Upper Bounds for Induced Operator Norms of Nonlinear Systems

Vahid Zahedzadeh, Student Member, IEEE,
Horacio Jose Marquez, Senior Member, IEEE, and
Tongwen Chen, Fellow, IEEE

Abstract—In this technical note, new methods are proposed to compute upper bounds on the \(L_p\), \(L_\infty\) and \(L_{\infty,\infty}\) induced operator norms of continuous-time nonlinear systems. These methods are based on the so-called \(\zeta_A\) representation of nonlinear systems, which was introduced earlier by the authors. Examples are provided in order to show the applicability of our approach. Moreover, a weighting technique is suggested to improve the upper bounds.

Index Terms—Continuous-time systems, induced operator norms, nonlinear systems.

I. INTRODUCTION

The complex structure of nonlinear systems is the major obstacle in the development of simple and efficient computational methods to test stability, compute system norms, etc. As a consequence, the majority of the computational techniques available in the literature are restricted to a narrow class of nonlinear systems for which a particular function, e.g., Lyapunov function or storage function, can be found by trial and error [4], [5].

In this technical note we consider the problem of computing the \(L_p\) operator norm of a nonlinear system, a problem which has remained a challenge in the systems literature. The importance of this problem originates from the fact that the influence of various inputs on the various signals inside the system can be quantified by such a measure. One of the applications of this measure is in control systems, where the attenuation of disturbance signals is required. The subject has attracted considerable attention for both linear and nonlinear systems. For linear systems, an upper bound on the \(\|\cdot\|_{\infty}\) has a well established solution; see, for example, [1]. For nonlinear systems, however, computation of the \(L_p\) operator norm continues to be a challenge. In [2], the \(L_{\infty,\infty}\) gain of nonlinear systems is characterized by means of the value function of an associated variational problem. The \(L_p\) gain, also referred to as \(\mathcal{H}_\infty\) gain of a nonlinear system, can be approximated using storage functions and the theory of dissipative systems. This approach is, however, conservative and finding storage functions is difficult; see also [3] for a numerical approximation of the \(\mathcal{H}_\infty\) norm. In [7], a computational method is proposed to compute the \(L_p\) induced norm for single-input linear systems with saturation.

In this technical note, we propose a method to compute an upper bound on the \(L_1\), \(L_2\) and \(L_{\infty,\infty}\) norms of a class of continuous-time nonlinear systems. Our method can be optimized based on some selected parameters. For systems not included in this class, a method is also provided for computing an upper bound of the \(L_{\infty,\infty}\) norm.

The remainder of the technical note is organized as follows: In Section II, we introduce the notation and present some preliminary results. In Section III, we propose a method to compute upper bounds on the induced norm of nonlinear systems and provide two illustrative examples. In Section IV, we introduce the weighting method, which can be used to reduce the intrinsic conservatism in the aforementioned method. An example is provided to illustrate the usage of the weighting technique.

II. NOTATION AND PRELIMINARIES

A. Notation

Let \(\mathbb{R}\) and \(\mathbb{Q}\) denote the fields of real and complex numbers, respectively. \(\mathbb{R}^n\) denotes the space of \(n \times 1\) real vectors. The \(q\)-norm in \(\mathbb{R}^n\) is denoted by \(\|\cdot\|_q\), i.e., \(\|x\|_q := \max \{ |x_1|^q, |x_2|^q, \ldots, |x_n|^q \}\), for \(q \leq \infty\). For \(q < \infty\), \(\|x\|_q := \left( \sum |x_i|^q \right)^{1/q}\). Let \(\mathbb{B}^n(r)\) denote the open ball with center \(c\) and radius \(r\), i.e., \(\mathbb{B}^n(c, r) := \{ x : \| x - c \|_2 < r \}\). For function \(F : \mathbb{R}^n \to \mathbb{R}^m\), we denote the gain by \(\gamma_q\) and define it as \(\gamma_q(F) := \sup_{x \in \mathbb{B}^n(1)} \| F(x) \|_q / \| x \|_q\). We also define regional gain as \(\gamma_q^r(F) := \sup_{x \in \mathbb{B}^n(r)} \| F(x) \|_q / \| x \|_q\), where \(D\) is an open ball in \(\mathbb{R}^n\). \(\mathcal{L}_p\) denotes Lebesgue p-space of n-vector valued functions on \([0, \infty)\), with norm defined as \(\| f \|_p := \left( \int_0^\infty \| f(t) \|_p^p \, dt \right)^{1/p}\) for \(1 \leq p < \infty\) and \(\| f \|_\infty := \esssup_{t \geq 0} \| f(t) \|_p\). In the aforementioned norm definitions, \(p\) and \(q\) are called the temporal and spatial norms, respectively. It is important to note that the definition of \(\mathcal{L}_p\) is independent of the spatial norm. Usually \(n\) is a finite integer. To simplify our notation we drop \(n\) and write \(\mathcal{L}_p\) instead of \(\mathcal{L}_p^n\).

To distinguish among various norm notation, we indicate the space as a subscript for the norm, such as \(\| f \|_{\mathcal{L}_p}\) or \(\| f \|_{\mathcal{L}_2}\). Whenever the space is not mentioned, norms with \(t\) argument denote the vector norm at \(t\) and without \(t\) denote the \(\mathcal{L}_p\) norm. Let \(\mathcal{T}\), denote the truncation operator: for \(f(t), 0 \leq t < \infty\), \(\mathcal{T}(f) := f(t)\) on \([0, T]\), and zero otherwise. We also denote the truncation of \(f(t)\) by \(f_r(t) := \mathcal{T}(f)(t)\). If the signal space can be clearly be understood from the context, we denote the norm of a truncated signal by \(\| f \|_r\), i.e., \(\| f \|_r := \| f_r \|_p\).

Let \(\mathcal{U} := \mathcal{L}_p\) and \(\mathcal{Y} := \mathcal{L}_q\) denote input and output signal spaces, respectively. A nonlinear time-varying system can be thought of as a possibly unbounded operator \(H : \mathcal{D}_h \to \mathcal{Y}\) where \(\mathcal{D}_h \subseteq \mathcal{U}\). The action of \(H\) on any \(u \in \mathcal{D}_h\) is denoted by \(H u\). A system \(H\) is called stable if \(\mathcal{D}_h = \mathcal{U}\). For an operator \(H : \mathcal{U} \to \mathcal{Y}\), let \(\gamma(H)\) stand for the induced norm (gain) of the operator defined as

\[ \gamma(H) := \sup_{u \neq 0} \frac{\| H u \|_r}{\| u \|_r} \]

where the supremum is taken over all \(u \in \mathcal{U}\) and all \(T \in \mathbb{R}^+\) for which \(u_T \neq 0\). Let \(\gamma_p(H)\) stand for \(\gamma(H)\) in \(\mathcal{L}_p\).

B. \(\zeta_A\) and \(\zeta_{AB}\) Representations

Our proposed method to compute the aforementioned norms is based on \(\zeta_A\) and \(\zeta_{AB}\) representations of nonlinear systems, which have recently been introduced in [8]. In this section, we briefly explain the \(\zeta_A\) and \(\zeta_{AB}\) representations; see [8] for further details.

Assume that the nonlinear system of interest, \(N\), is

\[ N : \dot{x}(t) = f(t, x(t)) \]

where \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\) is locally Lipschitz [4], [5]. We also assume that the initial condition of the system is finite. Let \(A \in \mathbb{R}^{n \times n}\),

\[ A := \begin{bmatrix} 0 & -I \end{bmatrix} \]

where \(I\) is the identity matrix.
and \( \Gamma \) and \( \Omega \) be defined in the same formulas as in (4). The nonlinear system is equivalent to the structure represented in Fig. 2(a). This representation of the nonlinear system is called the \( \zeta_{AB} \) representation with ordered operator set \([\Phi, \Gamma, \Omega]\) [8].

It is important to note that \[ \begin{bmatrix} A & I \\ I & 0 \end{bmatrix} \] and \[ \begin{bmatrix} A & I \\ I & 0 \end{bmatrix} \] are state-space realizations for \( \Gamma \) and \( \Theta \), respectively. Since \( \Lambda \) and \( \Gamma \) are chosen arbitrary, \( \zeta_{AB} \) and \( \zeta_{AB} \) representations are not unique. A useful choice for the \( \zeta_{AB} \) representation is \( B = 0 \), which implies \( \theta = 0 \) and simplifies the \( \zeta_{AB} \) structure as the structure shown in Fig. 2(b). For forced systems, this representation is also called \( \zeta_{A} \) representation.

### III. Upper Bounds for the Induced Operator Norms

**A. Proposed Method**

In this section, we obtain a computable upper bound for induced operator norms. We will use the structure shown in Fig. 2(b); namely, the \( \zeta_{A} \) representation for forced system. In this structure, it is trivial to show that

\[
||x||_{\ell_p} \leq ||u||_{\ell_p} + ||d||_{\ell_p} \\
\leq \gamma_p(\Gamma) \gamma_p(\Phi)||x||_{\ell_p} + ||d||_{\ell_p} \\
\leq \gamma_p(\Gamma) \gamma_p(\Phi)||x||_{\ell_p} + \gamma_p(\Theta)||x_0||. \tag{8}
\]

The computation of \( \gamma_p(\Gamma), \gamma_p(\Theta) \) and \( \gamma_p(\Phi) \) was discussed in [8].

**Lemma 3.1:** The following equation is true for \( x, u \in \mathcal{L}_2 \):

\[
\|x\|_{\ell_2} \leq \gamma_p(\Theta)||x_0|| \tag{9}
\]

Moreover, if \( x, u \in \mathcal{L}_2 \)

\[
\|x\|_{\ell_2} \leq \|x\|_{\ell_2} + \|u\|_{\ell_2}. \tag{10}
\]

**Proof:** The proof is trivial and is omitted. \( \square \)

The first part of this lemma, (9), is true for all Banach spaces; however, the second part is true when the temporal norm is \( \mathcal{L}_2 \) with the Euclidean 2-norm chosen as the corresponding spatial norm.

**Theorem 3.1:** Let \([\Phi, \Gamma, \Omega] \) be a \( \zeta_{A} \) representation for a forced system, \( N \). If

\[
\gamma_p(\Gamma) \gamma_p(\Phi) < 1 \tag{11}
\]

then

\[
\gamma_p(\Theta) \leq \frac{\gamma_p(\Gamma) \gamma_p(\Phi)}{1 - \gamma_p(\Gamma) \gamma_p(\Phi)}. \tag{12}
\]

**Proof:** Substituting (9) in (8) implies that

\[
\|x\|_{\ell_2} \leq \gamma_p(\Gamma) \gamma_p(\Phi) \|x\|_{\ell_2} + \gamma_p(\Theta) \|x_0\|. \tag{13}
\]

Thus

\[
(1 - \gamma_p(\Gamma) \gamma_p(\Phi)) \|x\|_{\ell_2} \leq \gamma_p(\Gamma) \gamma_p(\Phi) \|u\|_{\ell_2} + \gamma_p(\Theta) \|x_0\|. \tag{14}
\]
Since $\gamma_p(\Gamma \gamma_p(\Phi) < 1$,
\begin{equation}
||x|| \leq \frac{\gamma_p(\Gamma \gamma_p(\Phi)}{1 - \gamma_p(\Gamma \gamma_p(\Phi))} ||u|| + \frac{\gamma_p(\Omega)}{1 - \gamma_p(\Gamma \gamma_p(\Phi))} ||x_0||
\end{equation}
which implies (12).

Inequality (12) can be used as an upper bound for the $\mathcal{L}_p$ induced norm. It is important to note that since the $\zeta_s$ representation is not unique, the solution of the following minimization problem is the lowest upper bound that can be obtained by our method:

\begin{equation}
\gamma_p(N) \leq \min_a \frac{\gamma_p(\Gamma \gamma_p(\Phi)}{1 - \gamma_p(\Gamma \gamma_p(\Phi))}.
\end{equation}

where $\Gamma(s) = \begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}$ and $\Phi(x, u) = f(x, u) - Ax$. Unfortunately, there is no existing method to find $\Gamma$ which provides the lowest upper bound. A good strategy is to start with choosing $A$ as the linearity of $f$ and decrease the upper bound by adding or subtracting terms from each entry of $A$ by trial and error.

The method provided by Theorem 3.1 is general in the sense of the induced norm, $\gamma_p$. An interesting case occurs when the temporal norm is $\mathcal{L}_2$ with the Euclidean 2-norm chosen as the corresponding spatial norm. The reason is that a quite mature theory, namely, $\mathcal{H}_\infty$ optimization, has been developed for linear systems in this case. Suppose $\Gamma$ is a continuous-time linear time-invariant stable operator with impulse response $g(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^{m \times n}(g(t) : \mathbb{Z}^+ \rightarrow \mathbb{R}^{m \times n})$. Let $G(s)$ denote the Laplace transform of $g(t)$. We have

\begin{equation}
\gamma_2(\Gamma) := ||G(s)||_{\mathcal{H}_\infty}.
\end{equation}

In this case, the following theorem provides lower upper bounds for the induced norm $\gamma_2$ than Theorem 3.1.

**Theorem 3.2:** Let $[\Phi, \Gamma, \Theta]$ be a $\zeta_s$ representation for a forced system, $N$. If $\gamma_2(\Gamma \gamma_2(\Phi) < 1$ then

\begin{equation}
\gamma_2(N) \leq \frac{\gamma_2(\Gamma \gamma_2(\Phi)}{1 - \gamma_2(\Gamma \gamma_2(\Phi))}.
\end{equation}

**Proof:** Inequality (8) implies that

\begin{equation}
(||x|| - \gamma_2(\Omega)||x_0||)^2 \leq \left( \gamma_2(\Gamma \gamma_2(\Phi) \left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2.
\end{equation}

Using (10)

\begin{equation}
||x||^2 - 2\gamma_2(\Omega)||x_0||^2 + \gamma_2(\Omega)^2||x_0||^2 \leq \gamma_2(\Gamma \gamma_2(\Phi))^2 \left( ||x||^2 + ||u||^2 \right).
\end{equation}

For simplicity, let $\alpha := \gamma_2(\Gamma \gamma_2(\Phi)$

\begin{equation}
||x||^2 - 2\gamma_2(\Omega)\alpha ||x_0||^2 + \gamma_2(\Omega)^2\alpha ||x_0||^2 \leq \frac{\alpha^2}{1 - \alpha^2} ||u||^2.
\end{equation}

Hence

\begin{equation}
\left( ||x|| - \gamma_2(\Omega)\alpha ||x_0|| \right)^2 \leq \frac{\alpha^2}{1 - \alpha^2} ||x_0||^2 + \frac{\alpha^2}{1 - \alpha^2} ||u||^2.
\end{equation}

Since $\alpha^2 + \beta^2 \leq (a + b)^2$ for all $a, b \geq 0$, we have

\begin{equation}
||x|| - \gamma_2(\Omega)\alpha ||x_0|| \leq \frac{\alpha \gamma_2(\Omega)}{1 - \alpha^2} ||x_0|| + \frac{\alpha}{\sqrt{1 - \alpha^2}} ||u||.
\end{equation}

Consequently

\begin{equation}
||x|| \leq \frac{\gamma_2(\Gamma \gamma_2(\Phi)}{1 - \gamma_2(\Gamma \gamma_2(\Phi))^2} ||u|| + \frac{\gamma_2(\Omega)}{1 - \gamma_2(\Gamma \gamma_2(\Phi))} ||x_0||
\end{equation}
which implies (18).

Similarly, the solution of the following minimization problem is the lowest upper bound that can be obtained by our method:

\begin{equation}
\gamma_2(N) \leq \min_a \frac{\gamma_2(\Gamma \gamma_2(\Phi)}{1 - \gamma_2(\Gamma \gamma_2(\Phi)^2}.
\end{equation}

\begin{example} (RLC Circuit With Non-Ideal Inductor): The network of Fig. 3 represents a RLC circuit with a non-ideal inductor. The inductor has nonzero resistance and saturation characteristic as shown in Fig. 4(a), where $\lambda$ is the flux linkage. The relationship of the magnetic flux linkage to terminal voltage of an inductor is given by Faraday’s law; namely $v_L(t) = d(\lambda(t))/dt$. The state equations for this network may be written as

\begin{equation}
v_L = \lambda = \frac{d\lambda}{dt} = \frac{dL}{dt},
\end{equation}

\begin{equation}
\frac{di_L}{dt} = \frac{dL}{dt} = \frac{dV_C}{dt} = \frac{dC}{dt}.
\end{equation}

Defining $x_1 := i_L$, $x_2 := v_C$ and $u := i$

\begin{equation}
x_1 = (x_2 - R_2x_1) (\frac{d\lambda}{d\lambda})^{-1},
\end{equation}

\begin{equation}
x_2 = \frac{\tau}{\tau} - \frac{x_2}{\tau} = \frac{\tau}{\tau}.
\end{equation}

Let $R_1 = 1/2, R_2 = 1$ and $C = 2$. Assuming $A = \begin{bmatrix} -1 & 0.5 \\ -0.5 & -1 \end{bmatrix}$, we have

\begin{equation}
\Phi(x_1, x_2, u) = \begin{bmatrix} x_1 - 0.5x_2 + (x_2 - x_1) (\frac{d\lambda}{d\lambda})^{-1} \end{bmatrix}.
\end{equation}

We use the computational methods that has been introduced in [8]. Since there are three independent variables in $\gamma_p(\Phi)$, i.e. $x_1, x_2$ and $u$, we plot $||\Phi(x, u)||/||x||$ versus $||x||$ instead of plotting versus $x_1, x_2$ and $u$. As shown in Fig. 5, $\gamma_1(\Phi) \approx 0.50$, $\gamma_2(\Phi) \approx 0.50$ and $\gamma_\infty(\Phi) \approx 0.50$. Computation also shows that $\gamma_1(\Gamma) \approx 1.237$, $\gamma_2(\Gamma) \approx 1.00$ and $\gamma_\infty(\Gamma) \approx 1.237$. Theorems 3.1 and 3.2 imply that $\gamma_1(N) \leq 1.62, \gamma_2(N) \leq 0.577$ and $\gamma_\infty(N) \leq 1.62$, respectively.

There is no doubt that the condition $\gamma_p(\Gamma \gamma_p(\Phi) < 1$ in Theorems 3.1 and 3.2 is restrictive. For example, polynomial systems are excluded by the aforementioned condition. The following theorem might be used to overcome this shortcoming. The result provides an upper bound on system output for bounded input and initial state.

**Theorem 3.3:** Let $[\Phi, \Theta, \Gamma, \Omega]$ be a $\zeta_{BH}$ representation for a nonlinear system. Let $\eta > 0$ and $M_\eta > \gamma_\infty(\Omega) + \gamma_\infty(\theta)$ and

\begin{equation}
\dot{\hat{\eta}} := \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m}, \hat{\gamma}_{\eta}(\Phi) \leq \frac{M_\eta - \gamma_\infty(\Omega) - \eta \gamma_\infty(\theta)}{(M_\eta + \eta) \gamma_\infty(\Gamma)}.
\end{equation}
Let $\mathcal{D} := B^\infty(0, r_D)$ be an open ball inside $\hat{\mathcal{D}}$. Assume that $\mathcal{D}_x$ and $\mathcal{D}_u$ are the images of $\mathcal{D}$ under $F(m \times n) \to 0_{m \times n}$ and $0_{n \times m} \to I_{m \times m}$, respectively. Therefore, $\mathcal{D}_x$ and $\mathcal{D}_u$ are also open balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $r_x$ and $r_u$ denote respectively their radius, i.e. $\mathcal{D}_x = B^\infty(0, r_x)$ and $\mathcal{D}_u = B^\infty(0, r_u)$. Choose $\epsilon$ and $\delta$ such that $0 < \epsilon < r_x$ and

$$0 < \delta \leq \frac{1 - \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma)}{\gamma^\infty_\infty(\Omega) + \eta(\gamma^\infty_\infty(\Theta) + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma))}. $$

If $\|u\|_\infty < \min(\eta \delta, r_u)$ and $\|x_0\|_\infty \leq \delta$, then

$$\|x\|_\infty < \epsilon.$$  

(24)

**Proof:** It is trivial that $M_p - \gamma^\infty_\infty(\Omega) - \eta \gamma^\infty_\infty(\theta) < M_p + \eta$; therefore $\gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) < 1$. We use contradiction to prove the theorem.

$$\|x_r\| \leq \|d_1\| + \|u_r\|$$

$$\leq \|d_1\| + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \left(\|d_2\| + \|x_r\|\right)$$

$$\leq \gamma^\infty_\infty(\Omega) \|x_0\| + \gamma^\infty_\infty(\Theta) \|u_0\| + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \|x_R\|$$

$$+ \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \|u_R\|$$

$$\leq \gamma^\infty_\infty(\Omega) \|x_0\| + \left[ \gamma^\infty_\infty(\Theta) + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \right] \|x_R\|$$

$$+ \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \|u_R\|$$

$$\leq \gamma^\infty_\infty(\Omega) \|x_0\| + \left[ \gamma^\infty_\infty(\Theta) + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \right] \|x_R\|$$

$$+ \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \|u_R\|$$

$$= \left[ \gamma^\infty_\infty(\Omega) + \eta \left[ \gamma^\infty_\infty(\Theta) + \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \right] \right] \delta$$

$$+ \gamma^D_\infty(\Phi) \gamma^\infty_\infty(\Gamma) \|x_0\|.$$  

(25)
Then
\[
\begin{align*}
\epsilon &= \|x_t\| \\
&< \gamma_\infty(\Theta) + \eta \left[ \gamma_\infty(\Theta) + \gamma_\infty^P(\Phi) \gamma_\infty(\Gamma) \right] \|x_0\| \\
&\leq \frac{\gamma_\infty(\Theta) + \eta \gamma_\infty(\Theta) + \gamma_\infty^P(\Phi) \gamma_\infty(\Gamma)}{1 - \gamma_\infty^P(\Phi) \gamma_\infty(\Gamma)} \delta \\
&\leq \epsilon.
\end{align*}
\]
(26)

Which is a contradiction. Therefore, \(x(t) < \epsilon; \forall t \geq 0\), i.e., \(\|x\| < r_\epsilon\).

**Example 3.2:** Consider a multi-tank system depicted in Fig. 6(a). Suppose that a proportional controller is utilized to adjust the fluid level in the second tank \(H_2\) by input flow \(q\). The problem of interest is to find an upper bound on the gain of the closed-loop system shown in Fig. 6(b). The following mathematical model is taken from [6]:

The transfer function of the controller is

\[
\frac{\text{controller output}}{\text{input}} = \frac{1}{s^2 + \frac{1}{\omega_p^2}} (C_1 H_1^{**} - C_2 H_2^{**}).
\]

The state equations for the closed-loop system are

\[
\begin{align*}
\dot{x}_1 &= \frac{1}{\omega_p^2} (g_0 - K_p(x_2 + u) - C_1 (x_1 + H_1)^{**}) \\
\dot{x}_2 &= \frac{1}{\omega_p^2} (C_1 (x_1 + H_1)^{**} - C_2 H_2^*)
\end{align*}
\]

and \(x = (x_1, x_2) = f(x, u)\). Let \(A = \begin{pmatrix} -0.0072 & -0.0114 \\ 0.0094 & -0.0118 \end{pmatrix}\) and \(B = \begin{pmatrix} -0.0111 \\ 0 \end{pmatrix}\), which are linearized parts of \(f(x, u)\) at \(x = 0\) and \(u = 0\), i.e., \(A = \partial f(x, u)/\partial x|_{x=0, u=0}\) and \(B = \partial f(x, u)/\partial u|_{x=0, u=0}\). Therefore, \(\Phi(x, u) = [\Phi_1(x, u) \Phi_2(x, u)]\) where \(\Phi_1(x, u) = 0.00373 + 0.0072 x_1 - 0.00647 (x_1 + 0.15)^{0.25}\) and \(\Phi_2(x, u) = (5.66 \times 10^{-5} x_1 + 0.15)^{0.20} - 5.58 \times 10^{-5} (x_2 + 0.0934^{0.226}/(0.0067 + 0.0345 x_2) - 0.00948 + 0.01176 x_2).\) Computation with the methods proposed in [8] provides \(\gamma_\infty(\Gamma) < 151.3, \gamma_\infty(\Theta) < 0.9756,\) and \(\gamma_\infty(\Omega) = 1.036.\) Let \(\eta = 3.0382\) which gives \(M_P = 4 > \gamma_\infty(\Omega) + \eta \gamma_\infty(\Theta).\) Since \(\Phi\) is independent from \(u, \mathcal{D} \subset \mathbb{R}^2, \|\Phi(x)\|_{\infty}/\|x\|_{\infty}\) versus \(\|x\|_{\infty}\) is depicted in Fig. 7. Since \(\mathcal{D}\) is independent of \(u, r_u = \infty.\) Let us take \(\mathcal{D}\) as the region where \(\|\Phi(x)\|_{\infty}/\|x\|_{\infty}\) < 0.0023, i.e., \(\gamma_\infty^P(\Phi) = 0.0023.\) Consequently \(r_u = 0.0155.\) Let \(\epsilon = 0.015 \) and \(\delta = 0.0019 \leq (1 - \gamma_\infty^P(\Phi) \gamma_\infty^P(\Gamma))/\gamma_\infty(\Omega) + \eta (\gamma_\infty(\Theta) + \gamma_\infty^P(\Phi) \gamma_\infty(\Gamma)))\). According to Theorem 3.3, for any input \(u\) which satisfies \(\|u\|_{\infty} < \min(\eta, r_u) = 0.00587\) and any initial state satisfying \(\|x_0\|_{\infty} < \delta = 0.0019,\) \(x\) is bounded as \(\|x\|_{\infty} < \epsilon = 0.015.\)

**IV. Weighting Technique**

As shown in the previous section, the proposed methods are based on the \(\mathcal{C}_A\) representation. Adding some weighting on state or input vectors may tighten the calculated bounds. However, there is no general rule which provides useful weighting matrices; therefore, they should be chosen by trial and error. In this section, we study the effect of the weighting and we show the effectiveness by an example.

In the \(\mathcal{C}_A\) representation for continuous-time systems shown in Fig. 1, let \(\dot{x} := W_x x\) where \(W_x\) is nonsingular. Consequently

\[
\dot{x} = W_x^{-1} A x + W_x F (W_x^{-1} x) .
\]

Denoting \(\hat{x} := W_x^{-1} x, \hat{\Phi} := W_x F (W_x^{-1} x), \hat{\Gamma} := \begin{bmatrix} A & \frac{B}{X} \\ \frac{I}{X} & 0 \end{bmatrix}\) and \(\Omega(x(t)) := e^{\hat{\Gamma} t} x_0,\) it is easy to show that ordered operator set \([\hat{\Phi}, \hat{\Gamma}, \hat{\Omega}]\) is a \(\mathcal{C}_A\) representation for the weighted system, i.e., the system with initial state \(x_0 := W_x x_0\) and state \(\dot{x}\).

Similarly, in the \(\mathcal{C}_{AH}\) representation shown in Fig. 2(a) for continuous-time systems, let \(\dot{x} := W_x x\) and \(\dot{u} := W_x u\) where \(W_x \) and \(W_x\) are nonsingular. Consequently

\[
\dot{x} = W_x^{-1} A x + W_x B W_x^{-1} u + W_x F (W_x^{-1} x, W_x^{-1} u) .
\]
Denoting $\hat{A} := W_x A W_x^{-1}$, $\hat{B} := W_x B W_x^{-1}$, $\hat{\Phi}(x, u) := W_x \Phi(W_x^{-1} x, W_x^{-1} u)$, $\hat{\Gamma} := \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}$, $\hat{\Theta} := \begin{bmatrix} I & \hat{B} \end{bmatrix}$, and $\Omega(x(t)) := e^{\hat{\Gamma} t} x_0$, it is trivial to show that ordered operator set $[\hat{\Phi}, \hat{\Gamma}, \hat{\Theta}, \hat{\Omega}]$ is a $\zeta_{\hat{A}B}$ representation for the weighted system, i.e., the system with input $\hat{u}$, state $\hat{x}$ and initial state $\hat{x}_0$. A very similar argument can be made for forced system with $\zeta_{AB}$ representation.

It is important to note that the mapping $\hat{u} \to \hat{x}$ is different than $u \to x$. However, Theorems 3.1, 3.2 and 3.3 can be used to find corresponding upper bounds for the weighted system. Then, using the definitions of $\hat{x}$, $\hat{u}$ and $\hat{x}_0$, the corresponding bounds can be found for the main system. Suppose that the inequality found for the weighted system $\|\hat{x}\| \leq \gamma_{\hat{u}}\|\hat{u}\| + \gamma_{x_0}\|\hat{x}_0\|$ where $\gamma_{\hat{u}}, u$ and $\gamma_{x_0}$ are derived by either (15) or (19). Therefore

$$\|x\| \leq \|W_x\| \|\hat{x}\| \leq \|W_x\| \|\gamma_u\|\|u\| + \|W_x\| \|\gamma_{x_0}\|\|x_0\| \leq \|W_x\| \|\gamma_u\|\|u\| + \|W_x\| \|\gamma_{x_0}\|\|W_x\|\|x_0\|.$$ (31)

It is important to note that norms used for $\|W_x\|$ and $\|W_u\|$ are the corresponding induced norms. Similarly, if an upper bound obtained for the weighted system is $\gamma(\hat{N})$ then

$$\gamma(N) \leq \|W_x\| \|\gamma(\hat{N})\|\|W_u\|.$$ (32)

There is no method to compute $\|W_x\|$ and $\|W_u\|$ in general. However, in some special cases, such as the case where $2$-norm is used for the spatial norm or the case where weighting matrices are multiplication of a scalar by the identity matrix, $\|W_x\|$ and $\|W_u\|$ can be calculated. The following example illustrates the usage and effectiveness of the weighting technique.

**Example 4.1** Consider the following nonlinear system:

$$\begin{align*}
\dot{x}_1 &= -x_1 + x_2 + 0.5 \text{sat}(x_2) - 0.25 \sin(x_1) + 0.25 \text{sat}(u) \\
\dot{x}_2 &= -x_1 - x_2 + 0.5 \text{sat}(x_1) - 0.25 \sin(x_2) - 0.25 u
\end{align*}$$

where sat(·) is depicted in Fig. 8. Let $A = \begin{bmatrix} -0.9 & 0.9 \\ -0.9 & -1.1 \end{bmatrix}$; hence, $\Phi(x_1, x_2, u) = [\Phi_1(x_1, x_2, u) \Phi_2(x_1, x_2, u)]$, where $\Phi_1(x_1, x_2, u) = -0.1x_1 + 0.1x_2 + 0.5 \text{sat}(x_2) - 0.25 \sin(x_1) + 0.25 \text{sat}(u)$ and $\Phi_2(x_1, x_2, u) = -0.1x_1 + 0.1x_2 + 0.5 \text{sat}(x_1) - 0.25 \sin(x_2) - 0.25 u$. Let $W_u = 1.75$ and $W_x = 1.75$. Therefore, $\|W_x\| = 1$ and $\|W_u\| = 1.75$. We plot $\|\hat{\Phi}(\hat{x}, \hat{u})\|/\|\hat{x}\|$, $\|\hat{\Phi}(\hat{x}, \hat{u})\|/\|\hat{u}\|$ versus $\|\hat{x}\|$ and $\|\hat{u}\|$ in Fig. 9. As shown in Fig. 9, $\gamma(\hat{\Phi}) \approx 0.53$, $\gamma(\hat{\Phi}) \approx 0.5$ and $\gamma(\hat{\Phi}) \approx 0.35$. Computation also shows that $\gamma(\hat{\Gamma}) \approx 1.253$, $\gamma(\hat{\Gamma}) \approx 1.003$ and $\gamma(\hat{\Gamma}) \approx 1.253$. Therefore, $\gamma_1(\hat{N}) \leq 1.361$, $\gamma_2(\hat{N}) \leq 0.58$ and $\gamma_\infty(\hat{N}) \leq 3.029$. Using (32), $\gamma(\hat{N}) \leq 2.382$, $\gamma(\hat{N}) \leq 1.015$ and $\gamma(\hat{N}) \leq 5.301$. The results obtained for various values of $W_u$ are summarized in Table I. As can be seen, tighter bounds can be found by trying different values for the weighting matrices.

### Table I

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<th>$W_u$</th>
<th>$\gamma(\hat{N})$</th>
<th>$\gamma(\hat{N})$</th>
<th>$\gamma(\hat{N})$</th>
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<td></td>
</tr>
</tbody>
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**V. CONCLUSION**

This technical note offers a contribution to the calculation of upper bounds on the $L_1$, $L_2$ and $L_\infty$ induced operator norms of continuous-time nonlinear systems. Based on the $\zeta_{\hat{N}}$ representation of nonlinear systems, methods are presented to compute the aforementioned bounds.

The main limitation of the proposed methods is inequality (11) that restricts the usage of the method for a class of the nonlinear systems and the freedom on choosing the parameter $A$. To lessen the restrictions encountered in the computation of the $L_\infty$ norm of a system, a method is given to compute an upper bound on the $L_\infty$ norm of the system output with respect to the $L_\infty$ norm of the input. This method does not suffer from the previous limitations. In the last section, our methods are improved by the use of a weighting technique on the $\zeta_{\hat{N}}$ representation. An example is provided to show the effectiveness of the weighting technique.
REFERENCES


