(j\omega) \) denotes the Fourier transform of \( P^* \) defined in (3.3), then the system is stable. The proof is a straightforward application of Theorem 4.2 and Corollary 4.1.

**V. CONCLUSION**

A new stability criterion for systems containing parametric uncertainties and an odd, monotone nondecreasing nonlinearity in the feedback has been derived. The result was cast in the framework of the input-output stability theory and makes use of frequency domain conditions for strong positivity of a shifted family of plants cascaded with a noncausal multiplier.

**APPENDIX**

**Proof of Theorem 4.1:** First notice that the multiplier, \( M \), satisfies the factorization conditions. Denote \( A^* \), the adjoint of the operator \( A \in B \). Then from inequality (4.1), with \( M^{-1}f = u \)

\[
\begin{align*}
\langle f, NM^{-1} f \rangle &= \langle f, NM_u^{-1} M^{-1} f \rangle \\
&= \langle u, M_u^* N_u M_u^{-1} u \rangle \\
&\geq \beta^{-1} ||NM_u^{-1} f||^2 \\
&= \beta^{-1} ||N M_u^{-1} f||^2 \\
&\Rightarrow \langle u, M_u^* N_u M_u^{-1} u \rangle \\
&\geq \beta^{-1} ||N M_u^{-1} u||^2
\end{align*}
\]  

(4.10)

and since \((M_u^* N_u M_u^{-1})\), and \((N M_u^{-1})\) are both causal, we have

\[
\langle u, M_u^* N_u M_u^{-1} u \rangle \geq \beta^{-1} ||N M_u^{-1} u||^2.
\]  

(A.2)

In the same way, from inequality (4.2), with \( M_u^*f = x \)

\[
\begin{align*}
\langle f, MH_1 f \rangle &= \langle f, M_u H_1 f \rangle \\
&= \langle M_u^* f, M_u H_1 f \rangle \\
&= \langle x, M_u H_1 (M_u^*)^{-1} x \rangle \\
&\geq (\delta - \beta^{-1}) ||f||^2 \\
&\Rightarrow \langle x, M_u H_1 (M_u^*)^{-1} x \rangle \\
&\geq (\delta - \beta^{-1}) ||(M_u^*)^{-1} x||^2 \\
&\Rightarrow \langle x, M_u H_1 (M_u^*)^{-1} x \rangle \geq (\delta - \beta^{-1}) ||(M_u^*)^{-1} x||^2.
\end{align*}
\]  

(A.3)

Then, conditions (a) and (b) of Theorem 4.1 imply that conditions (A.2) and (A.3) are satisfied. To complete the proof, refer to Fig. 1(b) and define

\[
z = [M_u H_1(M_u^*)^{-1}]x = [M_u H_1(M_u^*)^{-1}][M_u^* N_u M_u^{-1}] z.
\]  

(A.4)

It is well known that the system of Fig. 1(a) is stable if and only if the system of Fig. 1(b) is stable [11]. Consider now the following inner product

\[
\langle [M_u H_1(M_u^*)^{-1}]r', [M_u^* N_u M_u^{-1}] z \rangle_T = [z, [M_u^* N_u M_u^{-1}] z]_T + \langle [M_u H_1(M_u^*)^{-1}]r', [M_u^* N_u M_u^{-1}] z \rangle_T.
\]  

Then, taking account of (A.2) and (A.3)

\[
\langle [M_u H_1(M_u^*)^{-1}]r', [M_u^* N_u M_u^{-1}] z \rangle_T \\
\geq \beta^{-1} ||N M_u^{-1} z||^2 \\
+ (\delta - \beta^{-1}) ||(M_u^*)^{-1} ||N M_u^{-1} z||^2 \\
\geq \delta ||N M_u^{-1} z||^2
\]  

(A.5)
where we have made use of the Schwarz inequality. Thus,

Thus,

But,

(\langle T \mathbf{x}, \phi \rangle) = \int_{-\infty}^{\infty} T(t) \{\int_{-\infty}^{\infty} x(t) \phi'[z(t)] dt\} \, dt.

Then by Lemma VI.9.40 of [4], and using the fact that \( N \) is odd,

Now \([1 - \|T\|] > 0\) by assumption. Since (2.5) implies that

Therefore, \( N \) satisfies condition (a) of Theorem 4.1. The rest of the proof is completed by recognizing that \( \text{Re}[M(j\omega)\tilde{G}(j\omega)] + \beta^{-1} \geq \delta > 0 \) implies that \( [M + \delta] \) is strongly positive, which corresponds to condition (b). Thus, all the conditions of Theorem 4.1 are satisfied and the system is closed loop stable.

REFERENCES