Brief paper

Razumikhin-type stability theorems for discrete delay systems

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Abstract

In this paper, by employing the Razumikhin technique and Lyapunov functions, Razumikhin-type theorems that guarantee the uniform stability, uniformly asymptotic stability and uniformly exponential stability for the general discrete delay systems are established, respectively. Moreover, Razumikhin-type uniformly exponential stability theorem gives the estimation of the convergence speed. As theoretic application, the Razumikhin-type uniformly exponential stability result is further studied and used to show some well-known stability results for some kinds of discrete delay systems. Finally, examples are also worked through to illustrate our results.

Keywords: Razumikhin-type theorem; Discrete delay system; Uniform stability; Uniformly asymptotic stability; Uniformly exponential stability; Lyapunov exponent

1. Introduction

Time delays are commonly encountered in many physical systems and control schemes due to the finite switching times, network traffic congestions, etc. Stability analysis of time-delay systems has attracted increasing attention for the last 3 decades (Hale, 1977; Hale & Lunel, 1993; Liao, 2001; Marquez, 2003; Niculescu, 2001). It is well known that time delays often lead to the failure of stability for a stable system. It is, therefore, very important to investigate the stability problem for systems with time delays.

Among the methods contributed to the study of stability problem for time-delay systems, the Lyapunov functional, comparison principle and Razumikhin technique (Hale, 1977; Hale & Lunel, 1993) are the three main methods. The Lyapunov functional method requires constructively a Lyapunov function that decreases on the whole state space. The comparison principle requires finding an additional system, with known stability properties, and then compare that to the original time-delay system. Recently, Knospe and Roozbehani (2003) and Zhang, Knospe, and Tsiotras (2001, 2003) used the comparison principle to investigate the stability conditions for linear continuous delay systems, in which they derived many previously reported stability criteria by establishing a comparison system free of delays. On the other hand, the Razumikhin technique has advantage that, when dealing with time delays, the Lyapunov function is not required to be decrescent on the whole state space.

The Razumikhin technique has been applied successfully by various authors to study of several stability problems for continuous delay systems, see, for instance, Hale (1977), Hale and Lunel (1993), Mao (1997), Teel (1998) and Teel, Nešić, and Kokotović (1998). Recently, there have appeared several papers devoted to the study of right-continuous impulsive delay system by using Razumikhin technique (Liu, Liu, Teo, & Wang, 2006; Liu & Ballinger, 2001; Shen & Yan, 1998; Stamova & Stamov, 2001). Razumikhin-type stability theorems of continuous delay systems and right-continuous delay systems are based on the fact that the solution of these type of systems is a continuous or right-continuous function. Unlike continuous systems and right-continuous systems, the solution of a discrete-time system is no longer a continuous or right-continuous function. This brings difficulties in the use Razumikhin technique to...
investigate the stability problem for discrete delay system. Zhang and Chen (1998) studied a class of discrete delay systems and established a backward Razumikhin-type uniformly asymptotic stability theorem. However, the main condition developed in Zhang and Chen (1998) is restrictive and difficult to test. Hence, finding stability criteria in which the conditions can be less restrictive and easily tested is of practical significance.

The aim of this paper is to establish Razumikhin-type stability criteria with less restrictive assumptions for general discrete delay systems. In order to overcome the shortcoming in Zhang and Chen (1998), we provide the forward Razumikhin-type uniform stability theorems, in which the less restrictive Razumikhin-type uniformly asymptotic stability theorem and the easily tested Razumikhin-type exponential stability criteria are established for discrete delay systems. To the best of our knowledge, no Razumikhin-type exponential stability theorem has been previously reported for discrete delay systems.

The rest of this paper is organized as follows. In Section 2, we introduce our notation and definitions. Then in Section 3, we develop forward Razumikhin-type uniform stability theorems for discrete delay systems. In Section 4, as the theoretic application, the Razumikhin-type uniformly exponential stability result is specialized to some kinds of discrete delay systems. Finally, in Section 5, we discuss some examples to illustrate our results.

2. Preliminaries

In the sequel, \( R \) denotes the field of real numbers, \( R^+ \) the subset of non-negative elements of \( R \), defined by \( R^+=[0, +\infty) \), and \( R^n \) the \( n \)-dimensional Euclidean space. \( N \) represents the natural numbers, \( N = \{0, 1, 2, \ldots \} \), \( N_- = \{0, -1, -2, \ldots \} \), and for some positive integer \( m \), let \( N_m = \{-m, \ldots, -1, 0, \ldots, m\} \).

A function \( \gamma: R^n \to R^+ \) is of class \( K \) (\( \gamma \in K \)) if it is continuous, zero at zero and strictly increasing. For a given positive real number \( r > 0 \), let \( C([-r, 0], R^n) = \{ \psi : [-r, 0] \to R^n, \psi \text{ is continuous} \} \). Given a positive integer \( m \), we define \( \| \phi \|_m = \max_{0 \leq t \leq m} (\| \phi(t) \|) \).

Consider the discrete delay system of the form
\[
\begin{align*}
x(n+1) &= f(n, x_n), & n \geq n_0, \\
x_{n_0} &= \phi,
\end{align*}
\]
where \( x \in R^n \), \( n_0 \in N \), \( f \in C(N \times C([-m, 0], R^n), R^n) \), \( \phi \in C([-m, 0], R^n) \), where \( m \in N \) represents the delay in system (1), and \( x_n \in C([-m, 0], R^n) \) is defined by \( x_n(s) = x(n+s) \) for any \( s \in [-m, 0] \).

We assume \( f(n, 0) \equiv 0 \) so that system (1) admits the trivial solution. We also assume that system (1) has an unique solution, denoted by \( x(n) = x(n, n_0, \phi) \), for any given initial data: \( n_0 \in N \) and \( \phi \in C([-m, 0], R^n) \).

Remark 2.1. System (1) considered in this paper is more general than that in Zhang and Chen (1998), in which the function \( f(\cdot) \) needs to satisfy: \( \| f(n, \phi) \| \leq L \| \phi \| \), for some positive constant \( L > 0 \) and any \( n \in N \).

Definition 2.1. The trivial solution of system (1) is said to be uniformly stable (US) if, for any given initial data: \( n_0 \in N \), \( x_{n_0} = \phi \), and for any \( \varepsilon > 0 \), there exists a \( \delta = \delta(\varepsilon) > 0 \) independent of \( n_0 \) such that when \( \| \phi \|_m \leq \delta \), the following inequality holds:
\[
\| x(n, n_0, \phi) \| \leq \varepsilon \quad \text{for any } n \geq n_0, \ n \in N.
\]

Definition 2.2. The trivial solution of system (1) is said to be uniformly exponentially stable (UES) if, for any given initial data: \( n_0 \in N \), \( x_{n_0} = \phi \), and any \( \eta > 0 \), there exist a positive real number \( \sigma = \sigma(\eta) > 0 \) and a positive integer \( K = K(\eta) > 0 \), where both \( \sigma \) and \( K \) are independent of \( n_0 \), such that when \( \| \phi \|_m < \sigma \) and \( n \geq n_0 + K \), the following inequality holds:
\[
\| x(n, n_0, \phi) \| < \eta,
\]
i.e.,
\[
\lim_{n \to \infty} \| x(n, n_0, \phi) \| = 0.
\]

Definition 2.3. The trivial solution of system (1) is said to be uniformly asymptotically stable (UAS) if there exist two positive numbers \( \varepsilon > 0 \), \( M > 0 \), where both \( \varepsilon \) and \( M \) are independent of \( n_0 \), such that for all \( n \geq n_0, \ n \in N \),
\[
\| x(n, n_0, \phi) \| \leq M \| \phi \|_m e^{-\varepsilon(n-n_0)}.
\]

Remark 2.2. Obviously, if (5) holds, then the Lyapunov exponent of system (1) is not greater than \(-\varepsilon\).

For simplicity, if the trivial solution of system (1) is US (UAS, UES), system (1) is also called as US (UAS, UES).

### 3. Razumikhin-type theorems for discrete delay systems

In this section, three types of stability (US, UAS and UES) for discrete delay system (1) are investigated.

Theorem 3.1. Assume that there exist functions \( c_1, c_2 \in K \) and a positive definite function \( V(n, x) \) such that the following conditions hold:

(i) \( c_1(\| x \|) \leq V(n, x) \leq c_2(\| x \|) \);
(ii) for any \( \phi \in C([-m, 0], R^n) \) and \( s \in N_{-m} \), then \( V(n, \phi(0)) \geq V(n+s, \phi(s)) \) implies \( V(n+1, f(n, \phi)) \leq V(n, \phi(0)) \);
(iii) for any \( \phi \in C([-m, 0], R^n) \), and some \( s \in N_{-m} - 0 \), then \( V(n, \phi(0)) \leq V(n+s, \phi(s)) \) implies \( V(n+1, f(n, \phi)) \leq \max_{0 \leq t \leq m} \{ V(n+t, \phi(t)) \} \).

Then, system (1) is US.
Theorem 3.2. Assume that there exist positive functions $c_1, c_2 \in \mathcal{K}$ and a positive definite function $V(n, x)$ such that the condition (i) of Theorem 3.1 holds. The conditions (ii)–(iii) of Theorem 3.1 are replaced by:

(ii)’ there exist positive scalar functions $p(\cdot), q(\cdot)$ satisfying $p(s) > s$ ($s > 0$) and $0 < q(s) < s$ ($s > 0$) such that, for any $\varphi \in C([-m, 0], R^n)$ and $s \in N_{-m}$, $V(n + s, \varphi(s)) \leq p(V(n, \varphi(0)))$ implies $V(n + 1, f(n, \varphi)) \leq V(n, \varphi) - q(V(n, \varphi(0)))$;

(iii)’ for any $\varphi \in C([-m, 0], R^n)$ and for some $s \in N_{-m} \setminus \{0\}$, then $V(n + s, \varphi(s)) \geq V(n, \varphi(0))$ implies $p(V(n + 1, f(n, \varphi))) \leq \max_{s \in N_{-m}} \{V(n + s, \varphi(s))\}$.

Then, system (1) is US.

Proof. By Theorem 3.1, it is easy to see that system (1) is US. So we just need to prove that system (1) is also uniformly attractive to the trivial solution. Without loss of generality, let $n_0 = 0$. Denote $x(n) = (x(n, n_0, \phi))$. For any fixed positive number $H > 0$, we choose a positive number $0 < \delta \leq H$ such that $c_2(\delta) = c_1(H)$.

Thus, from the uniform stability of system (1), for any $\phi \in C([-m, 0], R^n)$, if $\|\phi\|_m \leq \delta$, then we have

$$
\|x(n)\| \leq H, \quad n \geq -m
$$

and

$$
V(n, x(n)) < c_2(\delta), \quad n \geq -m.
$$

It follows from (10) that $\tilde{V}(n) \leq c_2(\delta)$ for any $n \leq 0$. In order to show the uniform attraction of trivial solution, we need to prove that, for any positive real number $\eta$ satisfying $0 < \eta \leq H$, there exists a positive integer $K = K(\eta, \delta)$ independent of $n_0$ such that when $\|\phi\|_m \leq \delta$ and $n \geq K + m$, we have

$$
\|x(n)\| = \|x(n, n_0, \phi)\| \leq \eta.
$$

If for any $n \geq K$, we have

$$
V(n, x(n)) \leq c_1(\eta),
$$

then, by condition (i), (11) can be induced from (12). Hence, in the following, we just prove that (12) holds.

For any positive real number $s$ with $c_1(\eta) \leq s \leq c_2(\delta)$, then there exists a positive number $a > 0$ such that $p(s) - s > a$.

Let $K_0 = \min\{u : u \in N, c_1(\eta) + u \cdot a \geq c_2(\delta)\}$, $r = \min\{1, q(s) \leq c_1(H)p(s)\}$, and let $M^{(0)}_i = [i\cdot c_2(\delta)/r]$, $M^{(j)}_i = M^{(j-1)}_i + m, i = 1, 2, \ldots, K_0, j \leq i$, where $[X]$ stands for the maximal integer less than $X$.

We claim that for $i = 1, 2, \ldots, K_0$,

$$
V(n, x(n)) \leq c_1(\eta) + (K_0 - i)a, \quad n \geq M^{(i-1)}_i.
$$

When $i = 1$, we first show that there exists a positive integer $n_1$ with $n_1 \leq M^{(0)}_1$ such that

$$
V(n_1, x(n_1)) \leq c_1(\eta) + (K_0 - 1)a.
$$

Otherwise, for any $n$ satisfying $0 \leq n \leq M^{(0)}_1$, we have

$$
V(n, x(n)) > c_1(\eta) + (K_0 - 1)a.
$$

Let $n = M^{(0)}_1$. It follows from (16) that

$$
V(M^{(0)}_1 + 1, x(M^{(0)}_1 + 1)) \leq c_2(\delta) - c_2(\delta) = 0.
$$

This is a contradiction with the positive definiteness of $V(n, x(n))$. Hence, (14) holds for some $n_1: n_1 \leq M^{(1)}_1$. Then, we show that for any $n \geq n_1$,

$$
V(n, x(n)) \leq c_1(\eta) + (K_0 - 1)a.
$$

Otherwise, there exists an $\tilde{n} \geq n_1$ such that $V(\tilde{n} + 1, x(\tilde{n} + 1)) > c_1(\eta) + (K_0 - 1)a \geq V(\tilde{n}, x(\tilde{n}))$. Thus, from the fact that $V(n + 1) \leq V(n)$ (see (7) in the proof of Theorem 3.1), there exists $s \in N_{-m}$ such that $V(\tilde{n} + s, x(\tilde{n} + s)) \leq V(\tilde{n} + 1, x(\tilde{n} + 1))$. Thus, by condition (iii)’, we get

$$
\tilde{V}(\tilde{n}) \geq \min\{V(\tilde{n} + 1, x(\tilde{n} + 1)) + a \geq c_2(\eta).
$$

This is a contradiction with $\tilde{V}(\tilde{n}) \leq c_2(\eta)$. Thus, (18) holds for all $n \geq n_1$. It follows from (18) and $M^{(0)}_1 \geq n_1$ that (13) holds when $i = 1$.

Now we assume that $i > 1$ and (13) holds for $i - 1$. Then, it follows from the induction assumption that $V(n, x(n)) \leq c_1(\eta) + (K_0 - i + 1)a$ for any $n \geq M^{(i-2)}_1$, which leads to

$$
V(n, x(n)) \leq c_1(\eta) + (K_0 - i + 1)a, \quad n \geq M^{(i-1)}_i.
$$

Similar to the case $i = 1$, first, we show that there is $n_2 \in [M^{(i-1)}_i, M^{(i-1)}_i]$ such that $V(n_2, x(n_2)) \leq c_1(\eta) + (K_0 - i)a$. Otherwise, for any $n \in [M^{(i-1)}_i, M^{(i-1)}_i]$, we have
Thus, condition \( (K_0 - i)a < V(n, x(n)) \leq c_1(\eta) + (K_0 - i + 1)a \). It leads to, for any \( n \in [M_{i-1}^{(i-1)}], M_i^{(i-1)} \) and any \( s \in N_{-m}, p(V(n, x(n))) > V(n, x(n)) + a \geq c_1(\eta) + (K_0 - i + 1)a \geq V(n + s, x(n + s)). \)

Thus, by condition (ii)' and \( c_1(H) = c_2(\tilde{\delta}) \), for any \( n \in [M_{i-1}^{(i-1)}, M_i^{(i-1)}] \), we have that
\[
V(n + 1, x(n + 1)) - V(n, x(n)) \leq -r, \tag{20}
\]
which implies that
\[
V(M_i^{(i-1)} + 1, x(M_i^{(i-1)} + 1)) \leq V(M_i^{(i-1)}, x(M_i^{(i-1)} - 1)) - r(M_i^{(i-1)} + 1 - M_i^{(i-1)})
\leq c_2(\tilde{\delta}) - r(M_i^{(0)} + 1 - M_i^{(0)}) = c_2(\tilde{\delta} - c_2(\tilde{\delta}) = 0. \tag{21}
\]
This is a contradiction. Hence, there is \( n_2 \in [M_{i-1}^{(i-1)}, M_i^{(i-1)}] \) such that \( V(n_2, x(n_2)) \leq c_1(\eta) + (K_0 - i)a \). Then, we show that \( V(n, x(n)) \leq c_1(\eta) + (K_0 - i)a \) holds for any \( n \geq n_2 \). Otherwise, there exists an \( \tilde{n}_2 \geq n_2 \) such that for any \( n_2 \leq n \leq \tilde{n}_2 \),
\[
V(\tilde{n}_2 + 1, x(\tilde{n}_2 + 1)) > c_1(\eta) + (K_0 - i)a \geq V(n, x(n)). \tag{22}
\]
From (22) and the fact that \( V(n + 1) \leq \tilde{V}(n) \), there exists an \( s \in N_{-m} \) such that \( V(\tilde{n}_2 + s, x(\tilde{n}_2 + s)) \geq V(\tilde{n}_2 + 1, x(\tilde{n}_2 + 1)) > V(\tilde{n}_2, x(\tilde{n}_2)) \). Thus, by condition (iii)' and \( \tilde{n}_2 \geq n_2 \geq M_{i-1}^{(i-1)} = M_{i-1}^{(i-2)} + m \), we get
\[
\tilde{V}(\tilde{n}_2) \geq p(V(\tilde{n}_2 + 1, x(\tilde{n}_2 + 1)))
> c_1(\eta) + (K_0 - i + 1)a \geq \tilde{V}(\tilde{n}_2). \tag{23}
\]
This is a contradiction. Hence, \( V(n, x(n)) \leq c_1(\eta) + (K_0 - i)a \) holds for any \( n \geq n_2 \), which implies that \( V(n, x(n)) \leq c_1(\eta) + (K_0 - i)a \) holds for any \( n \geq M_{i-1}^{(i-1)} \). Thus, (13) holds for the case \( i \). Hence, by the induction principle, (13) holds for all \( i = 1, 2, \ldots, K_0 \).

Let \( K = M_{K_0}^{(K_0)} \), then, for any \( n \geq K \), it follows from (13) that \( V(n, x(n)) \leq c_1(\eta) \). Thus, (12) holds and hence (11) holds. The proof is complete. \( \square \)

**Remark 3.1.** It should be noticed that, in *Zhang and Chen* (1998), condition (ii)' is changed into:

\( \ast \) \( V(n + s, x(n + s)) \leq p(V(n, x(n)))^{\ast} \)

\( V(n, x(n)) \leq V(n - 1, x(n - 1)) - q(||x(n)||) \).

Hence, Razumikhin-type asymptotic stability theorem is backward, i.e., comparing between \( V(n, x(n)) \) and its backward item \( V(n - 1, x(n - 1)) \) after having compared \( V(n, x(n)) \) and \( V(n - 1, x(n - 1)), \ldots, V(n - m, x(n - m)). \) Thus, condition (\( \ast \)) puts one restriction on \( V(\cdot) \) or \( p(\cdot) \) and \( q(\cdot) \):
\[
V(n, x(n)) \leq p(V(n, x(n))) - q(||x(n)|| \} \) or \( p(s) \geq s + q(c_1(s)) \) for all \( s \in R^7 \). In Theorem 3.2, it is forward, i.e., comparing between \( V(n, x(n)) \) and its forward item \( V(n + 1, x(n + 1)), \) and hence there is no such kind of restriction on \( V(\cdot) \) or \( p(\cdot) \) and \( q(\cdot) \). In addition, in *Zhang and Chen* (1998), functions \( p(\cdot) \) and \( q(\cdot) \) need to satisfy other restrictive conditions: \( p(\cdot) \) is needed to be a non-decreasing function and \( q(\cdot) \in K \). Here, in Theorem 3.2, it is not necessary for them. Hence, Theorem 3.2 is forward Razumikhin-type theorem being less restrictive than that in *Zhang and Chen* (1998).

**Theorem 3.3.** Assume that there exist a positive definite function \( V(n, x) \) and constants \( r > 0, p > 1, c_1 > 0, c_2 > 0, 1 > \lambda > 0 \) such that the following conditions hold:

(i) \( c_1 ||x||^2 \leq V(n, x) \leq c_2 ||x||^2 \);

(ii) for any \( \phi \in C([-m, 0], R^n) \) and \( s \in N_{-m}, \) then \( V(n + s, \phi(s)) \leq pV(n, \phi(0)) \) implies \( V(n + 1, f(n, \phi)) \leq \lambda V(n, \phi(0)) \);

(iii) for any \( \phi \in C([-m, 0], R^n) \) and for some \( s \in N_{-m} \), then \( V(n + 1, f(n, \phi)) \leq (1/p) \max_{s \in N_{-m}} V(n + s, \phi(s)), \) where \( p = \min(\ln(1/\lambda), (\ln p)/(m + 1)) \).

Then, system (1) is UES and its Lyapunov exponent should not be greater than \(-\lambda / r\).

**Proof.** Let \( x(n) = x(n, n_0, \phi) \) be a solution of system (1). Just as in Theorem 3.2, without loss of generality, let \( n_0 = 0 \). Define
\[
U(n) = \max_{\theta \in N_{-m}} \{e^{(n+\theta)}V(n, \theta, x(n + \theta))\}, \quad n \in N.
\]
In the following, we prove that
\[
U(n + 1) \leq U(n), \quad n \in N. \tag{24}
\]
For every fixed \( n \in N \), we define
\[
\tilde{\theta}_n = \max\{\theta \in N_{-m} : e^{(n+\theta)}V(n, \theta, x(n + \theta)) = U(n)\}.
\]
Then, we have \( U(n) = e^{(n+\tilde{\theta}_n)}V(n, \tilde{\theta}_n, x(n + \tilde{\theta}_n)) \). When \( \tilde{\theta}_n \leq -1 \), then, for any \( \theta \in N_{-m} \), \( e^{(n+\theta+\theta)}V(n + 1 + \theta, x(n + 1 + \theta)) \leq e^{(n+\theta)}V(n, \theta, x(n + \theta)) \), which implies that
\[
\max_{\theta \in N_{-m} \cap [0]} \{e^{(n+\theta+\theta)}V(n + 1 + \theta, x(n + 1 + \theta))\} \leq U(n). \tag{25}
\]
We claim that
\[
e^{(n+1)}V(n + 1, x(n + 1)) \leq e^{(n+\tilde{\theta}_n)}V(n + \tilde{\theta}_n, x(n + \tilde{\theta}_n)). \tag{26}
\]
By the definition of \( \tilde{\theta}_n \), we get \( e^{(n+\tilde{\theta}_n)}V(n + \tilde{\theta}_n, x(n + \tilde{\theta}_n)) > e^{(n)}V(n, x(n)) \), which implies that \( V(n + \tilde{\theta}_n, x(n + \tilde{\theta}_n)) > e^{-\tilde{\theta}_n}V(n, x(n)) \geq e^{(n)}V(n, x(n)) \). It follows from condition (iii) and \( \tilde{\theta}_n \leq -1 \) that
\[
\max_{s \in N_{-m}} \{V(n + s, x(n + s))\} \geq pV(n + 1, x(n + 1)) \geq e^{(n+1)}V(n + 1, x(n + 1)). \tag{27}
\]
Thus, by (27), we have
\[
e^{2(n+1)}V(n+1, x(n+1)) \\
\leq e^{-2m}e^{2n} \max_{s \in N_{-m}} \{ V(n+s, x(n+s)) \} \\
\leq \max_{s \in N_{-m}} \{ e^{2(n-m)}V(n+s, x(n+s)) \} \\
\leq e^{2(n+\tilde{\theta}_n)}V(n+\tilde{\theta}_n, x(n+\tilde{\theta}_n)). \tag{28}
\]
Hence, (26) holds and hence from (25) and (26), we get that \( U(n+1) \leq U(n) \), for \( \tilde{\theta}_n \leq -1 \).

When \( \tilde{\theta}_n = 0 \), then, for any \( \theta \in N_{-m} \),
\[
e^{x(n+\theta)}V(n+\theta, x(n+\theta)) \leq e^{2n}V(n, x(n)),
\]
which implies that \( V(n+\theta, x(n+\theta)) \leq e^{2n}V(n, x(n)) \). Thus, by condition (ii), we get
\[
V(n+1, x(n+1)) \leq 2V(n, x(n)). \tag{29}
\]

It follows from (29) that
\[
e^{2(n+1)}V(n+1, x(n+1)) \leq e^{2n}e^{2}V(n, x(n)) \\
\leq e^{2n}V(n, x(n)) = U(n). \tag{30}
\]

Moreover, by \( \tilde{\theta}_n = 0 \), we have, for \( \theta \in N_{-m} \), - (0),
\[
e^{x(n+1+\theta)}V(n+1+\theta, x(n+1+\theta)) \leq U(n). \tag{31}
\]

Hence, we obtain that \( U(n+1) \leq U(n) \), for \( \tilde{\theta}_n = 0 \). Therefore, (24) holds and it leads to, for any \( n \in N \),
\[
U(n) \leq U(0) \leq \max_{\theta \in N_{-m}} \{ V(\theta, x(\theta)) \}. \tag{32}
\]

Thus, by the definition of \( U(n) \) and (32), we get that
\[
V(n, x(n)) \leq e^{-2n} \max_{\theta \in N_{-m}} \{ V(\theta, x(\theta)) \}. \tag{33}
\]

It follows from (33) and condition (i) that
\[
\|x(n)\| \leq \left( \frac{c_2}{c_1} \right)^{1/r} e^{-(s/r)n} \|\phi\|_{m^n}, \quad n \in N. \tag{34}
\]

Thus, the conclusion of the theorem is true. \( \square \)

**Remark 3.2.** (1*) Theorem 3.3 can easily be used to test exponential stability. Moreover, for those practical systems which have high requirement on convergence speed such as synchronization of network controlled system (Liu, Liu, Chen, & Wang, 2005), the exponential stability is more significant than stability or asymptotic stability. Hence, Theorem 3.3 may be of more significance in practical applications than Theorems 3.1 and 3.2.

(2*) The convergence speed is estimated in Theorem 3.3. Here, \( \alpha = \min \{ \ln(1/\tilde{\lambda}), (\ln p)/(m+1) \} \) is the main factor that affects heavily the convergence speed. From here, one can see that if the maximum time delay \( m \) is sufficiently large such that \( \alpha = (\ln p)/(m+1) \), then the larger \( m \) leads to the slower convergence speed.

4. Application to a class of discrete delay systems

In this section, we focus our attention on a special class of discrete delay systems. This class of discrete delay systems is often encountered in practical applications such as network control systems with time delays, see Zhigoglyadov and Middleton (2003) and Lu and Chen (2004).

Consider the discrete delay systems of the form
\[
\{ x(n+1) = f(n, x(n), x(n-h_1), \ldots, x(n-h_{m_0})), \quad x_0 = \phi, \}
\]
where \( f \in C(N \times R^n \times \cdots \times R^n, R^n) \), and \( h_i(n) \in \{1, 2, \ldots, m_i\} \), for any \( n \in N \), and \( i = 1, 2, \ldots, m_0 \).

**Theorem 4.1.** Assume that condition (i) of Theorem 3.3 holds, while conditions (ii)–(iii) of Theorem 3.3 are replaced by the following condition (ii)*:

(ii)* there exist positive constants \( 0 < \tilde{\lambda} < 1, 0 < \tilde{\lambda}_i < 1, i = 1, 2, \ldots, m_0 \), such that \( V(n+1, x(n+1)) \leq \tilde{\lambda}V(n, x(n)) + \sum_{i=1}^{m_0} \tilde{\lambda}_i V(n-h_i(n), x(n-h_i(n))) \).

If \( \tilde{\lambda} + \sum_{i=1}^{m_0} \tilde{\lambda}_i < 1 \), then system (35) is UES and its Lyapunov exponent is less than or equal to \( -\ln p/(m+1) \), where \( p > 1 \) is the unique root of the equation
\[
1 - \tilde{\lambda} - p \sum_{i=1}^{m_0} \tilde{\lambda}_i = \frac{\ln p}{m+1}. \tag{36}
\]

**Proof.** If \( 1 - \tilde{\lambda} > \sum_{i=1}^{m_0} \tilde{\lambda}_i \), then we see that Eq. (36) has a unique root satisfying \( 1 < p < 1 - \tilde{\lambda}/\sum_{i=1}^{m_0} \tilde{\lambda}_i \). Thus, for any \( n \in N_{-m} \), if \( V(n-h_i(n), x(n-h_i(n))) \leq pV(n, x(n)) \), then, by (ii)*, we have
\[
V(n+1, x(n+1)) \leq \left( \tilde{\lambda} + p \sum_{i=1}^{m_0} \tilde{\lambda}_i \right) V(n, x(n)).
\]

It follows from the fact \( \tilde{\lambda} + p \sum_{i=1}^{m_0} \tilde{\lambda}_i < 1 \) that the condition (ii) of Theorem 3.3 is satisfied.

Denote \( \tilde{\lambda} = \tilde{\lambda} + p \sum_{i=1}^{m_0} \tilde{\lambda}_i \). From (36), we get that \( \ln p/(m+1) = 1 - \tilde{\lambda} \). It is easy to see from the properties of continuous function that for any \( 0 < \tilde{\lambda} < 1 \), we have \( 1 - \tilde{\lambda} < -\ln \tilde{\lambda} \), which implies that \( \ln p/(m+1) < -\ln \tilde{\lambda} \) holds. Hence, \( \alpha = \min \{ \ln(1/\tilde{\lambda}), (\ln p)/(m+1) \} \) is the main factor that affects heavily the convergence speed. From here, one can see that if the maximum time delay \( m \) is sufficiently large such that \( \alpha = (\ln p)/(m+1) \), then the larger \( m \) leads to the slower convergence speed.

\[
\dot{c}e^{-\alpha} + \sum_{j=1}^{m_0} \tilde{\lambda}_j < \frac{-\tilde{\lambda}}{p} + \frac{-\ln p}{p(m+1)} < \frac{1}{p}. \tag{38}
\]
Thus, if for some \(-h_i(n) \in N_{-m} = \{0\}, V(n - h_i(n), x(n - h_i(n))) > e^2V(n, x(n)),\) then, by condition (ii)* and (38), it follows that \(V(n + 1, x(n + 1)) \leq (e^{-\lambda} + \sum_{j=1}^{m_0} \lambda_j) \max_{1 \leq j \leq m_0} \{V(n - h_j(n), x(n - h_j(n)))\} \leq (1/p)\bar{V}(n)\).

Therefore, the condition (iii) of Theorem 3.3 also holds. Thus, the conclusion of the theorem is true. \(\square\)

**Theorem 4.2.** Assume that condition (i) of Theorem 3.3 holds, while conditions (ii)–(iii) of Theorem 3.3 are replaced by the following condition:

(ii)** there exists a positive constant \(0 < \lambda < 1\) such that \(V(n + 1, x(n + 1)) \leq \lambda\bar{V}(n)\).

Then, system (35) is UES and its Lyapunov exponent is less than or equal to \(\log \lambda/r(m + 2)\).

**Proof.** Let \(p = (1/\lambda)(m+1)/(m+2)\), then, from \(\lambda < 1\), we get that \(1 < p < 1/\lambda\), and \(\log p/(m + 1) = \log 1/\lambda p\). Thus, for any \(-h_i(n) \in N_{-m}\), if \(V(n - h_i(n), x(n - h_i(n))) \leq pV(n, x(n))\), then, by (ii)**, we have \(V(n + 1, x(n + 1)) \leq \lambda\bar{V}(n) \leq \lambda pV(n, x(n))\). It follows from the fact \(\lambda p < 1\) that the condition (ii) of Theorem 3.3 is satisfied.

Let \(z = \log p/(m + 1) = \log 1/\lambda p\). If for some \(-h_i(n) \in N_{-m}\), \(V(n - h_i(n), x(n - h_i(n))) > e^2V(n, x(n))\), then, by condition (ii)**, it follows that \(V(n + 1, x(n + 1)) \leq \lambda\bar{V}(n) < (1/p)\bar{V}(n)\). Hence, the condition (iii) of Theorem 3.3 also holds. Thus, the conclusion of the theorem is true. \(\square\)

**Remark 4.1.** Theorem 4.1 can be seen as the counterpart of the Halanay-type inequality on continuous delay systems (Niculescu, 2001). And Theorem 4.2 is a well-known stability result for discrete delay system (Liao, 2001). But its proof in Liao (2001) is complicated. Here, by using Theorem 3.3, we can easily derive the result. Moreover, the Lyapunov exponent can also be estimated. Furthermore, Theorem 4.1 can be derived from Theorem 4.2.

## 5. Examples

To illustrate our theorems obtained in the previous section, we now consider some illustrative examples.

**Example 5.1.** Consider the discrete delay system:

\[
\begin{align*}
&x(n + 1) = A x(n) + F(n, x(n), x(n - h(n))), \\
&x_0 = \phi,
\end{align*}
\]

where \(m = 2, h(n) = 1\), or 2, \(A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0.1 & 0.5 \end{pmatrix}\) and \(F(n, x(n), x(n - h(n))) = \frac{1}{2}((x_1(n - h(n)) + (1 + \sin^2 n + \|x(n)\|^2), x_2(n - h(n)) \sin(x_3(n)), x_3(n - h(n)) \cos(x_3(n)))^T\).

Let \(V(n, x) = x^T x\), then \(V(n + 1, x(n + 1)) \leq 0.8670 V(n, x(n)) + 0.0973 V(n - h(n), x(n - h(n))),\) which implies that conditions of Theorem 4.1 hold. Thus, by Theorem 4.1, system (39) is UES with its Lyapunov exponent less than or equal to \(-0.0137\). The numerical simulation is given in Fig. 1.

**Example 5.2.** Consider the discrete delay system in the form of

\[
x(n + 1) = A(n)x(n) + \sum_{j=1}^{m_0} B_j(n)x(n - h_j(n)),
\]

where \(x \in \mathbb{R}^n, A(n), B_j(n) \in \mathbb{R}^n\) and \(-h_j(n) \in N_{-m}\). Let \(A^* = \sup_{n \in \mathbb{N}} \|A(n)\|, B_j^* = \sup_{n \in \mathbb{N}} \|B_j(n)\|\), \(j = 1, 2, \ldots, m_0\). Denote \(\frac{\lambda^*}{A^*} + B_1^* + B_2^* + \cdots + B_{m_0}^*\). Let Lyapunov function be \(V(n, x(n)) = \|x(n)\|\), then it is easy to get that \(V(n + 1, x(n + 1)) \leq \lambda\bar{V}(n, x(n))\). Hence, if \(\lambda < 1\), then by Theorem 4.2, we obtain that system (40) is UES with its Lyapunov exponent less than or equal to \(\log \lambda/m + 1\).

**Remark 5.1.** Example 5.2 is discussed in Zhang and Chen (1998). However, the UAS of system (40) is not easily tested by the results in Zhang and Chen (1998). Here, not only the UAS, but also the UES of system (40) can be easily derived and the convergence speed also can be estimated.

## 6. Conclusions

In this paper, Razumikhin-type stability theorems have been investigated for the general discrete delay systems. By using the Razumikhin technique and Lyapunov function, forward Razumikhin-type theorems of uniform stability, uniformly asymptotic stability and uniformly exponential stability are derived, respectively. Moreover, Razumikhin-type uniformly exponential stability theorem gives the estimation of the convergence speed, in which the time delay is one of the main factors that affects the convergence speed. As theoretical application, the Razumikhin-type uniformly exponential stability result is used to derive some well-known stability results for some classical discrete delay systems. Finally, some relevant examples have
been solved so as to illustrate the theoretical results obtained in this paper.

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References


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