Finite horizon robust model predictive control with terminal cost constraints

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Abstract: A finite horizon model predictive control (MPC) algorithm that is robust to modelling uncertainties is developed along with the construction of a moving average system matrix to capture modelling uncertainties and facilitate the future output prediction. The authors’ main focus is on the step tracking problem. Using linear matrix inequality techniques, the design is converted into a semi-definite optimisation problem. Closed-loop stability, known to be one of the most challenging topics in finite horizon MPC, is treated by adding extra terminal cost constraints in the semi-definite optimisation. A simulation example demonstrates that the approach can be useful for practical applications.

1 Introduction

Since the first version of model predictive control (MPC), known as dynamic matrix control, was published in 1978, various MPC algorithms have been developed in the past two decades [1–3]; for example, generalised predictive control designed for stochastic systems, predictive function control capable of handling non-linear and unstable processes and internal model predictive control which guarantees closed-loop stability. All such schemes are featured with a critical property of combining input/output constraints with MPC formulation explicitly, which enables MPC to be widely accepted in petro-chemical, automotive, food processing, metallurgy and other industries [4]. Most versions of MPC adopt a common assumption, namely, setting system models precisely (so-called nominal models) and neglecting all internal and external ubiquitous uncertainties. The assumption simplifies MPC formulation dramatically, but may impair the controller performance and/or closed-loop stability. Normally, the standard MPC process is composed of three steps: future state/output prediction, objective function optimisation and control signal implementation. The accuracy in the three steps is highly dependent on the model precision. A small parameter perturbation may lead to constraint violation or unstable regulation. To overcome such a limitation, it is necessary to study robust model predictive control (RMPC) by incorporating model uncertainties into the design.

The barriers of the extension of tradition MPC strategies to robust cases lie in two aspects: one is state/output predictions and the other is closed-loop stability. For the former, researchers tend to utilise the uncertainty configuration to facilitate future state/output expression, and for the latter, employ the invariant set theorem to guarantee the state/output convergence in the presence of system uncertainties. The min–max optimisation, as a quite mature technique, has been used to analyse RMPC problems in the late 1990s [5–7]. Briefly speaking, given a structured objective function, which is usually defined in the form of the weighted two-norm summation, maximise the objective based on the definition of system uncertainties, derive an upper bound of the objective and then minimise the bound with respect to manipulated inputs. Campo and Morari [8], Lee and Yu [9], Bemporad and Morari [7] and Lee and Cooley [10] independently introduced min–max into the MPC formulation. In 1996, Kothare et al. [11], based on the min–max strategy, established a successful infinite horizon robust model predictive control (IH-RMPC) algorithm, which dramatically decreased the computational complexity and increased the implementation efficiency. Two structured uncertainty frameworks, namely, polytopic and structured feedback uncertainties [12], were covered in this paper. The IH-RMPC design was converted to a standard semi-definite optimisation problem [13], which was featured with linear objective functions and linear matrix inequality (LMI) constraints. The key point of this approach is the derivation of an auxiliary quadratic functions of predicted states and an upper bound of the objective. Consequently, future state/output predictions are avoided skillfully. From the characteristics of LMIs, fast convergence and polynomial complexity, IH-RMPC is known as one of the most effective regulations in the robust process control area. Since then IH-RMPC has drawn considerable attention in the literature: Rodrigues and Odloak developed an output-tracking IH-RMPC algorithm [14]; Wan and Kothare derived an off-line IH-RMPC formulation problem [15] and Hu and Limnemann extended the IH-RMPC scheme into non-linear cases [16]. They all inherited the effectiveness of LMI techniques and guaranteed closed-loop stability. Moreover, associated by terminal constraints or deriving a state invariant set, the upper bound of the objective function was employed as a Lyapunov function and enforced to converge while MPC iteration. Although IH-RMPC possesses superiority as viewed from stability and efficiency, it limits the system tuning freedoms, and feasibility is another potential problem [17].

Compared with traditional MPC schemes, IH-RMPC cannot use prediction horizon \( N_p \) and control horizon \( N_u \) as tuning parameters to achieve the trade-off between
system stability and performance (actually for nominal cases, these two parameters are quite effective tuning algorithms) [18]. On the other hand, IH-RMPC always presumes that there exists a unique control policy which leads to the expected performance in all possible uncertain situations throughout entire infinite horizons. The condition may result in low-performance solutions and even infeasibility problems [6, 19]. Therefore the development of FH-RMPC is necessary as well as natural. From the above discussion, we believe that the main obstacle of FH-RMPC comes from the computational complexity of future state/output predictions. For systems with some uncertain terms in state space matrices, specifically in matrix $A$ of linear discrete state space models, when performing state predictions, it can be seen that high-order terms of uncertainties will appear in the expression of the future signals. It is extremely hard to generalise the effects of these uncertain terms on MPC online optimisation. Therefore for a successful FH-RMPC algorithm, it is essential to describe the characteristics of these uncertain factors. Researchers have constructed several novel frameworks to study this issue. Park and Jeong modified the system parameter perturbations of these uncertain factors. Researchers have constructed several novel frameworks to study this issue. Park and Jeong modified the system parameter perturbations into the structured uncertainties with a bounded increment rate [20]; Langston et al. proposed an uncertain ‘tube’ to maintain the controlled trajectories [21] and Fukushima and Bitmead constructed an additional comparison model for the worst-case analysis [22]. Although all of these algorithms can obtain acceptable control performance, suffering from the computational complexity, their applications were limited to slow-rate systems. Furthermore, as a popular approach to MPC stability analysis, the theorem of optimality is not available for ‘min–max’ suboptimal problems any more, so that we cannot directly use the objective as a Lyapunov function to conclude closed-loop stability, which poses a new challenge for the FH-RMPC design [19].

Preserving the efficiency of IH-RMPC using LMIs, in this paper, we will develop an FH-RMPC to achieve robust tracking control. A moving average system matrix [23] is used to capture modelling uncertainties and extend IH-RMPC using LMIs to FH-RMPC cases. By imposing two extra terminal cost constraints in the form of LMIs, closed-loop asymptotical stability is also achieved. Besides $N_p$ and $N_s$, terminal weighting $Q_{N_s}$ as another tuning parameter, is capable of adjusting system closed-loop stability and performance. The robust LMI theorem, namely, LMIs of uncertain matrices [24, 25], is introduced in FH-RMPC. The moving average system matrix, called uncertainty block in the paper, is weighted and norm-bounded by one, which is consistent with the conditions of the robust LMI theorem. Parallelising the system nominal model with the system nominal model for robustness problems, we will develop an FH-RMPC to achieve robust state predictions as well. Finally, based on the properties of robust LMIs, the FH-RMPC design is recast into a semi-definite optimisation problem which can be solved numerically using existing software packages. From a simulation example, we can see that the algorithm is efficient, flexible and reliable.

Notation: Throughout the paper, let $\bar{x}$ denote the nominal value if the corresponding vector or scalar $x$ is uncertain, and $\hat{x}$ for the backward difference, that is, $\hat{x}(k) = x(k) - x(k-1)$. For any symmetric matrices $Y$ and $Z$, $Y \geq Z$ (or $Y > Z$) means $Y - Z \geq 0$ (or $Y - Z > 0$), namely, $Y - Z$ is positive semi-definite (or positive definite). $N_p$ and $N_s$ are both positive integers. $N_p$ represents the prediction horizon and $N_s$ the control horizon. We assume that $0 < N_u \leq N_p$. The maximal singular value of a matrix $M$ is denoted by $\sigma(M)$.

2 LMI for the nominal MPC

Consider a nominal model of the controlled system

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$  

(1)

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector and $y \in \mathbb{R}^p$ is the output vector. $A$, $B$ and $C$ are constant matrices with compatible dimensions. To obtain the nominal MPC for the step tracking scheme, the objective function of input $u(-1)$ and state measurement $x(k)$ over a horizon starting at instant $k$ is defined by

$$J = \sum_{i=1}^{N_y} \|r - y(k + i)\|^2_Q + \sum_{i=0}^{N_y-1} \|u(k + i)\|^2_R$$

(2)

where $r$ is the given reference signal, $y(-1)$, $y(-2)$ are the predicted input and output signals over the control horizon and prediction horizon starting at instant $k$ and $Q$ and $R$ are the output, input and terminal weightings, respectively. The norms in $J$ are defined as

$$\|y(k + i)\|^2 = (r - y(k + i))^T Q (r - y(k + i))$$

similarly for the other ones. Based on the model in (1), the predicted states can be calculated by

$$x(k + i) = \begin{bmatrix} A^i x(k) + A^{i-1} Bu(k) + \cdots + Bu(k + i - 1)k) & \text{if} \ 1 \leq i \leq N_u \\
A^i x(k) + A^{i-1} Bu(k) + \cdots + A^{N_p-1} Bu(k + N_p - 2k) & \text{if} \ N_p < i \leq N_p \\
(A^{N_p-1} B + \cdots + B) u(k + N_p - 1k) & \text{if} \ N_u < i \leq N_p \end{bmatrix}$$

(3)

Rewrite the objective function in (2) into the augmented matrix form described in [3]

$$J = (R - \gamma(k))^T Q (R - \gamma(k)) + U^T(k) R U(k)$$

(4)

where the augmented vectors are given as follows

$$U(k) = [u^T(kk) \ u^T(k + 1k) \ \cdots \ u^T(k + N_p - 1k)]^T$$

$$\gamma(k) = [y^T(k + 1k) \ y^T(k + 2k) \ \cdots \ y^T(k + N_p - 1k)]^T$$

$$T = [r^T \ r^T \ \cdots \ r^T]^T$$

(5)

and the augmented weightings are

$$Q = \diag(Q_1, Q, \ldots, Q, Q_{N_p})$$

$$R = \diag(R, R, \ldots, R)$$

(6)

Inserting the predicted states in (3) into (1) from $i = 1$ to $N_p$ and utilising the augmented vectors and weightings in (5) and (6), predicted output sequence $\gamma(k)$ can be expressed in state measurement $x(k)$

$$\gamma(k) = CAx(k) + CBU(k)$$

(7)

where

\[
A = \begin{bmatrix}
A_N^1 & 0 & \cdots \\
\vdots & \ddots & \vdots \\
A_{N_t}^1 & \cdots & A_{N_t}\end{bmatrix}, \\
B = \begin{bmatrix}
B^1 & 0 & \cdots \\
\vdots & \ddots & \vdots \\
B_{N_t}^1 & \cdots & B_{N_t}\end{bmatrix}
\]

Substituting (7) into (4), and defining an auxiliary positive scalar \( t \), the nominal MPC can be solved by the following programming

\[
J_0 = \min_{t, U(k)} t \quad \text{subject to} \quad t \geq J
\]

\[
J = (T - (C^T A x(k) + C R u(k)))^T Q (T - (C^T A x(k) + C R u(k))) + R^T(k) R U(k)
\]

with \( J_0 \) being the optimal value of objective \( J \) and scalar \( t \) can be regarded as an upper bound of \( J \). Applying Schur complements to the constraint in (9), the nominal MPC problem is converted into a semi-definite optimisation problem.

For the nominal step tracking MPC, optimal control sequences \( U(k) \) over a horizon starting at instant \( k \), if exist, can be calculated by solving the following semi-definite optimisation problem

\[
J_0 = \min_{t, U(k)} t \quad \text{subject to} \quad t > 0 \quad \begin{bmatrix}
T - (C^T A x(k) + C R u(k))^T U^T(k) \\
Q^{-1} \\
0 \\
R^{-1}
\end{bmatrix} \begin{bmatrix}
U(k) \\
0 \\
R
\end{bmatrix} \geq 0
\]

where symbol ‘+’ indicates symmetric terms in a matrix and \( x(k) \) is the state measurement at instant \( k \).

### 3 Robust MPC

Modelling uncertainties are ubiquitous, and how to configure a system framework to represent the influence of modelling uncertainties on controller design as well as capture system dynamics is the first step in the robust MPC synthesis. In this paper, we focus on the finite horizon robust model predictive control (FH-RMPC), that is, setting both the prediction and control horizons as finite integers, therefore it is inevitable to perform state/output predictions. Equation (3) provides an approach to future state calculation of nominal MPC systems. In the same fashion, we can perform state predictions in the presence of modelling uncertainties. However, if there exist uncertain terms in matrix \( A \), the high-order factors of these uncertain terms will appear in the expression of predicted states, which are notorious for MPC formulation. Such a fact motivates us to introduce a new framework to represent the uncertain factors in matrix \( A \) technically or a new prediction method for future state computation. In this paper, we manage to develop a new framework, and consequently, a moving average system matrix is constructed to represent modelling uncertainties, which differs from the conventional uncertainty frameworks for IH-RMPC schemes, for example, polytopic and structured feedback-loop uncertainty [12].

#### 3.1 Framework for modelling uncertainties

Fig. 1 shows the framework adopted by this paper. It is composed of the nominal model paralleling the modelling uncertainty block. Here we assume that \( C \) is known precisely and the states are fully measurable. The whole system is in the form of input to state, and then to output. In Fig. 1, \( \Delta_k \) stands for the modelling uncertainties over the prediction horizon starting at instant \( k \). It is weighted and norm-bounded by one, and \( W \) and \( P \) are weighting matrices, that is

\[
\Delta_k = \begin{bmatrix}
\Delta_k(k, k) & 0 & \cdots \\
\Delta_k(k + 1, k) & \Delta_k(k + 1, k + 1) & \cdots \\
\vdots & \vdots & \ddots \\
0 & 0 & \cdots & \Delta_k(k + N_p, k + N_p)
\end{bmatrix}
\]

with \( \| \Delta_k \| = \sigma(\Delta_k) \leq 1 \). To simplify the FH-RMPC formulation, we assume that predicted state \( x(k + i|k) \) is independent of the previous modelling uncertainties due to the monotonicity of the prediction horizon. Taking advantages of such an assumption, the controller design may be significantly simplified.

#### 3.2 FH-RMPC formulation

Based on the uncertainty block defined in (11), we do the state predictions. The key point in the FH-RMPC scheme is to exploit the monotonicity of the prediction horizon, say, at every prediction horizon starting at instant \( k \), predictions are not influenced by the previous horizon uncertainty block \( \Delta_{k-1} \). Here the nominal model is given by

\[
\hat{x}(k + i|k) = A \hat{x}(k + i|k) + B u(k + i|k)
\]

and the uncertain term \( \delta(k) \) caused by modelling uncertainties can be computed from

\[
\delta(k + i|k) = \sum_{j=0}^{k-i} \Delta(k + i, j) u(j|k)
\]
where the uncertainty matrix \( \hat{\Delta} \) is defined, for convenience, to be

\[
\hat{\Delta} = P \Delta W
\]

From (12) and (13), we have

\[
x(k + i + 1|k) = \tilde{x}(k + i + 1|k) + \delta(k + 1 + i|k) \\
= A \tilde{x}(k + i|k) + Bu(k + i|k) \\
+ \sum_{j=k}^{k+i} \Delta(k + 1 + i, j)u(j|k)
\]

(14)

It is obvious that

\[
x(k + i|k) = \tilde{x}(k + i|k) + \delta(k + i|k)
\]

\[
= \tilde{x}(k + i|k) + \sum_{j=k}^{k+i} \Delta(k + i, j)u(j|k)
\]

(15)

Substitute \( \tilde{x}(k + i|k) \) in (15) into (14), and derive

\[
x(k + 1 + i|k) = Ax(k + i|k) + Bu(k + i|k) \\
+ \sum_{j=k}^{k+i} \Delta(k + 1 + i, j)u(j|k) \\
- A \sum_{j=k}^{k+i} \Delta(k + i, j)u(j|k)
\]

(16)

The predicted output satisfies

\[
y(k + i|k) = Cx(k + i|k)
\]

To illustrate the procedure of the state predictions, we implement the first two steps, that is, calculations of \( x(k + 1|k) \) and \( x(k + 2|k) \)

\[
x(k + 1|k) = Ax(k|k) + Bu(k|k) + \sum_{j=k}^{k+1} \Delta(k + 1, j)u(j|k)
\]

\[
- A \Delta(k, k)u(k|k)
\]

(18)

\[
x(k + 2|k) = Ax(k + 1|k) + Bu(k + 1|k) \\
+ \sum_{j=k}^{k+2} \Delta(k + 2, j)u(j|k)
\]

\[
- A \sum_{j=k}^{k+1} \Delta(k + 1, j)u(j|k)
\]

(19)

Substituting (18) into (19), we have

\[
x(k + 2|k) = A^2 x(k|k) + ABu(k|k) + Bu(k + 1|k) \\
+ \sum_{j=k}^{k+2} \Delta(k + 2, j)u(j|k)
\]

\[
- A^2 \Delta(k, k)u(k|k)
\]

(20)

Without loss of generality, we can presume that uncertainty block \( \Delta_k \) is strictly causal, hence the first element of uncertainty block \( \Delta_k(k, k) = 0 \), consequently, \( \Delta(k, k) = 0 \) (weightings \( P \) and \( W \) are block diagonal matrices). So we can derive the common expression of the state predictions,

\[
x(k + i|k) = \left\{
\begin{array}{l}
A' x(k) + A'^{-1} Bu(k|k) + \cdots \\
+ Bu(k + i - 1|k) \\
+ \sum_{j=k}^{k+i} \Delta(k + i, j)u(j|k), \\
\text{if } 1 \leq i \leq N_a - 1 \\
A' x(k) + A'^{-1} Bu(k|k) + \cdots \\
+ A^{i-N_a+1} Bu(k + N_a - 2|k) + \cdots \\
+ (A^{i-N_a} B + \cdots + B) u(k + N_a - 1|k) \\
+ \sum_{j=k}^{k+N_a-1} \Delta(k + i, j)u(j|k) \\
+ \sum_{j=k+N_a}^{k+i} \Delta(k + i, j)u(j|k) \\
\text{if } N_a \leq i \leq N_p
\end{array}
\right.
\]

(21)

Rewrite the predicted states into an augmented matrix form

\[
\begin{bmatrix}
x(k + 1|k) \\
\vdots \\
x(k + N_a|k) \\
x(k + N_p|k)
\end{bmatrix} =
\begin{bmatrix}
A \\
\vdots \\
A^{N_a} \\
A^{N_p}
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\vdots \\
A^{N_a-1} B \\
A^{N_p-1} B
\end{bmatrix}
\begin{bmatrix}
B \\
\vdots \\
A^{N_a-1} B \\
A^{N_p-1} B
\end{bmatrix}
\begin{bmatrix}
u(k) \\
\vdots \\
u(k + N_a - 1|k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta(k + 1, k) \\
\Delta(k + 1, k + 1) \\
\vdots \\
\Delta(k, k + N_a) \\
\Delta(k + 1, k) \\
\Delta(k + 1, k + 1) \\
\vdots \\
\Delta(k + N_p, k + 1)
\end{bmatrix}
\begin{bmatrix}
u(k|k) \\
\vdots \\
\Delta(k + 1, k) \\
\Delta(k + 1, k + 1) \\
\vdots \\
\Delta(k, k + N_a) \\
\Delta(k + 1, k) \\
\Delta(k + 1, k + 1) \\
\vdots \\
\Delta(k + N_p, k + 1)
\end{bmatrix}
\]

(22)
Here we define two auxiliary matrices $M_1$ and $M_2$ as the left- and right-multipliers of uncertainty block $\Delta$, namely,

$$M_1 = \begin{bmatrix} 0 & I_1 & 0 & \cdots & 0 \\
0 & 0 & I_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_1 \\
\end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} I_2 & 0 & 0 & \cdots & 0 \\
0 & I_2 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & I_2 & 0 \\
0 & 0 & \cdots & 0 & I_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_2 \\
\end{bmatrix}$$

where both $I_1 \in \mathbb{R}^{n \times n}$ and $I_2 \in \mathbb{R}^{m \times m}$ are identity matrices. In terms of $M_1$ and $M_2$, uncertainty block $\Delta_k$ defined in (11) can represent the uncertain terms in (22). Using the notation defined in (5) and (8), we can rewrite (17) and (21) from $i = 1$ to $N_p$ in the augmented matrix form as follows

$$X(k) = \mathcal{A}X(k) + \mathcal{B}U(k) + M_1P_1X(k) + M_2U(k)$$

$$Y(k) = \mathcal{C}X(k)$$

where $X(k)$ is the augmented, predicted state vector

$$X(k) = [x^T(k + 1|k), x^T(k + 2|k), \ldots, x^T(k + N_p|k)]^T$$

Motivated by the approach to nominal MPC, we now extend this approach to the case of FH-RMPC.

A finite horizon robust MPC system can be represented by its corresponding nominal model in parallel with a weighed unity-norm uncertainty block. Based on such a framework, robust step tracking control, say, step tracking in the presence of modelling uncertainties, can be achieved by solving a robust semi-definite optimisation problem (if solutions exist) whose constraints contain uncertain matrices

$$J_a = \min_{t \leq t(k)} t$$

subject to

$$t > 0 \quad (\max \Delta_k) J \leq t \ (\text{with } \|\Delta_k\| = \sigma(\Delta_k) \leq 1)$$

$$J = (T - \mathcal{J}(k))^TQ(T - \mathcal{J}(k)) + U^T(k)RU(k)$$

$$X(k) = \mathcal{A}X(k) + \mathcal{B}U(k) + M_1P_1X(k) + M_2U(k)$$

$$Y(k) = \mathcal{C}X(k)$$

(25)

where $T$ is the augmented reference input, that is

$$T = [r_1^T, r_2^T, \ldots, r_{T-1}^T]^T$$

and $t$ is an upper bound of the objective $J$.

From (24), the objective $J$ can be represented by

$$J = (T - (\mathcal{C}AX(k) + CBU(k) + CM_1P_1X(k) + M_2U(k)))^TQ(T - (\mathcal{C}AX(k) + CBU(k) + CM_1P_1X(k) + M_2U(k)))$$

$$+ U^T(k)RU(k)$$

(26)

and then in the same fashion as in the derivation in (9) and (10), condition (26) can be created.

3.3 FH-RMPC algorithm based on LMIs

We have converted the FH-RMPC problem into the robust semi-definite optimisation. Because of the presence of modelling uncertainties, (27) comprises the uncertain terms of $\Delta_k$. Therefore we cannot apply Schur complements and use existing software packages to solve it numerically. In order to overcome such a barrier, we first introduce the following lemma.

**Lemma 1** [16, 17]: Let $T_1 = T_1^T$ and $T_2, T_3, T_4$ be real matrices of appropriate sizes. Then $\det(I - T_4\Delta) \neq 0$ and $T_1 + T_2\Delta(I - T_4\Delta)^{-1}T_3 + T_5T_1^T(I - T_4\Delta)^{-1}\Delta T_4^T \geq 0$ for every $\Delta$, $\|\Delta\| = \sigma(\Delta) \leq 1$, if and only if $\|T_4\| < 1$ and there exists a scalar $\tau \geq 0$ such that

$$
\begin{bmatrix} T_1 - \tau T_2 T_1^T \\
T_3 - \tau T_3 T_2^T \\
\tau I - T_4 T_4^T \end{bmatrix} \geq 0
$$

**Proof**: Here we assume $T_2$ and $T_3$ are non-zero (the result is straightforward if either of them is zero). Pre- and post-multiplying $z^T$ and $z$ to (28), we have

$$z^T T_1 z + z^T T_2 \Delta(I - T_4\Delta)^{-1}T_3 z$$

$$+ z^T T_5^T(I - T_4\Delta)^{-1}\Delta T_4^T z \geq 0$$

(29)

where $z$ is a non-zero vector with a proper dimension. Define by

$$\xi = (I - T_4\Delta)^{-1}\Delta T_5^T z$$

(30)

Therefore (29) can be rewritten as

$$
\begin{bmatrix} z^T \\
\xi^T \\
\xi^T \end{bmatrix} \begin{bmatrix} T_1 & T_3^T \\
T_3 & 0 \\
\tau I - T_4 T_4^T \end{bmatrix} \begin{bmatrix} z \\
\xi \\
\xi \end{bmatrix} \geq 0
$$

(31)

Pre-multiplying both sides of (30) by $(I - T_4\Delta)^T$, we obtain

$$\xi = \Delta^T (T_4^T z + T_5^T z)$$

Set $p = T_4^T z + T_5^T z$ for simplicity, that is, $\xi = \Delta^T p$. According to the condition $\|\Delta\| = \sigma(\Delta) \leq 1$, we can say

$$\xi^T \xi = p^T \Delta^T p \leq p^T p$$

Thus

$$(T_4^T z + T_5^T z)(T_4^T z + T_5^T z) - \xi^T \xi \geq 0$$

(32)

Using the $S$-procedure, we know that (32) is satisfied if and only if

$$\begin{bmatrix} T_1 & T_3^T \\
T_3 & 0 \\
\tau I - T_4 T_4^T \end{bmatrix} \geq 0$$

(33)

where $\tau$ is a positive scalar. Rewrite (33), and then lemma 1 is proven.

The key idea of the above lemma is to employ an auxiliary positive scalar $\tau$ to convert robust LMIs, namely, LMIs with some uncertain matrices, into standard LMI constraints. Taking advantages of such a property, we can transform FH-RMPC for robust step tracking control into a standard semi-definite optimisation problem.
Theorem 1: The robust step tracking performance for the MPC system in Fig. 1 is achievable if the following semi-definite optimisation problem is solvable

\[ J_o = \min_{t, \hat{l}(k), \tau} t \]

subject to

\[ t > 0, \quad \tau \geq 0 \]

and

\[
\begin{bmatrix}
T & (T - CAx(k) - CB\hat{l}(k))^T & U(k)^T \\
* & Q^{-1} - \tau CM^TP(CM^P)^T & 0 \\
* & 0 & R^{-1} \\
* & 0 & \alpha I
\end{bmatrix}
\geq 0
\]

where \( I \) is an identity matrix. Augmented reference input \( T \), predicted input sequence \( \hat{l}(k) \) and weighting matrices \( Q, R \), are defined in (5) and (6). The constant augmented matrices \( A, B, C \) and left-, right-matrices \( M_i, M_r \) of uncertainty block \( \Delta_i \) in Fig. 1 are introduced in (8) and (23).

\[ \text{Proof:} \quad \text{Applying Schur complements and rewriting constraints in (26), we have} \]

\[
\begin{bmatrix}
T & (T - CAx(k) - CB\hat{l}(k))^T & U(k)^T \\
* & Q^{-1} & 0 \\
* & 0 & R^{-1} \\
* & 0 & \alpha I
\end{bmatrix}
\geq 0
\]

Separating the certain and uncertain terms of (35)

\[ T_1 - \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\geq 0
\]

and recasting (36), we have

\[ T_1 - \begin{bmatrix}
(WM_i\hat{l}(k)) \Delta_i & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\geq 0
\]

and putting (38) into the form of (28), we can take advantages of the property described in Lemma 1, that is,

\[ T_1 - T_2^T \Delta_i^T T_2^T - T_2 \Delta_i T_3 \geq 0 \iff \begin{bmatrix}
T_1 & -T_2^T \Delta_i^T & T_3^T \\
T_2 & \Delta_i & 0 \\
0 & 0 & \alpha I
\end{bmatrix} \geq 0 \]

(39)

Therefore it is not difficult to convert FH-RMPC for step tracking into a standard semi-definite optimisation problem. Theorem 1 is then proved.

Theorem 1 provides an effective approach for solving FH-RMPC problems for robust step tracking control. By adjusting the length of prediction horizon \( N_p \) and/or control horizon \( N_u \), different requirements on the pre-specified performance may be satisfied. From the previous theoretical analysis, if \( N_p \) and \( N_u \) are large enough (e.g. \( N_p = N_u = \infty \)) and optimal control sequences do exist, we can find a Lyapunov function to guarantee closed-loop stability of RMPC without any terminal constraints. However for the FH-RMPC case, the situation is different. If both \( N_p \) and \( N_u \) are finite, terminal cost constraints have to be imposed to facilitate the robust stability analysis.

4 Terminal cost constraints

MPC strategies belong to feedback control areas. Although feedback helps to attenuate the influence of modelling uncertainties, feedback can also lead to system instability. In 1988, Keerthi and Gilbert first proposed a method which employed the objective function of MPC systems as a Lyapunov function to solve the nominal stability problem [26]. Later the same approach was used for non-linear systems [27]. In this paper, we will employ a similar idea and develop terminal cost constraints to guarantee robust stability of FH-RMPC systems.

4.1 LMs for terminal cost constraints

Without loss of generality, here we set \( N_p = N_u \), otherwise we can enforce the terminal input \( u(k + N_p - 1 | k) = 0 \) to resume the following derivation. For ease of notation, we denote \( e(k + i | k) \triangleq y(k + i | k) - r(k + i | k) \). Consider a quadratic function

\[
V(x(k + i | k)) = e(k + i | k)^T \Phi e(k + i | k)
\]

\[
= \|Cx(k + i | k) - r\|_P^2, \quad \Phi > 0
\]

(40)

of state measurement \( x(k), k > 0 \). In the prediction horizon starting at instant \( k \), we set

\[
V(x(k + i + 1 | k)) - V(x(k + i | k)) < -\|e(k + i | k)\|_P^2 + \|u(k + i | k)\|_P^2
\]

(41)

consequently

\[
V(x(k + N_p | k)) - V(x(k + N_p - 1 | k)) < -\|e(k + N_p - 1 | k)\|_P^2 + \|u(k + N_p - 1 | k)\|_P^2
\]

(42)

Summing (41) and (42) from \( i = 0 \) to \( N_p \), we obtain

\[
V(x(k + N_p | k)) - V(x(k | k)) < -J - \|e(k)\|_P^2 + \|e(k + N_p | k)\|_P^2
\]

In the sequel, we will employ $V(x(k))$ as a Lyapunov function satisfying

$$V(x(k)) > t + \|e(k)\|^2_Q - \|e(k + N_p[k])\|^2_{Q_p}$$

$$+ V(e(k + N_p[k]))$$

where $t$ is the upper bound of objective $J$ defined in (26). Then $\tilde{V}(k) = \mathbb{R}^n \rightarrow \mathbb{R}$, the difference of Lyapunov functions of $x(k+1)$ and $x(k)$, can be defined as:

$$\tilde{V}(k) := V(x(k+1)) - V(x(k))$$

$$< V(x(k+1)) - t - \|e(k)\|^2_Q + \|e(k + N_p[k])\|^2_{Q_p}$$

$$- V(x(k + N_p[k]))$$

In order to derive closed-loop asymptotic stability, we should guarantee the right-hand side of (44) is negative, that is

$$0 < \|e(k + N_p[k])\|^2_{Q_p} - \|e(k)\|^2_Q + \|e(k + N_p[k])\|^2_{Q_p}$$

From (21), we know that if $u(k)$, the first element of input sequence $\tilde{u}(k)$ is sent to the real process, the state measurement at instant $(k + 1)$ can be expressed as

$$x(k + 1) = Ax(k) + Bu(k) + \Delta(k + 1, k)u(k)$$

consequently

$$e(k + 1) = CAx(k) + CBu(k) + \Delta(k + 1, k)u(k) - r$$

Introduce two constant matrices $E_1$ and $E_2$ such that

$$\Delta(k + 1, k) = E_1 \Delta E_2 = E_1 M_1 P_\Delta I E_2$$

where

$$E_1 = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}^T$$

Inserting (46) and (47) into (45), we obtain

$$\|CAx(k) + CBu(k)\|_Q - r$$

$$+ CE_1 M_1 P_\Delta WM_2 u(k)\|_Q - t$$

$$- \|Cx(k) - r\|^2_Q + e^T(k + N_p[k])$$

$$\times (Q_{N_p} - \Phi)e(k + N_p[k]) < 0$$

So if the following two inequalities

$$\|Cx(k) - r\|^2_Q + t - \|CAx(k) + CBu(k)\|_Q$$

$$- r + CE_1 M_1 P_\Delta WM_2 u(k)\|_Q > 0$$

$$\Phi - Q_{N_p} \geq 0$$

hold simultaneously, we can guarantee the condition in (48). Applying Schur complements and the property of robust LMIs (lemma 1), we can recast (49) into

$$\begin{bmatrix} \|Cx(k) - r\|^2_Q + t & \|CAx(k) + CBu(k)\|_Q - r & WM_2 u(k)\|_Q \end{bmatrix}$$

$$\begin{bmatrix} * & * & \Phi - Q_{N_p} \end{bmatrix} > 0$$

where $X = \Phi^{-1}$ and $\lambda_1$ is a positive scalar. Then left- and right-multiplying $X$ to both sides of inequality (50) and defining a small non-negative scale $\kappa$, which is selected as a tuning scalar of $(Q_{N_p} + \kappa I)$, we have

$$X - X(Q_{N_p} + \kappa I)X \geq 0$$

It is obvious that if $\kappa \rightarrow 0$, (52) is equivalent to (50). Apply Schur complements to (52) and derive

$$\begin{bmatrix} X & X \end{bmatrix} > 0$$

Combined with (53), (51) composes a sufficient condition to (45), which is designed for asymptotical stability of the closed-loop FH-RMPC system.

Meanwhile, in order to use $V(x(k))$ as a Lyapunov function candidate, in the sequel, we will manage to derive another LMI to guarantee (43). To this end, taking advantages of the condition in (50), we can derive a sufficient condition to (43)

$$\|e(k)\|^2_{Q_p} - t - \|e(k)\|^2_Q + \|e(k + N_p[k])\|^2_{Q_p} > 0$$

From (21), $x(k + N_p[k])$ is expressed as:

$$e(k + N_p[k]) = CA^N x(k) + CE_3 \tilde{u}(k)$$

$$+ CE_1 M_1 P_\Delta WM_2 \tilde{u}(k) - r$$

where $E_1 = [0 \cdots 0 0]$ and $I$ is an identity matrix with a proper dimension. Substituting (55) into (54), applying Schur complements and using the property of robust LMIs, we obtain

$$\begin{bmatrix} \|CAx(k) - r\|_Q^2 - \|CAx(k) - r\|^2_Q + t & \|CAx(k) + CBu(k)\|_Q - r & WM_2 u(k)\|_Q \end{bmatrix}$$

$$\begin{bmatrix} * & * & \Phi - Q_{N_p} \end{bmatrix} > 0$$

where $\lambda_2$ is a positive scalar.

**Theorem 2:** To achieve step tracking performance for the FH-RMPC system defined in Fig. 1, manipulated input $u_t(k) = E_2 \tilde{u}(k)$, $k > 0$, can be obtained by minimising the following optimisation problem:

$$J_o = \min_{\tilde{u}(k)} t$$

subject to (34), (51), (53) and (56), where $X$, $\lambda_1$ and $\lambda_2$ are variables of LMIs for terminal cost constraints, and $E_4$ is a truncation matrix, given by

$$E_4 = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$$
The closed-loop system is guaranteed asymptotically stable if the optimal input sequences

\[ U_{0}(k) = \begin{bmatrix} u_{0}(k) & u_{0}(k+1) & \cdots \end{bmatrix}^T, \quad k > 0 \]

do exist.

**Proof:** From theorem 1 we know that the robust semi-definite optimisation problem in (26) can be solved by minimising the linear objective in (34). Meanwhile, combined with constraints (51), (53) and (56), the quadratic function of \( e(x(k)) \)

\[ V(x(k)) = e^T(k)\Phi e(k) \]

can be regarded as a Lyapunov function, which is convergent with MPC iteration. Therefore by adding additional constraints (51), (53) and (56) into the optimisation problem defined in (34), we can guarantee the resulting FH-RMPC regulator is asymptotically stable, associated with the Lyapunov function \( V(x(k)) \).

### 5 Simulation example

Consider a classical angular positioning system proposed by Kwakernaak and Sivan in 1972 [28]. The system model is written as

\[
\begin{bmatrix}
  x_1(k+1) \\
  x_2(k+1)
\end{bmatrix} =
\begin{bmatrix}
  1 & 0.1 \\
  0 & 1 - 0.1\alpha
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  u(k)
\end{bmatrix} + \begin{bmatrix}
  0 \\
  0.787
\end{bmatrix}
\]

\( y(k) = [1 \ 0] x(k) \) (57)

where \( \alpha \in [0.1, 10] \) reflects the uncertain coefficient of viscous friction in system’s physical structure. Based on approaches discussed in [7], an IH-RMPC controller for the modelling uncertainties in the form of the structured feedback loop is first designed. Comparing with the FH-RMPC controllers considered in this paper, it can be seen that the FH-RMPC controllers have the better tracking performance and smaller overshoot of the optimal input sequences (Fig. 2). Here the tuning parameters are selected as: \( r = 1, Q = I, Q_N = I, R = 0.00002I, P = I, N_u = 3 \) and \( W = 0.1 \). The simulation length equals to 50. For the simulation results presented in Fig. 2, \( \alpha = 0.7 \) (nominal value \( \bar{\alpha} = 0.495 \)). In order to reconfigure the system in (57) into the framework of Fig. 1, we can take advantage of the method described in Fig. 3, that is, using the difference between the nominal model and real process to derive uncertainty block \( \Delta \).

![Fig. 2 IH-RMPC controller (dash-dotted) and FH-RMPC controllers: \( N_p = 3 \) (solid) and \( N_p = 6 \) (dotted)](image)

![Fig. 3 Modelling uncertainty reconfiguration](image)
Let us increase and decrease uncertain term $\alpha$ oppositely till its left and right bounds, that is, setting $\alpha = 10$ and $0.1$, respectively. In the same fashion, we first design the IH-RMPC controller. However, we find that it takes very long time to reach the steady-state value and serious ripples occur and therefore was not presented. Fig. 4 shows the simulation results based on FH-RMPC controllers with the different control horizons. FH-RMPC can achieve the prespecified tracking even under extreme conditions. From this point, the FH-RMPC algorithm proposed in this paper has a better robustness property. Similar to nominal MPC controllers, FH-RMPC controllers also possess the property that if increasing the difference between $N_p$ and $N_u$, the overshoot of performance decreases; meanwhile, system responses become slower.

As discussed above, closed-loop stability is one of serious problems in FH-RMPC controller design. By imposing several extra terminal cost constraints, we can guarantee that the resulting system is closed-loop stable. Fig. 5 demonstrates the influence of the imposed terminal cost constraints on the system performance with the different prediction horizons $N_p$. Here we set $\alpha = 0.8$ and the control horizon $N_u = 3$. It can be seen that the terminal cost constraints manage to attenuate the input and output peaks, meanwhile derive slower responses. Fig. 6 demonstrates the influence of the terminal cost constraints on the system performance with the different terminal weightings.
We reset $\alpha = 0.9$, $N_p = 3$ and keep $N_u = 3$. In the figures, solid lines (no cost constraints) are derived from theorem 1, and dash-dotted lines from theorem 2. It can be seen that for some systems, even though we do not impose extra terminal cost constraints, the FH-RMPC algorithm can still come to closed-loop stability.

All the simulations were performed on a PC with a Pentium 4 processor, 512 MB RAM, using the software LMI Control Toolbox [29] in the Matlab environment to compute solutions of the linear minimisation problem. Table 1 shows that the on-line computational cost can be reduced by choosing FH-RMPC controllers. In the table, the numbers parenthesised are average time to compute $u^*(k)$ over every prediction horizon, and the other is for the total time with the simulation length equal to 50 ($N_p = N_u = 3$).

### Table 1: Time cost of the online computation

<table>
<thead>
<tr>
<th>Uncertainty factor $\alpha$</th>
<th>0.7</th>
<th>0.8</th>
<th>0.1</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>IH-RMPC controller (s)</td>
<td>4.366 (0.087)</td>
<td>5.049 (0.101)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>FH-RMPC controller without</td>
<td>3.956 (0.079)</td>
<td>3.986 (0.079)</td>
<td>3.695 (0.074)</td>
<td>3.736 (0.074)</td>
</tr>
<tr>
<td>terminal cost constraints (s)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6 Conclusion

In this paper, we have discussed FH-RMPC problems. Two topics were covered: how to achieve robust step tracking control based on a FH-RMPC scheme and the closed-loop stability analysis of the resulting FH-RMPC system. Taking advantages of the property of robust LMI, whose constraints have uncertain terms, the conventional min–max problem was converted into a standard semi-definite optimisation problem. Comparing with the infinite horizon MPC, the final simulation results demonstrate that FH-RMPC has more tuning freedoms, better control performance and faster online implementation. The whole algorithm development is based on the assumption of fully measurable states. How to remove this is left to the future.

7 References