Effects of network communications on a class of learning controlled non-linear systems†

Ya-Jun Pana*, Horacio J. Marquezb, Tongwen Chenb and Long Shenga

aDepartment of Mechanical Engineering, Dalhousie University, 1360 Barrington street, Halifax, Nova Scotia, Canada B3J 2X4; bDepartment of Electrical and Computer Engineering, University of Alberta, Edmonton, ECERF Building, Alberta, T6G 2V4 Canada

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In this article, an iterative learning control approach is proposed for a class of sampled-data non-linear systems over network communication channels. The effects of constant time delays and stochastic packet loss are discussed and demonstrated by simulation results. The focus of this article is to study the remote control problems when the environment is periodic or repeatable over iterations in a fixed finite interval. Because of the existence of time delays and packet loss in input and output signal transmissions, it is not trivial to accomplish the remote stabilisation task of any system. Moreover, to track a desired trajectory through a remote controller is even more difficult. Previous cycle-based learning method is incorporated into the network-based control for a class of non-linear systems which satisfies a global Lipschitz condition. The convergence property of this approach is proven. Furthermore, the convergence in the iteration domain is also discussed when there exists packet loss in both transmission channels of the system. Finally, one single-link rigid robot is given as an example to show the effectiveness of the proposed approach.

Keywords: time delays; packet loss; Iterative learning control; remote control systems; sampled-data systems; nonlinearity; global lipschitz condition

1. Introduction

In the remote control systems, one feature is that the control loops are closed through a real-time communication channel which transmits signals from the sensors to the controller and from the controller to the actuators (Nilsson 1998; Zhang, Branicky and Phillips 2001). As a result, it could reduce weight, cost, system wiring and power requirements, and is easy for maintenance, etc. However, the introduction of communication networks make the analysis and control design more complicated than classical feedback loops. Two main issues occur in the networked control systems. The first is the network-induced delays, namely, sensor-to-controller delay and controller-to-actuator delay, that occur while exchanging data among devices connected to the shared medium. Such delays, either constant or time varying, may destabilise the system (Dugard and Verriest 1998; Niculescu 2001; Pan, Marquez and Chen 2006), or degrade the performance of control systems designed without considering the delays. The second is that some packets not only suffer transmission delay but, even worse, can be lost in the transmission channel. Hence, packet dropout should also be considered in how it affects the performance of the resultant closed-loop system (Badaik 2003).

Besides the stability problem, designing a remote controller to achieve tracking of non-linear systems is a challenging problem. Fortunately, for periodic systems, iterative learning control (ILC) offers a systematic design that can improve the tracking performance by iterations in a fixed time interval (Arimoto, Kawamura and Miyazaki 1984; Xu 1997). It has been shown to be one of the most effective methodologies for repeated tracking control tasks for deterministic systems. Control objectives can be achieved iteratively through updating the control input in the iteration domain while the system executes the same motion (Moore 1993; Sun and Wang 2001). In the literature, there are many learning control approaches proposed for different classes of systems: linear or non-linear systems, time delay systems, cascade systems, etc. These approaches include: P-type learning, D-type learning, anticipatory learning, robust learning and optimal learning (Lee and Bien 1997; Frueh 2000), etc. Specifically, previous cycle based learning (PCL) can be well incorporated into the remote controller design since the nature of the PCL

*Corresponding author. Email: yajun.pan@dal.ca
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includes delay properties itself. As a result, the PCL with anticipatory learning can be applied to the remote control system for the periodic tracking control tasks. This is one of the motivations of this article. Another motivation is to investigate the effect of packet loss on the PCL based learning control systems. For a linear system with input time delay only, in Park, Bien and Hwang (1998), the authors proposed a sampled-data ILC to ensure the stability of the closed loop system which can be unstable if the control law is designed in the continuous time domain. For the application of the ILC in the remotely controlled non-linear systems, to achieve tracking control tasks, no results have been available in the literature yet, which also motivates the study of this article. Furthermore, in practice, most networked control systems are implemented by the digital control techniques (Chen and Francis 1995). Hence, the proposed control is investigated as a sampled-data approach.

In this article, for a non-linear system controlled over a network, a sampled-data PCL based learning control approach is proposed. The purpose is to deal with control problems when the environment is periodic over iterations in a finite interval. Due to the existence of time delays and packet loss in input and output signal transmissions, tracking a desired trajectory through a remote controller is not an easy task. The proposed control law is realised assuming that: (i) the sensor-to-controller time delay is measurable under the ideal condition that the clock can be synchronized in the sensor and controller sides; (ii) according to (i), we have partial knowledge on the controller-to-actuator time delay such that we can compensate the delay effect on the system performance by anticipatory steps ahead in the previous cycle learning control design. The convergence property of this approach is then proven, i.e. the tracking error is zero as the number of iterations increases. Furthermore, the effects of packet loss is also discussed.

The article is organised as follows. In Section 2, the problem formulation is presented for a remotely controlled general non-linear system. In Section 3, a PCL based learning controller is designed with the proof of convergence over iterations. In Section 4, the tracking accuracy is further analysed in the case of packet dropout in data transmission. Section 5 shows the simulation results of a single-link rigid robots. Section 6 draws the conclusions.

**Notations:** $\|x\|_\infty$ is the sup-norm defined as $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ and the induced matrix norm is $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^m |a_{ij}|$ for the matrix $A \in \mathbb{R}^{n \times m}$; $\lambda_i(A)$ denotes the $i$-th eigenvalue of the matrix $A$; $D(A) = \frac{\partial f(x)}{\partial x}$ is a row vector, where $f(\cdot)$ is a scalar function.

2. **Problem formulation**

Consider a class of non-linear systems as

$$
\begin{align*}
\dot{x}(t) &= f(x(t), t) + Bu(t - \tau_2) \\
y(t) &= Cx(t),
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output vector, $u(t - \tau_2) \in \mathbb{R}^q$ is the input vector where $\tau_2$ is the time delay from the controller to the actuator and the subscript $i$ denotes the operation cycle. $f(x, t)$ is a non-linear function with respect to the corresponding arguments. Note that parameters in the function $f(x, t)$ can be unknown while it satisfies Assumption 2. $B$ and $C$ are known constant matrices. $t \in [0, T]$ is the finite time for the periodic operation of the system.

The set up of the control system in (1) is illustrated as in Figure 1. The sensor, actuator and the non-linear system are remotely controlled by an iterative learning controller that interchanges measurement output and control signals through a communication network. In the sensor and controller sides, they are time driven and in the actuator side it is event-driven. In sending packets from one side to another, all information in one sampling is packaged in one packet. The objective of the controlled system is to track the desired trajectory $y_d(t)$ in a finite time period $[0, T]$ which can be divided into $K$ equal intervals with a certain sampling interval $T_s$. $y_d(t)$ is realisable with a unique input bounded as $\|u(t)\| \leq \beta_{iad}$, where $\beta_{iad}$ is a positive constant. In this article, we assume the system parameters $B$ and $C$ are known as same as many learning control schemes in the literature. The main task is to deal with the time delay and packet loss problem when a non-linear system is in a periodic environment. The desired trajectory can be of the form

$$
\begin{align*}
\dot{x}_d(t) &= f(x_d(t), t) + Bu_d(t - \tau_2) \\
y_d(t) &= Cx_d(t).
\end{align*}
$$

The following assumptions are first made in this article.
Assumption 1: The resetting condition \( x_i(0) = x_i(0) \) is satisfied for every i-th iteration where \( i \) is a positive integer.

Assumption 2: The function \( f(x, t) \) is assumed to be globally uniformly Lipschitz in \( x \) on the finite period \([0, T]\), or

\[
\|f(x_1(t), t) - f(x_2(t), t)\|_\infty \
\leq c_f \|x_1(t) - x_2(t)\|_\infty, \quad \forall t \in [0, T],
\]

where \( c_f \) is a positive constant. The function \( f(x, t) \) is assumed to be bounded as \( \|f(x(t), t)\|_\infty \leq \beta_f, \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+ \equiv [0, T] \), where \( \beta_f \) is a positive constant.

Lemma 1 Bellman–Gronwall Lemma (II) (Ioannou and Sun 1996): Let \( \lambda(t), k(t) \) be non-negative piecewise continuous function of time \( t \) and let \( \lambda(t) \) be differentiable. If the function \( y(t) \) satisfies the inequality \( y(t) \leq \lambda(t) + \int_{t_0}^{t} \lambda(s)y(s)\,ds, \quad \forall t \geq t_0 \geq 0 \), then \( y(t) \leq \lambda(t_0)e^{\int_{t_0}^{t} \lambda(s)\,ds} + \int_{t_0}^{t} \lambda(s)e^{\int_{s}^{t} \lambda(s)\,ds} \, ds, \quad \forall t \geq t_0 \geq 0 \).

Assumption 3: Clock synchronization is assumed to be satisfied in the sensor and controller sides. The transmission time delay from the sensor to the controller, namely \( \tau_1 \), can be measured. The time delay from the controller to the actuator \( \tau_2 = h_2T_s \) is a constant, where \( h_2 > 0 \) is a known integer constant and \( T_s \) is the sampling time.

In Section 3, we first present the iterative controller design when there is no packet loss in transmission. Section 4 further considers the case when packet loss occurs.

### 3. Sampled-data ILC design

In this section, we consider the controller design when there is no packet loss in signal transmission, i.e., the network load is not heavy. The following PCL-based learning controller with delay compensation is designed:

\[
u_{i+1}(k) = u_i(k) + \Gamma(k)e_i(k + h_2 + 1), \quad k \in [0, 1, \ldots, K - h_2 - 1]. \tag{3}
\]

where \( u_i(k) \equiv u_i(kT_s), e_i(k + h_2 + 1) \equiv e_i((k + h_2 + 1)T_s) = y_i((k + h_2 + 1)T_s) - y_i((k + h_2 + 1)T_s) \), and \( \Gamma(k) \equiv \Gamma(kT_s) \) is the control gain to be designed. The convergence property of the closed-loop system is shown as in the following theorem.

Theorem 1: Assume that the system in (1) satisfies Assumptions 1–3. Under the learning control law in (3), if the following condition

\[
u(k) = \|I - T_s\Gamma(k)CB\|_\infty \leq \rho < 1, \tag{4}
\]
is satisfied, then \( \lim_{k \to \infty} x_i(kT_s) = x_i(kT_s), \lim_{k \to \infty} y_i(kT_s) = y_i(kT_s), \forall k \in \{h_2 + 1, h_2 + 2, \ldots, K\} \).

Proof: Denotes \( f_i = f(x_i(t), t) \). At the ith iteration cycle and at the sampling instant \( t = (k + h_2 + 1)T_s \), the tracking error \( e_i(k + h_2 + 1) \) can be represented as

\[
\begin{align*}
e_i(k + h_2 + 1) &= y_i(k + h_2 + 1) - y_i(k + h_2 + 1) \\
&= Cx_i(k + h_2 + 1) - Cx_i(k + h_2 + 1) \\
&= e_i(k + h_2) + C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} [x_i(s) - x_i(s)]\,ds \\
&= e_i(k + h_2) + C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} (f_i - f_i)\,ds \\
&+ CBT_s \Delta u_i(k).
\end{align*}
\]

We have \( u_i(t - \tau_2) = u_i(k), \forall t \in [(k + h_2)T_s, (k + h_2 + 1)T_s] \). Denote \( \Delta u_i = w_i - w_i \), where \( w \) is a variable. The equation in (5) becomes

\[
\begin{align*}
e_i(k + h_2 + 1) &= e_i(k + h_2) + C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} (f_i - f_i)\,ds \\
&+ CBT_s \Delta u_i(k).
\end{align*}
\]

Substituting (6) into the controller in (3) and deriving

\[
\begin{align*}
\Delta u_{i+1}(k) &= \Delta u_i(k) - \Gamma(k)e_i(k + h_2 + 1) \\
&= \Delta u_i(k) - \Gamma(k)\left\{e_i(k + h_2) \\
&+ C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} (f_i - f_i)\,ds + CBT_s \Delta u_i(k)\right\} \\
&= [I - T_s\Gamma(k)CB]\Delta u_i(k) - \Gamma(k) \\
&\times \left[ e_i(k + h_2) + C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} (f_i - f_i)\,ds \right].
\end{align*}
\]

Using the Lipschitz and bounded properties in Assumption 2, \( \|C\|_\infty \leq \beta_L \) and \( \|\Gamma(k)\|_\infty \leq \beta_T \),

\[
\begin{align*}
\|\Delta u_{i+1}(k)\|_\infty &\leq \nu(k)\|\Delta u_i(k)\|_\infty + \|\Gamma(k)\|_\infty \|e_i(k + h_2) + C \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} (f_i - f_i)\,ds \|_\infty \\
&\leq \nu(k)\|\Delta u_i(k)\|_\infty + \beta_T \beta_L \|\Delta x_i(k + h_2)\|_\infty \\
&+ \beta_T \beta_L c_f \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} \|\Delta x_i(s)\|_\infty ds_\infty. \tag{7}
\end{align*}
\]
where
\[ \nu(k) = \|I - T_1\Gamma(k)CB\|_\infty. \]

Now we derive the relationship between \(\|\Delta x(k)\|_\infty\) and \(\|\Delta u(k)\|_\infty\). From system (1),
\[ \Delta x(t) = \int_0^t [\Delta f + B_2\Delta u(s - \tau_2)]ds. \]

It follows that
\[ \|\Delta x(t)\|_\infty \leq \int_0^t [c_f\|\Delta x(s)\|_\infty + \beta_B\|\Delta u(s - \tau_2)\|_\infty]ds. \]

Using \(u(t) = 0\) and \(\varepsilon_2(t) = 0\) \(\forall t < 0\), the constant control input \(u(t - \tau_2) = u(k)\) in each interval \(\forall t \in [(k + h_2)T_n, (k + h_2 + 1)T_n]\), the Bellman–Gronwall Lemma II, \(\forall t \leq (k + h_2 + 1)T_n\), we have
\[ \|\Delta x(t)\|_\infty \leq \sum_{j=0}^{k} \beta_3\rho_4(i)\|\Delta u(j)\|_\infty. \]

Lemma II, \(\forall t \leq (k + h_2 + 1)T_n\), we have
\[ \|\Delta x(t)\|_\infty \leq \sum_{j=0}^{k} \beta_3\rho_4(i)\|\Delta u(j)\|_\infty. \]

Substituting (9) and (10) into the inequality (7), we have
\[ \|\Delta u_{i+1}(k)\|_\infty < \rho_1(k)\|\Delta u_i(k)\|_\infty + \rho_2\|\Delta u(k - 1)\|_\infty + \rho_3 \sum_{j=0}^{k-2} \rho_4(k, j)\|\Delta u(j)\|_\infty, \]

where
\[ \rho_1(k) = \nu(k) + \beta_B(e^{\varepsilon_1T_n} - 1)c_f, \]
\[ \rho_2 = \beta_T\beta_c\beta_B(e^{\varepsilon_2T_n} - 1)c_f + \beta_T\beta_cT_2\beta_B(e^{\varepsilon_3T_n} - e^{\varepsilon_4T_n}, \]
\[ \rho_3 = \beta_T\beta_B(e^{\varepsilon_4T_n} - e^{\varepsilon_5T_n})c_f, \]
\[ \rho_4(k, j) = \beta_3\varepsilon_1\|\Delta u(k - 1)\|_\infty + \beta_3\varepsilon_2\|\Delta u(k - 2)\|_\infty. \]

From \(k = 0\) to \(k = K - h_2 - 1\) where \(K = T_nT_n\), (11) can be expressed as
\[ \phi_{i+1} = H_i\phi_i, \]

where \(\phi_i = [\|\Delta u_0(k)\|_\infty, \|\Delta u_1(k)\|_\infty, \ldots, \|\Delta u_{K - h_2 - 1}(k)\|_\infty]^T\)

and
\[ H_i = \begin{bmatrix}
  \rho_1(0) & 0 & 0 & \cdots & 0 \\
  \rho_2 & \rho_1(1) & 0 & \cdots & 0 \\
  \rho_3\rho_4(2, 0) & \rho_2 & \rho_1(2) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  \rho_3\rho_4(K - h_2 - 1, 0) & \rho_3\rho_4(K - h_2 - 1, 1) & \cdots & \rho_1(K - h_2 - 1)
\end{bmatrix}. \]

Since \(H_i\) is a lower triangular matrix, all of its eigenvalues are the diagonal elements \(\rho_1(k), \forall k = 0, 1, \ldots, K - h_2 - 1\). Selecting the gain matrix \(\Gamma\) such that all the eigenvalues of \(H_i\) are within the unit circle, i.e. \(\rho_1(k) < 1\), the asymptotic convergence property of \(\|\Delta u(k)\|_\infty\) can be guaranteed along the iteration axis. Then \(\|\Delta u(k)\|_\infty \to 0\) as \(i \to \infty\), \(\forall k = 0, 1, \ldots, K - h_2 - 1\). Note that if \(0 < \nu(k) \leq \rho < 1\) is satisfied, it is possible to choose the sampling time \(T_n\) to be small such that \(e^{\varepsilon_1T_n} - 1 > 0\) to make \(\rho_1(k) < 1\), \(\forall k = 0, 1, \ldots, K - h_2 - 1\). Furthermore, from the controller structure in (3), \(\|\Delta u(k)\|_\infty \to 0\) implies that: \(u(k) \to u_d(k)\). Then \(x_i(k) \to x_d(k), y_i(k) \to y_d(k)\) as \(i \to \infty\), \(\forall k = h_2 + 1, \ldots, K\).

**Remark 1:** It is known that the conventional PCL method lacks robustness to initial setting condition errors, measurement noises, parametric uncertainties and external disturbances. But in the proposed approach, we introduce anticipatory steps ahead in the conventional PCL method, which is likely to possess robustness to initial setting condition errors, measurement noises, parametric uncertainties and external disturbances. Detailed analysis is beyond the scope of this article and is therefore left for future discussions.
Remark 2: When the sampling time $T_s$ is very small while the time delay $\tau_2$ and finite learning period $T$ are finite, $h_2$ and $K$ then become larger. However, the ratio $\frac{h_2}{T_s}$ is finite. As long as the time delay is far less than the learning period, the controller will not be a failure.

4. Effects of packet loss

In this section, we further discuss the situation when packet loss happens during the transmission. This kind of phenomena occurs when the network is busy and under heavy load. The following analysis is composed of three main parts: (1) single packet loss happens from controller to actuator side only; (2) single packet loss happens from sensor to controller side only; (3) discussions on general situations with multiple packet loss.

4.1. Single packet loss from controller to actuator side

Here we start by assuming that there is one packet lost at time $t = kT_s$ during the $i$-th iteration. In this case, the actuator side will not receive this packet. On the actuator side, since it is event driven, the packet sent at time $t = (k-1)T_s$ from the controller side will continue to be applied in the system before the packet sent at time $t = (k+1)T_s$ arrives. The learning controller is the same as in (3). For this case, we have

$$\mathbf{u}(t - \tau_2) = \mathbf{u}(k-1), \quad \forall t \in [(k+h_2-1)T_s, (k+h_2+1)T_s).$$

Hence, $e_t(k+h_2+1)$ and $e_t(k+h_2+2)$ are changed because the packet of $\mathbf{u}(k)$ is lost in the transmission channel from the controller to the actuator. The convergence property of this case can be concluded in the following theorem.

Theorem 2: Assume that there is one packet loss at time $t = kT_s$ during the $i$-th iteration in the iteration domain. If the condition in (4) holds, then we still have

$$\lim_{i \to \infty} x_t(kT_s) = x_t(kT_s), \quad \lim_{i \to \infty} y_t(kT_s) = y_t(kT_s),$$

$$\forall k \in \{h_2 + 1, h_2 + 2, \ldots, K\}.$$

Proof: The proof can be composed of the following three parts due to packet loss effects on $e_t(k+h_2+1)$, $e_t(k+h_2+2)$ and afterwards during the $i$-th iteration.

Part A: The effect on $\|\Delta \mathbf{u}_{l+1}(k)\|_\infty$ of the packet loss at time $t = kT_s$ during the $i$-th iteration

According to the derivation in (5) and (6), at $t = (k+h_2+1)T_s$ during the $i$-th iteration cycle, the tracking error $e_t(k+h_2+1)$ is

$$e_t(k+h_2+1) = e_t(k+h_2) + \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} C\Delta \mathbf{f}_d ds$$

$$+ CBT_s \Delta \mathbf{u}_t(k - 1).$$

Substituting (14) into the controller (3) and deriving $\Delta \mathbf{u}_{l+1}(k) = \mathbf{u}_t(k) - \mathbf{u}_{l+1}(k)$ as

$$\Delta \mathbf{u}_{l+1}(k) = \Delta \mathbf{u}_t(k) - \Gamma(k) \left[ e_t(k+h_2) + \int_{(k+h_2)T_s}^{(k+h_2+1)T_s} C\Delta \mathbf{f}_d ds \right]$$

$$+ CBT_s \Delta \mathbf{u}_t(k - 1).$$

Using the global Lipschitz condition and boundness property in Assumption 2,

$$\|\Delta \mathbf{u}_{l+1}(k)\|_\infty \leq \|\Delta \mathbf{u}_t(k)\|_\infty + \beta_T \beta_c \|\Delta \mathbf{x}_t(k + h_2)\|_\infty$$

$$+ \beta_T \beta_c \frac{\Delta \mathbf{f}_d}{\Delta \mathbf{u}_t(k - 1)} \|\Delta \mathbf{x}_t(\delta)\|_\infty,$$ (15)

where all parameters are the same as in (7). By using the constant control input $\mathbf{u}_t(t - \tau_2) = \mathbf{u}(j)$ in each intervals $t \in [(j+h_2)T_s, (j+h_2+1)T_s)$ when $j \leq k - 2$, (8) becomes

$$\|\Delta \mathbf{x}_t(t)\|_\infty \leq \beta_B \int_{j}^{(k+h_2+1)T_s} e^{\gamma(k+h_2+1)T_s - s} \|\Delta \mathbf{u}_t(s - \tau_2)\|_\infty ds$$

$$= \beta_B \sum_{j=0}^{k-2} \left\| \Delta \mathbf{u}_t(j) \right\|_\infty \int_{j(h_2+1)T_s}^{(j+h_2+1)T_s} e^{\gamma(k+h_2+1)T_s - e^{\gamma j} T_s} ds$$

$$+ \beta_B \left\| \Delta \mathbf{u}_t(k-1) \right\|_\infty \int_{(k+h_2-1)T_s}^{(k+h_2+1)T_s} e^{\gamma(k+h_2+1)T_s - e^{\gamma j} T_s} ds$$

$$= \beta_B \sum_{j=0}^{k-2} \left[ e^{\gamma(k-j-1)T_s} \left( e^{2\gamma T_s} - e^{\gamma j T_s} \right) \right] \left\| \Delta \mathbf{u}_t(j) \right\|_\infty$$

$$+ \beta_B \left( e^{2\gamma T_s} - e^{\gamma T_s} \right) \sum_{j=0}^{k-2} \left\| \Delta \mathbf{u}_t(k-1) \right\|_\infty$$

$$< \beta_B \frac{e^{2\gamma T_s} - e^{\gamma T_s}}{\gamma} \sum_{j=0}^{k-2} \left\| \Delta \mathbf{u}_t(j) \right\|_\infty$$

Equation (10) also holds for this case. Substituting (10) and (16) into the inequality (15), we have

$$\|\Delta \mathbf{u}_{l+1}(k)\|_\infty < \|\Delta \mathbf{u}_t(k)\|_\infty + \rho_1 \|\Delta \mathbf{u}_t(k - 1)\|_\infty$$

$$+ \rho_3 \sum_{j=0}^{k-2} \rho_4(k, j) \|\Delta \mathbf{u}_t(j)\|_\infty,$$ (16)
where $\rho'_2 = \beta_T \beta_B \beta_c T_s (e^{\gamma j T} - 1)$, $\rho_3$ and $\rho_4(k, j)$ are the same as in (11).

**Part B:** The effect on $\|\Delta u_{i+1}(k+1)\|_\infty$ of the packet loss at time $t = kT_s$ during the $i$-th iteration

Omitting some details, we present a brief derivation as follows:

$$
e_i(k+h_2 + 2) = e_i(k+h_2 + 1) + \int_{(k+h_2+1)T_s}^{(k+h_2+2)T_s} C \Delta f_i d\tau + T_s CB \Delta u_i(k + 1).$$

Then

$$||\Delta u_{i+1}(k+1)||_\infty$$

$$= ||I - T_s \Gamma(k+1) CB I\Delta u_i(k+1) - \Gamma(k+1)||_\infty$$

$$\times \left[ e_i(k+h_2 + 1) + \int_{(k+h_2+1)T_s}^{(k+h_2+2)T_s} C \Delta f_i d\tau \right]$$

$$\leq \nu(k+1)||\Delta u_i(k+1)||_\infty + \beta_T \beta_c ||\Delta x_i(k+h_2 + 1)||_\infty$$

$$+ \beta_T \beta_c \int_{(k+h_2+1)T_s}^{(k+h_2+2)T_s} ||\Delta x_i(s)||_\infty d\tau.$$  (17)

Furthermore, we have

$$||\Delta x_i(t)||_\infty$$

$$\leq \beta_B \sum_{j=0}^{k-2} ||\Delta u_i(j)||_\infty \int_{(j+h_2)T_s}^{(j+h_2+1)T_s} e^{\tau j(k+h_2)T_c - \gamma \tau} d\tau$$

$$+ \beta_B \sum_{j=0}^{k-2} ||\Delta u_i(k-1)||_\infty \int_{(j+h_2+1)T_s}^{(j+h_2+2)T_s} e^{\tau j(k+h_2)T_c - \gamma \tau} d\tau$$

$$+ \beta_B \sum_{j=0}^{k-2} ||\Delta u_i(j)||_\infty \int_{(j+h_2)T_s}^{(j+h_2+1)T_s} e^{\tau j(k+h_2+1)T_c - \gamma \tau} d\tau$$

$$+ \beta_B \sum_{j=0}^{k-2} ||\Delta u_i(k-1)||_\infty \int_{(j+h_2+1)T_s}^{(j+h_2+2)T_s} e^{\tau j(k+h_2+1)T_c - \gamma \tau} d\tau.$$  (18)

**Part C:** Summary

From the above two steps, we know that the mapping matrix $H_i$ is changed in the $(k+j)$-th rows $(j \geq 1)$ if there is one packet loss at time $t = kT_s$ during the $i$-th iteration. From $k = 0$ to $k = K-h_2-1$, the mapping matrix $H_i$ in (12) becomes $H_{i,k}$:

$$\phi_{i+1} \leq H_{i,k} \phi_i.$$
Note that $H_{i,k}$ is also a lower triangular matrix, all its eigenvalues are the diagonal elements, with one element being '1' at $(k+1, k+1)$ and other $\rho_{1}(k) < 1$, $\forall k = 0, 1, \ldots, K-h_2-1$. The asymptotic convergence property of $\|\Delta u(k)\|_{\infty}$ can also be guaranteed along the iteration axis $i$ because all eigenvalues of the mappings satisfy

$$
\lambda_m \left[ \prod_{j=1}^{i-1} H_j \right] H_i \left( \prod_{j=i+1}^{\infty} H_j \right) \to 0, \\
m = 0, 1, \ldots, K-h_2-1. \quad (22)
$$

Then $\|\Delta u(k)\|_{\infty} \to 0$ as $i \to \infty$, $\forall k = 0, 1, \ldots, K-h_2-1$. Furthermore, from the controller structure in (3), $\|\Delta u(k)\|_{\infty} \to 0$ also implies that $u(k) \to u_i(k)$, $\forall k = 0, 1, \ldots, K-h_2-1$. Then $x_i(k) \to x_i(k)$ and $y_i(k) \to y_i(k)$ as $i \to \infty$, $\forall k = h_2+1, \ldots, K$.

4.2. Single packet loss from sensor to controller side

The difference between the packet loss in the two channels is that from sensor to controller side the measured output signal is lost, while from controller to actuator side, the calculated control signal is lost. Hence, the following analysis in this subsection is different from the previous subsection. For convenience, assuming that the packet of $y_i(k+h_2+1)$ during the $i$-th iteration is missing in the memory of the controller side due to the data dropout in the transmission from sensor to controller. In this case, the controller side will continue to use the output signal $y_i(k+h_2)$ closest available signal previous to the current time stamp in the memory. Hence, from (3), the controller is changed accordingly,

$$
u_{i+1}(k) = u_i(k) + \Gamma(k)e_i(k+h_2), \quad k \in \{0, 1, \ldots, K-h_2-1\}. \quad (23)
$$

**Theorem 3:** Assume that there is one packet loss at time $t = (k + h_2 + 1)T_s$ during the $i$-th iteration in the iteration domain. If the condition in (4) holds, then we still have

$$
limit_{i \to \infty} x_i(kT_s) = x_i(kT_s), \quad limit_{i \to \infty} y_i(kT_s) = y_i(kT_s), \forall k \in \{h_2 + 1, h_2 + 2, \ldots, K\}.
$$

**Proof:** $e_i(k+h_2)$ can be derived through replacing $k$ by $k-1$ in (6). Using (23) and $e_i(k+h_2)$, we have

$$
\|\Delta u_{i+1}(k)\|_{\infty} = \|\Delta u_i(k) - \Gamma(k)e_i(k+h_2)\|_{\infty} \\
= \|\Delta u_i(k) - \Gamma(k)\left\{e_i(k+h_2 - 1) + \int_{T(s)-1}^{T(s)} C\Delta f ds + T_s CB\Delta u_i(k-1)\right\}\|_{\infty} \\
\leq \|\Delta u_i(k)\|_{\infty} + \beta_1 \beta_i \|\Delta x_i(k+h_2 - 1)\|_{\infty} \\
+ \beta_1 \beta_i \beta g T_s \|\Delta u_i(k-1)\|_{\infty}, \quad (24)
$$

where all parameters are the same as in (7). Replacing $k$ by $k-1$ in (9) and (10), and then substituting them into the inequality (24), we have

$$
\|\Delta u_{i+1}(k)\|_{\infty} < \|\Delta u_i(k)\|_{\infty} + \beta_1 \|\Delta u_i(k-1)\|_{\infty} + \rho_2 \|\Delta u_i(k-2)\|_{\infty} + \rho_3 \sum_{j=0}^{k-3} \tilde{\rho}_4(k,j) \|\Delta u_i(j)\|_{\infty}, \quad (25)
$$

where $\rho_2$ and $\rho_3$ are as same as in (11), $\tilde{\rho}_1 = \beta_1 \beta_i \beta g T_s$, $\tilde{\rho}_4(k,j) = \beta_i T_s e^{\alpha(k-2-j)T_s} + \beta_1 e^{\alpha(k-3-j)T_s}$. Then the mapping matrix $H_i$ is changed in the $(k+1)$th row. From $k = 0$ to $k = K-h_2-1$, the mapping matrix $H_i$ in (12) becomes $\tilde{H}_{i,k}$ in which the $(k+1)$st row is different with $H_i$.

$$
\tilde{H}_{i,k} = \left[
\begin{array}{cccccccc}
\rho_1(0) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\rho_2 & \rho_1(1) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots \\
\rho_3 \rho_4(k-1,0) & \cdots & \cdots & \rho_1(k-1) & \cdots & \cdots & \cdots & \cdots \\
\rho_3 \tilde{\rho}_4(k,0) & \cdots & \cdots & \tilde{\rho}_1 & \cdots & \cdots & \cdots & \cdots \\
\rho_3 \rho_4(k+1,0) & \cdots & \cdots & \rho_2 & \rho_1(k+1) & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\
\rho_3 \rho_4(K-h_2-2,0) & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_1(K-h_2-2) & 0 \\
\rho_3 \rho_4(K-h_2-1,0) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_1(K-h_2-1)
\end{array}\right]. \quad (26)
$$
Note that $\hat{H}_{ik}$ is also a lower triangular matrix, all its eigenvalues are the diagonal elements, with one element being ‘1’ at $(k,k)$ and other $\rho_i(k) < 1$, $\forall k = 0, 1, \ldots, K - h_2 - 1$. Similar as in the proof of Theorem 2, the asymptotic convergence property of $\|\Delta u_i(k)\|_\infty \rightarrow 0$ can also be guaranteed along the iteration axis $i \rightarrow \infty$. From the controller structure in (3), $\|\Delta u_i(k)\|_\infty \rightarrow 0$ also implies that $u_i(k) \rightarrow u_i(k)$. Then $x_i(k) \rightarrow x_i(k)$ and $y_i(k) \rightarrow y_i(k)$ as $i \rightarrow \infty$, $\forall k = h_2 + 1, \ldots, K$.

4.3. Discussions on multiple packet loss

From the observations in Theorems 2 and 3, we can find that multiple packet loss in both channels together will result in more ‘1’ in the diagonal elements of the mapping matrix $H_i (i = 1, \ldots, \infty)$, i.e. there is multiple packet loss at $t=jT$, in the $i$-th iteration as $j \in \{0, 1, \ldots, K - h_2 - 1\}$. The structure of $H_i$ remains the lower triangular form. This can be concluded from the proofs in Theorems 1 and 2 that only the past information previous to the time stamp of the lost packet will be used when the packet loss happens. Now review the structure of the serial product of the mapping matrix $H_i$ in the $i$-th iteration axis:

$$\Pi_{i=1}^{\infty} H_i = \begin{bmatrix}
\Pi_{i=1}^{\infty} \rho_{1,i}(0) & 0 & 0 \\
* & \Pi_{i=1}^{\infty} \rho_{2,i}(0) & 0 \\
* & * & \Pi_{i=1}^{\infty} \rho_{3,i}(0) \\
\vdots & \vdots & \vdots \\
* & * & * \\
* & * & * \\
\Pi_{i=1}^{\infty} \rho_{1,i}(K - h_2 - 2) & 0 \\
* & * & \Pi_{i=1}^{\infty} \rho_{2,i}(K - h_2 - 1)
\end{bmatrix}$$

(27)

From (27), $\Pi_{i=1}^{\infty} H_i$ is also a lower triangular matrix. The entries denoted as (*) are multiple combination of the elements in $H_{ij}$ and they are finite. However it is difficult to represent them mathematically to be a general form when $i \rightarrow \infty$. Hence, we have Corollary 1, which further shows the impact of multiple packet loss on the system convergence rate.

**Corollary 1:** Define the packet loss rate as $0\% < P_{\text{loss}} \ll 100\%$ along the infinite iteration domain. Assume that the packet loss probability in the time period $\forall t \in [0, T]$ is uniformly distributed, we have the following convergence property: as $i \rightarrow \infty$, $\forall k = h_2 + 1, \ldots, K$, $y_i(k)$ approaches $y_i(k)$ with a slower convergence rate than the case without any packet loss.

**Proof:** From (27), the diagonal elements $\Pi_{i=1}^{\infty} \rho_{1,i}(k)$ with $k = 0, \ldots, K - h_2 - 1$ directly determine the convergence rate of the process $\|\Delta u_i(k)\|_\infty \rightarrow 0$, which can be defined as $r(k, P_{\text{loss}})$. Note that $r(k, P_{\text{loss}})$ is the convergence rate function of the packet loss rate and the time stamp $k$. Such a function does exist though we cannot express it explicitly. Assume that the packet loss probability in the time period $\forall t \in [0, T]$ is uniformly distributed and $0\% < P_{\text{loss}} \ll 100\%$, it ensures that we do not have the extreme case when $\Pi_{i=1}^{\infty} \rho_{1,i}(k) = 1$ with $i \rightarrow \infty$. Hence, $\Pi_{i=1}^{\infty} \rho_{1,i}(k) \rightarrow 0$ holds because $\rho_{1,i}(k) > 0$ is either $1$ or a value less than 1. Equivalently, $\lambda_m[\Pi_{i=1}^{\infty} H_i] \rightarrow 0$.

The above mentioned analysis further means that $\|\Delta u_i(k)\|_\infty \rightarrow 0$ as $i \rightarrow \infty$. When there is no packet loss, no ‘1’ appears in the serial product $\Pi_{i=1}^{\infty} \rho_{1,i}(k)$, which approaches zero faster than the case with packet loss. Thus, the convergence rate with packet loss satisfies $r(k, P_{\text{loss}}) < r(k, 0)$. Similarly as in Theorems 1–3, $\|\Delta u_i(k)\|_\infty \rightarrow 0$ also implies that $u_i(k) \rightarrow u_i(k)$ with $k = 0, \ldots, K - h_2 - 1$. Thus, $x_i(k)$ approaches $x_i(k)$ and $y_i(k)$ approaches $y_i(k)$ as $i \rightarrow \infty$ with $k = h_2 + 1, \ldots, K$.

**Remark 3:** When there is no control input $u$, the system is then solely dependent on the non-linear function $f(\cdot)$ which satisfies Assumption 2. In the assumption, the function is assumed to be globally uniformly Lipschitz with respect to the state $x$. As long as $T$ is finite and $f(\cdot)$ satisfies the Global Lipschitz Condition, there is no finite escape phenomena.

**Remark 4:** In the case of packet loss, the control law is not updated and remains the same as the control signal in the previous time stamp. It is shown that the convergence property of the whole system can also be retained.

5. Illustrative examples

Consider the following single-link rigid robot controlled through some network

$$\ddot{y} = -(0.5 mgl + Mgl/\sin(\theta)) + u,$$

(28)

where $\theta$, $m = 1.5$ kg, $M = 3$ kg, $g = 9.8$ m/s$^2$, $l = 0.5$ m and $J = Ml^2 + \frac{1}{4} l^2 = 0.8333$ kg m$^2$ denote the rotating angle, mass of the load, mass of the rigid link, gravitational acceleration, length and inertia of the robot link, respectively. $u$ is the torque control input.
The desired trajectory is \( \theta_d = (\pi^2 t^2 - (\pi t^3) / 27 \text{ rad.} \)
The trajectory is one-fourth circle in the vertical plane with the horizontal starting position and the vertical ending position. The single-link rigid robot in (28) can be rewritten in the state space form by defining \( x_1 = \theta \) and \( x_2 = \dot{\theta} \):

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\
-(0.5mg/l + Mg/\sin(x_1)) \end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u,
\]

and \( y = [x_1, x_2]^T \) which means both states are measurable, where \( x_1(0) = x_2(0) = 0 \) is the initial condition.

According to the controller designed in (3), the iterative learning controller is as follows

\[
u_{i+1}(k) = u(k) + \gamma_1[\theta_d(k + h_2 + 1) - \theta_k(k + h_2 + 1)] \\
+ \gamma_2[\dot{\theta}_d(k + h_2 + 1) - \dot{\theta}_k(k + h_2 + 1)],
\]

where the control gain \( \Gamma = [\gamma_1, \gamma_2]^T \) is designed to satisfy the condition in (4): \( \nu = |1 - TsT_\beta| \leq \rho < 1 \). In this simulation, \( \Gamma = [0.5, 0.5]^T \) is chosen. The controller is set to \( u_{i+1}(k) = u(k) + \Gamma(k)\epsilon(k) \) when \( k > K - h_2 - 1 \).

As shown in Figure 2, the maximum tracking error goes to zero asymptotically as iteration increases under numerical time delays \( \tau_1 = \tau_2 = 4T_\nu \text{ i.e. } h_2 = 4 \).

The maximum tracking error versus iterations is as shown in Figure 3 when there are constant delays and random packet loss at a 5% loss rate. From the simulation results compared in Figure 4, the system position error converges from iteration to iteration. The convergence is not monotonic but is asymptotic while there is a little difference in the convergent profiles due to packet loss.

Under the constant time delays in both channels, when random packet loss rate through the time duration is increased to \( P_{\text{loss}} = 10\% \) throughout the iteration domain (as shown in Figure 5), and to \( P_{\text{loss}} = 20\% \) (as shown in Figure 6), the convergence property is still retained while the convergence rate is slower than the case without any packet loss, which further verifies the results in Theorems 2 and 3 and Corollary 1.
6. Conclusions

For periodic control tasks of a class of non-linear systems whose dynamics satisfy the global Lipschitz condition, a sampled-data ILC approach is proposed with network transmission. By virtue of a PCL-based learning law and partially known information on the time delays, the tracking error approaches zero as the number of the iterations increases. This property may also hold when there are packet dropouts at certain rates in the data transmission. The proposed approach can eliminate the influence of time delays by compensation in the learning law, ensuring convergence.

Acknowledgements

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Notes on contributors

Ya-Jun Pan received the BE degree in Mechanical engineering from Yanshan University, P.R. China, in 1996, the M.E. degree in Mechanical engineering from Zhejiang University, PR China, in 1999 and the PhD degree in Electrical and Computer engineering from National University of Singapore, in 2003. After receiving the PhD degree, she was a post-doctoral fellow of CNRS in the Laboratoire d’Automatique de Grenoble, France from 2003 to 2004. In 2004, she held a post-doctoral position in the Department of Electrical and Computer Engineering at the University of Alberta, Canada. In January 2005, she joined the Faculty of the Mechanical Engineering Department at Dalhousie University, Canada and currently she is an Assistant Professor. Her research interests are in the fields of non-linear systems, networked control systems, intelligent transportation control systems and tele-robotics. She is currently an Associate Editor of the Journal of the Franklin Institute. She is a member of IEEE and a registered Professional Engineer in Nova Scotia, Canada.

Horacio J. Marquez received the BSc degree from the Instituto Tecnologico de Buenos Aires (Argentina), and the MScE and PhD Degrees in electrical engineering from the University of New Brunswick, Fredericton, Canada, in 1987, 1990 and 1993, respectively. From 1993 to 1996 he held visiting appointments at the Royal Roads Military College, and the University of Victoria, Victoria, British Columbia. Since 1996 he has been with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada, where he is currently Professor and Department Chair. In 2008, he was a guest research professor at Nancy University-Henri Poincare, France. Dr Marquez is currently an Area Editor for the International Journal of Robust and Nonlinear Control and Associate Editor of the Journal of the Franklin Institute. He is the Author of “Nonlinear Control Systems: Analysis and Design” (Wiley 2003). He is the recipient of the 2003–2004 McCalla Research Professorship awarded by the University of Alberta. His current research interests include non-linear dynamical systems and control, non-linear observer design, robust control and applications.

Tongwen Chen received the BEng degree in Automation and Instrumentation from Tsinghua University (Beijing) in 1984, and the MASc and PhD degrees in Electrical Engineering from the University of Toronto in 1988 and 1991, respectively. From 1991 to 1997, he was an Assistant/Associate Professor in the Department of Electrical and Computer Engineering at the University of Calgary, Canada. Since 1997, he has been with...
the Department of Electrical and Computer Engineering at the University of Alberta, Edmonton, Canada, and is presently a Professor. His research interests include computer and network-based control systems, process control, multirate digital signal processing, and their applications to industrial problems. Dr Chen is an IEEE Fellow and a Fellow of the Engineering Institute of Canada. He received a Killam Professorship for 2006–2007 from the University of Alberta, and a Fellowship from the Japan Society for the Promotion of Science for 2004. He has served as an Associate Editor for several international journals, including IEEE Transactions on Automatic Control, Automatica, Systems and Control Letters and Journal of Control Science and Engineering. He is a registered Professional Engineer in Alberta, Canada.

Long Sheng received the BS degree in control engineering from the School of Electrical Information and Control Engineering, Beijing University of Technology, Beijing, China, in 2003 and the MS degree in applied science from Saint Mary’s University, Nova Scotia, Canada, in 2006. He is currently a PhD candidate at the Department of Mechanical Engineering, Dalhousie University, Canada. His main research focus is on the networked control systems with issues arising from communication channels, and on the consensus problem for multiagent systems.

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