mixers with local oscillator frequency doubler, and frequency mixers with AGC.

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Stability Conditions for Systems With Parametric Uncertainties and Nonlinear Feedback

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Abstract—The purpose of this paper is to use the Passivity Theorem to derive new stability conditions for a class of systems that consist of a linear subsystem with parametric uncertainty and nonlinear feedback. To address the inherent conservativeness in the application of the Passivity Theorem, causal multipliers are introduced into the analysis and it is shown that a Popov-like stability condition results. The stability results are then extended for use with noncausal multipliers in Section IV. The main shortcoming of the Passivity Theorem is that it provides sufficient but not necessary conditions for closed-loop stability and consequently leads to conservative results. This conservatism can be significantly alleviated by analyzing an equivalent feedback system which incorporates multipliers. The use of multipliers has not been exploited in previous work. Furthermore, since previous work has imposed strict positive realness conditions on certain transfer functions to ensure that passivity conditions are met [5], [7]–[10], only causal systems may be considered.

The purpose of this paper is to use the Passivity Theorem to derive new absolute stability conditions using both causal and noncausal multipliers for a class of systems that consist of a linear subsystem with parametric uncertainty and nonlinear feedback. The purpose of this paper is to use the Passivity Theorem to derive new absolute stability conditions using both causal and noncausal multipliers for a class of systems that consist of a linear subsystem with parametric uncertainty and nonlinear feedback. The remainder of the paper is organized as follows. In Section II we introduce notations and amalgamate various preliminary results. In Section III we use causal multipliers to derive stability conditions for a class of nonlinear systems and show that a Popov-like stability condition can be obtained as a special case. The stability results are then extended for use with noncausal multipliers in Section IV.

II. PRELIMINARIES

A. Definitions and Notation

The general input-output formulation will be adopted here. We collect some basic facts and clarify the notation to be used and refer the reader to references [1]–[4] for details. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the real and nonnegative real numbers. A function, $x(t)$, defined on $t \in \mathbb{R}^+$ belongs to $L_2$ if and only if $\int_0^\infty |x(t)|^2 dt < \infty$. Define the usual inner product in $L_2$, and the $L_2$ norm: $(x(.), g(.)) = \int_0^\infty x(t)y(t) dt, \|x\|^2 = (x(.), x(.))$. Denote $x_{\leq T}(.)$ a truncation of the function $x(.)$ to the interval $(0, T)$. Then $x_{\leq T}(t) = x(t)$ for $t < T$, and $x_{\leq T}(t) = 0$ elsewhere. The extension of the space $L_2$, denoted by $L_{2e}$, is defined as follows. A function belongs to $L_{2e}$ if every truncation of $x(.)$ belongs to $L_2$. Thus, $L_{2e} = \{x(.): x_{\leq T}(.) \in L_2 \forall T \in \mathbb{R}^+\}$.

Let $H : L_2e \rightarrow L_{2e}$ be an operator in $L_{2e}$. Then $H$ is said to be strongly passive if there exist a fixed $\delta > 0$ such that

$$\langle x(\cdot), (Hx)(\cdot) \rangle \geq \delta \langle x, x \rangle \quad \forall x(\cdot) \in L_2, \quad (2.1)$$

and $H$ is said to be passive if it satisfies (2.1) with $\delta = 0$. In the same way $H : L_2e \rightarrow L_{2e}$ is said to be strongly passive if there exist a fixed $\delta > 0$ such that

$$\langle x(\cdot), (Hx)(\cdot) \rangle \geq \delta \langle x, x \rangle \quad \forall x(\cdot) \in L_2 \quad (2.2)$$

and $H$ is said to be passive if it satisfies (2.2) with $\delta = 0$.
An important role will be played by the concept of causality.

$H: L_2 \rightarrow L_2$ is said to be causal if

$$H(x(.)_\tau) = (Hx_T(.)_\tau) \quad \forall \omega \in L_2$$

$$\forall \tau \in R^+ \quad (2.3)$$

It is easy to show that if $H$ is causal then the definition of positivity and passivity (strong positivity and strong passivity) are equivalent. See, for example, [1] Lemma VI.9.2.

Let $H(s)$ be a rational function. Then $H(s)$ is said to be strictly positive real, SPR, if:

1. $H(s)$ has no poles in the closed right-half plane, and
2. $\text{Re}[H(j\omega)] > 0, \forall \omega \in (-\infty, \infty). \quad (2.4)$

For causal linear time-invariant systems the concepts of strict positive realness and passivity are related as follows. Let $H(s)$ be a causal linear time-invariant system defined by, $y(t) = \int_0^\infty h(t-u)x(u)du$, where $h(.)$ represents the output, $x(.)$ the input, and $h(.)$ is the impulse response of $H$. Assume $H$ is causal and also strictly passive. Namely, assume that

$$\int_0^{\infty} h(t)z(t)dt = \int_0^\infty h(t)\tilde{z}(t)dt \quad \forall \omega \in (-\infty, \infty)$$

These definitions imply that if $H$ is causal and strictly passive, then $H$ is strictly passive and moreover, since (2.4) and (2.5) are satisfied, $H(s)$ is SPR. The converse of this result is not true, i.e., if $H(s)$ is SPR, then $H$ is passive, but in general not strictly passive. The following Theorem shows that, if $H(s)$ is SPR and biproper (i.e., both $H(s)$ and $H(s)^{-1}$ are proper), then $H$ is strictly passive.

**Theorem 2.1:** Let $H: L_2 \rightarrow L_2$ be a causal linear time-invariant operator defined by, $(Hx(.)_\tau) = \int_0^\infty h(t-u)x(u)du$, where $h(.)$ is the impulse response of $H$. Denote by $\tilde{H}(s)$ the Laplace transform of $h(t)$. Under these conditions, if $\tilde{H}(s)$ is SPR and $H(s)$ is biproper, i.e., $\lim_{\omega \rightarrow \infty} \tilde{H}(s) = k > 0$, then $H$ is strictly passive.

**Proof of Theorem 2.1:** To prove Theorem 2.1 we reason by contradiction. Namely, assume that $H(s)$ is SPR but $H$ is not strictly passive. $H(s)$ is biproper by assumption, therefore $\text{Re}[\tilde{H}(j\omega)] - k > 0$, as $\omega \rightarrow \infty$. Since $\tilde{H}(s)$ is SPR, $H$ is not strictly passive if and only if $\inf \text{Re}[\tilde{H}(j\omega)] = 0$ for some $\omega^* \in R$. It follows that,

$$\inf \text{Re}[\tilde{H}(j\omega)] = 0 \quad \omega \in \Omega = [0, \omega^*]$$

But the interval $\Omega = [0, \omega^*]$ is compact and restricted to this interval, $\inf \text{Re}[\tilde{H}(j\omega)] = \min \text{Re}[\tilde{H}(j\omega)] = 0$. It follows that $\text{Re}[\tilde{H}(j\omega)] = 0$ for some $\omega^* \in R$, and $\tilde{H}(s)$ is not SPR, which contradicts the assumption.

**B. Factorization of Positive Operators**

Let $W$ be a convolution operator defined by $(WX)(t) = (W * x)(t)$, where the symbol $*$ denotes convolution and $W$ is the impulse response of $W$. In addition let $W$ satisfy the following conditions

1. $W: L_2 \rightarrow L_2$
2. $W^{-1}: L_2 \rightarrow L_2$
3. $\gamma(W) < \infty$

The convolution operators satisfying the conditions (1)-(3) form a commutative Banach algebra $B = B_s \cup B_n$, where $B_s$ and $B_n$ are the Banach algebras of causal and anticausal operators [4]. Let $P_s$ be the projection on $B$ defined as follows

$$(Wu)(t) = \int_0^\infty W(t-\tau)u(\tau)d\tau$$

then

$$(P_sw)(t) = \int_0^\infty W(t-\tau)u(t-\tau)d\tau$$

It follows that $P_sW$ is causal. Let $P_n = I - P_s$, where $I$ is the identity in $B$. Denote $B_+ = \{P_s\}$ and $B_+ = \{P_n\}$.

If $W$ is an arbitrary strongly positive operator in $B$, then there exist elements $W_+ = \exp[P_s \log W]$ and $W_- = \exp[P_n \log W]$ such that: (a) $W_+ \in B_+$ and $W_- \in B_-$, (b) $W \equiv W_+ W_-$, (c) $W_+$ and $W_-$ are invertible with $W_+^{-1} \in B_+$ and $W_-^{-1} \in B_-$. Notice that the only conditions for the existence of such a factorization are (1)-(3) above.

**C. Interval Polynomials**

Let $\Delta$ be the real polynomial of $n$-th degree, $\Delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n$. We say that $\Delta(s)$ belong to the family, $FA$, of interval polynomials if each coefficient, $\delta_i$, can take values between an upper and lower bound, i.e., $\delta_i \in [x_i, y_i]$. Consider the plant, $P(s) = n(s)/d(s)$, where $n(s)$ and $d(s)$ belong to the families, $FA_N$ and $FA_D$, respectively. Denote $K_{\delta_0}, K_{\delta_1}, i, j = 1, 2, 3, 4$ the $i$th and $j$th Kharitonov polynomials of the families $FA_N$ and $FA_D$. Then the $32$ Kharitonov Segments, $K_s$, are defined by

$$K_N = S_{\delta_0}^N/K_{\delta_1}^N \quad \text{or} \quad K_N = S_{\delta_0}^N/K_{\delta_1}^N$$

where $S_{\delta_0}^N$ represents the line segment that joins the Kharitonov polynomials $K_{\delta_0}$ and $K_{\delta_1}$, i.e., $S_{\delta_0}^N = [\{1-\lambda\}K_{\delta_0} + \lambda K_{\delta_1}]$, $(i,j) \in (1,2), (1,3), (2,4), (3,4)$. $S_{\delta_0}^N$ is similarly defined. Notice that the set of $32$ Kharitonov Segments is comprised of the two subsets, $K_{DS}$ and $K_{NS}$. Each subset contains $16$ rational functions dependent on the parameter, $\lambda$.

**Theorem 2.2:** Consider the rational functions $M(s) = n_m(s)/d_m(s)$ and $P(s) = n_p(s)/d_p(s)$, where $n_m(s)$ and $d_m(s)$ are fixed polynomials and $n_p(s)$ and $d_p(s)$ belong to the families, $FA_N$ and $FA_D$, respectively, and let $k \in R$. Then the function $H(s) = M(s)(P(s) + k)$ is strictly positive real if and only if $\gamma(H(s))$ is SPR when $P(s)$ is replaced by the $16$ Kharitonov Segments, $K_{DS}$.

**Corollary 2.1:** Under the assumptions of Theorem 2.1, $M(s)P(s)$ is SPR if and only if $M(s)P(s)$ is SPR when $P(s)$ is replaced by the $16$ Kharitonov Systems, $P_\delta = K_{\delta_0}/K_{\delta_1}$, $i, j = 1, 2, 3, 4$.

**Proof of Theorem 2.2 and Corollary 2.1:** See references [10] and [5], respectively.

**III. STABILITY RESULTS WITH CAUSAL MULTIPLIERS**

Consider the system of Fig. 1, where $H_1$ is a linear time-invariant plant with parametric uncertainty associated with its transfer function.
and \( H_2 \) is a nonlinear operator. The Passivity Theorem states that
the closed-loop system is \( L_2 \) bounded if \( H_1 \) and \( H_2 \) are both passive, and one of them is strongly passive and has finite gain [1], [2]. To relax these requirements a multiplier is introduced into the analysis. The following result provides stability conditions for feedback systems with parametric uncertainties and a nonlinearity in the forward path. It is the least conservative result that may be obtained with a causal multiplier.

**Theorem 3.1:** Consider the system, S, of Fig. 1. Assume \( H_1 \); \( L_2 \rightarrow L_1 \) is nonlinear and \( H_2 ; L_2 \rightarrow L_2 \) is a convolution operator defined by \( z(t) = (H_2 x)(t) = \int_0^t g(t - u)x(u)du \). Denote \( \hat{G}(s) = n(s)/d(s) \) the Laplace transform of \( g(t) \) and assume \( n(s) \in F_a, d(s) \in F_a \) and all members of \( F_a \) have roots in the open left half plane. Let \( K(s) \) be any element contained in \( K_{DS} \), the 16 Kharitonov segments defined in (2.10), i.e., \( K(s) = n^*(s)/d^*(s) \), where \( n^*(s) = K_{i,j,k}(s) \), \( d^*(s) = S_{j}^d(s) \), for some \( i, j, k = 1, 2, 3, 4 \) and some \( \lambda_i \in [0, 1] \), and let \( g_{DS} = L^{-1}[K(s)] \). Let \( W : L_2 \rightarrow L_2 \) be a causal linear time-invariant multiplier introduced in the loop such that \( H'_1 = H_1 W, H'_2 = W^{-1} H_2, \) and \( u' = W^{-1} u \). If conditions (a) through (c) below are satisfied and \( W^{-1} u \in L_2 \), then \( e \in L_2 \), and \( y \in L_2 \).

\[
\begin{align*}
(a) \quad & \langle x_T, (H_1 W x_T) \rangle \geq \beta^{-1} \| \hat{G}(s) x_T \|^2 \quad \forall x \in L_2, \\
(b) \quad & \langle x_T, (W^{-1} g_{DS}, x_T) \rangle \geq (\delta - \beta^{-1}) \| x_T \|^2 \quad \forall x \in L_2, \\
(c) \quad & \gamma (W^{-1} H_2)^{-1} < \infty
\end{align*}
\]

**Proof of Theorem 3.1:** We first notice that the addition of the multiplier, \( W \), does not affect the stability of the closed-loop system. Condition (3.3) requires that \( W^{-1} g_{DS} \) be (1) proper and (2) stable for all possible combinations of the parameters of the plant. Therefore, (2), is satisfied if and only if the four Kharitonov polynomials of the denominator of the plant have roots in the open left half plane. Condition (3.2) is equivalent to \( W^{-1} g_{DS} + \beta^{-1} \) being strongly passive and since \( W^{-1} g_{DS} \) is causal this is equivalent to \( \hat{W}(s)^{-1} [K(s) + \beta^{-1}] \) being SPR. Thus, by Theorem 2.2, \( \hat{W}(s)^{-1} G(s) + \beta^{-1} \) is SPR for all possible combinations of the parameters of the plant. Moreover, \( \hat{W}(s)^{-1} G(s) + \beta^{-1} \) is biproper, from (3.2). From Theorem 3.1, \( W^{-1} H_1 + \beta^{-1} \) is strongly passive for all possible combinations of the parameters of the plant and satisfies, \( \langle x_T, (W^{-1} H_1 x_T) \rangle \geq (\delta - \beta^{-1}) \| x_T \|^2 \). Also \( e = W^{-1} u - W^{-1} H_2 H_1 W e \). Thus

\[
\begin{align*}
& \langle (W^{-1} u), (H_1 W e) \rangle \\
& = \langle (W^{-1} H_1 H_1 W e) \rangle + \langle (W^{-1} H_1 H_2 H_1 W e) \rangle \\
& \geq \beta^{-1} \| H_1 W e \|^2 + (\delta - \beta^{-1}) \| (H_1 W e) \|^2 \\
& \geq \delta \| (H_1 W e) \|^2 \\
& \Rightarrow \| (H_1 W e) \|^2 \leq \delta^{-1} \| (W^{-1} u) \|^2
\end{align*}
\]

Therefore, if \( W^{-1} u \in L_2 \), \( H_1 W e = y \in L_2 \). Also \( e = W^{-1} u - W^{-1} H_2 H_1 W e = W^{-1} u - W^{-1} H_2 y \)

\[
\Rightarrow \| e \| \leq \| W^{-1} u \| + \| W^{-1} H_2 y \| \leq \| W^{-1} u \| + \gamma (W^{-1} H_2) \| y \|
\]

Thus \( W^{-1} u \in L_2 \Rightarrow e \in L_2 \).

The next result shows that the Popov criteria is a special case of Theorem 3.1

**Corollary 3.1:** Consider the system, S, of Fig. 1. Assume \( H_1 = \phi \) is the memoryless nonlinearity defined by, \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \phi \) is continuous with \( 0 < \alpha \leq \phi(s)/\sigma < \beta < \infty \) and \( \phi(0) = 0 \). Let \( H_2 : L_2 \rightarrow L_2 \) be a convolution operator defined by \( z(t) = (H_2 x)(t) = \int_0^t g(t - u)x(u)du \). Denote \( \hat{G}(s) = n(s)/d(s) \) the Laplace transform of \( g(t) \) and assume \( G(s) \) is strictly proper, \( n(s) \in F_a, d(s) \in F_a \) and all members of \( F_a \) have roots in the open left half plane. Let \( K(s) \) be any element contained in \( K_{DS} \), i.e., \( K(s) = n^*(s)/d^*(s) \), where \( n^*(s) = K_{i,j,k}(s), d^*(s) = S_{j}^d(s) \), for some \( i, j, k = 1, 2, 3, 4 \) and some \( \lambda_i \in [0, 1] \). Let \( g_{DS} = L^{-1}[K(s)] \). Under these conditions if there exists \( q, \delta \in \mathbb{R}^+ \) such that the following condition is satisfied, then the system is \( L_2 \) stable.

\[
\Re[[1 + q \omega j \omega]K(j \omega)] + \beta^{-1} = \delta > 0 \quad \forall \omega \geq 0
\]

**Proof of Corollary 3.1:** Let \( W(s) = 1/(1 + q s), q \in \mathbb{R}^+ \) in Theorem 3.1. Then condition (3.1) is satisfied for the nonlinearity, \( \phi \), defined in Corollary 3.1 [2], [3]. Finally, the proof is completed by recognizing that condition (3.3) can be expressed as (3.4) and that \( W^{-1} H_2 \) is stable by assumption.

**IV. STABILITY RESULTS WITH NONCAUSAL MULTIPLIERS**

The multipliers used in Section III were assumed to be causal. Since every noncausal operator can be written as the sum of a causal and an anticausal operator [1] the use of noncausal multipliers will in general produce less conservative results and are guaranteed to never produce more conservative results than the use of causal multipliers. The objective of this section is to show how noncausal multipliers may be incorporated into the analysis. The use of noncausal multipliers requires special consideration. Notice that conditions (2.6) and (2.7) were derived under the assumption that \( K \) was causal. If an operator is not causal then strong positivity does not imply SPR.

The following example will further clarify this point. Consider the noncausal multiplier, \( W \), defined by its bilateral Laplace transform, \( \tilde{W}(s) \).

\[
\tilde{W}(s) = \left[ s^3 - s^2 + 3s + 7 \right]/\left[ s^3 + s^2 + 3s + 10 \right] \equiv \left[ s + 1 \right]/\left[ s + 2 \right] + \left[ 1 - s \right]/\left[ s^2 - s + 3 \right] \equiv M_+ + M_-
\]

\( \tilde{W}_+ \) and \( \tilde{W}_- \) are the causal and anticausal part of \( \tilde{W}(s) \), respectively. It is easily verified that \( \tilde{W} \) is strongly positive. However, \( \tilde{W}(s) \) is not SPR. Note that if a rational function, \( H(s) \), is SPR then we can always find a network containing passive elements which has \( H(s) \) as a driving point function. Clearly if a function, \( \tilde{W}(s) \), is noncausal such a network cannot exist.

The extension of the stability results for systems containing parametric uncertainties is nontrivial. Notice that the existence of a noncausal multiplier in the loop rules out the application of Theorem 2.2 and Corollary 2.1.

Theorem 4.1: Consider the system, S, of Fig. 1. Let \( H_1, H_2 \) be causal maps from \( L_2 \) into itself and let \( (H_1 x)(t) = \int_0^t h(t - u)x(u)du \). Denote \( \tilde{H}(s) = n(s)/d(s) \) the Laplace transform of \( h(t) \) and assume \( n(s) \in F_a, d(s) \in F_a \) and all members of \( F_a \) have roots in the open left half plane. Denote by
Consideration of either Kharitonov systems or uncertain coefficients and the other nonlinear. The conditions require was considered, i.e., stability in the presence of noncausal multipliers, \( H_i \). Similarly, condition (2) implies \( \{W^+, H_i W^{-1}\} \) passive. Thus \( W^+ \) strongly positive implies that this is true for all possible combinations of the parameters \( K_{ij}/K_{kj} \)

\[
\begin{align*}
(1) & \quad \gamma(Wh) < \infty \\
(2) & \quad \exists \delta: \langle x, Wh x \rangle \geq \delta(x,x) \quad \forall x \in L_2 \\
(3) & \quad \langle x, H_i W^{-1} x \rangle \geq 0 \quad \forall x \in L_2
\end{align*}
\]

Comments: Theorem 4.1 is one of the most general and least conservative available in stability theory. No assumptions were made on the type of nonlinearity other than satisfying condition (3) and very few restrictions were placed on the class of multipliers allowed. In particular, \( W \) includes, but is not limited to the multiplier, \( H_i \), used in the paper [11]. As a final observation note that it is not necessary to find \( W^+ \) and \( W^- \) in the application of Theorem 3.1. Condition (2) can be checked directly by computing the Fourier Transform of \( Wh \).

Proof: Condition (1) ensures that the finite gain condition of the Passivity Theorem is satisfied. Since \( W \) satisfies the factorization conditions, one may write \( W = W^+ W^- \), where \( W^+ \) and \( W^- \) and their inverses belong to \( B_* \) and \( B_* \). Thus,

\[
\langle x, Wh x \rangle = \langle x, W^- W^+ h x \rangle = \langle W^- x, W^+ h x \rangle
\]

where \( W^- \) is the adjoint of \( W^- \). Define \( W_+ x = y \), thus \( x = \langle W_+^{-1} y, W^- \rangle \). Then \( Wh \) strongly positive \( \Rightarrow [W^+ h W^-^{-1}] \) strongly positive. But \( [W^+ h W^-^{-1}] \) is causal [1]. Thus, \( Wh \) strongly positive \( \Rightarrow [W^+ h W^-^{-1}] \) strongly passive and the transfer function \( [H_i (s) H_i (s) W^-^{-1}] \) is SPR and biproper. From Corollary 2.1, \( [H_i (s) H_i (s) W^-^{-1}] \) SPR for the 16 Kharitonov systems of \( H_i \) implies that this is true for all possible combinations of the parameters of \( H_i \). Similarly, condition (2) implies \( \{W^+, H_i W^{-1}\} \) passive. Thus all of the conditions of the Passivity Theorem are satisfied and the system, \( S \), is \( L_2 \) stable.

V. CONCLUSION

In this paper we have derived stability conditions for a class of systems comprised of two subsystems; one which is linear with uncertain coefficients and the other nonlinear. The conditions require consideration of either 16 Kharitonov Systems or 16 of the 32 Kharitonov Segments associated with the linear subsystem and the selection of a suitable noncausal multiplier. Since the general case was considered, i.e., stability in the presence of noncausal multipliers, the results of this paper are of the most general available for stability analysis.

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Analog Circuit For Solving Assignment Problems

Kiichi Urahama

Abstract—A novel analog electronic circuit for solving assignment problems is presented. Total length of wiring in the proposed circuit amounts to at most \( O(n^2) \) with \( n \) being the number of variables in contrast to \( O(n^4) \) required for previously developed circuits based on the Hopfield neural networks. Moreover its power dissipation is extremely small by virtue of subthreshold operation of MOS transistors.

I. INTRODUCTION

The assignment problem is one of fundamental polynomial class combinatorial optimization problems [1]. The problem appears in a wide range of practical tasks requiring real time solutions, for example the crossbar switch control in communication networks [2]. Analog electronic circuits have been devised for solving the assignment problem in real time that implement neural network-like solution methods [2–4]. All of these circuits developed thus far requires entangled connection wiring of \( O(n^3) \) length with a being the number of variables in the problem. Such entangled global wiring makes the fabrication of these circuits difficult. In this letter a novel circuit with wiring complexity of \( O(n^2) \) is presented. This reduction of wiring complexity is achieved by direct implementation of the Lagrange formulation of the problem instead of the penalty method employed in the previous approaches.

II. OPTIMUM SOLUTION OF ASSIGNMENT PROBLEM

This letter addresses the linear assignment problem which can be formulated as a zero-one integer program [1]:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}
\]