Brief paper

Robust $H_{\infty}$ observer design for sampled-data Lipschitz nonlinear systems with exact and Euler approximate models☆

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Abstract

An LMI approach is proposed for the design of robust $H_{\infty}$ observers for a class of Lipschitz nonlinear systems. Two type of systems are considered, Lipschitz nonlinear discrete-time systems and Lipschitz nonlinear sampled-data systems with Euler approximate discrete-time models. Observer convergence when the exact discrete-time model of the system is available is shown. Then, practical convergence of the proposed observer is proved using the Euler approximate discrete-time model. As an additional feature, maximizing the admissible Lipschitz constant, the solution of the proposed LMI optimization problem guarantees robustness against some nonlinear uncertainties. The robust $H_{\infty}$ observer synthesis problem is solved for both cases. The maximum disturbance attenuation level is achieved through LMI optimization.

Keywords: Lipschitz nonlinear systems; Robust observers; $H_{\infty}$ filtering; Euler discretization; LMI optimization

1. Introduction

Design of discrete-time nonlinear observers has been the subject of significant attention in recent years. See for example Califano, Monaco, and Normand-Cyrot (2003), Xiao, Kazantzis, Kravaris, and Krener (2003), Kazantzis and Kravaris (2001) and Wang and Unbehauen (2000) as well as the sampled data nonlinear observers of Moraal and Grizzle (1995), Biyik and Arcak (2006) and Laila and Astolfi (2006). The study of the nonlinear discrete-time observers is important at least for two reasons. First, most continuous-time control system designs are implemented digitally. Given that in most practical cases it is impossible to measure every state variable in real time, these controllers require the reconstruction of the states of the discrete-time model of the true continuous-time plant. Second, there are systems which are inherently discrete-time and do not originate from discretization of a continuous-time plant. Of those, discrete-time observers of continuous-time systems are particularly challenging. The reason is that exact discretization of a continuous-time nonlinear model is usually not possible to obtain. Approximate discrete-time models, on the other hand, are affected by the consequent approximation error. In this paper, we address both problems. First, we consider a class of nonlinear discrete-time systems with exact model. A nonlinear $H_{\infty}$ observer design algorithm is proposed for these systems based on an LMI approach. Then, the nonlinear sampled data system, sampled using a zero-order hold device, with Euler approximate model is considered. The Euler approximation is important because not only it is easy to derive but also it maintains the structure of the original nonlinear model. We show that by appropriate selection of one of the parameters in our proposed LMIs (the only design parameter in our algorithm), the practical convergence of the observer via Euler approximation is guaranteed as well as the robust $H_{\infty}$ cost. Our approach is based on the recent results of Arcak and Nešić (2004). See Assoudi, Yaagoubi, and Hammouri (2002) and Busawon, Saif, and Leon-Morales (1999) for other approaches. We emphasize that while the algorithms in Assoudi et al. (2002) and Busawon et al. (1999) are specifically designed...
for Euler discretization, our proposed algorithm can be applied either to the nominal exact discrete-time model or to its Euler approximation.

The LMI based observer design for uncertain discrete-time systems has been addressed in several works e.g. Xu (2002) and Lu and Ho (2004). In all these studies, the proposed LMIs are nonlinear in the Lipschitz constant and thus it cannot be considered as one of the LMI variables. In the algorithm proposed here, first the problem is addressed in the general case, then, having a bound on the Lipschitz constant, the LMIs become linear in the Lipschitz constant and we can take advantage of this feature to solve an optimization problem over it. Provided that the optimal solution is larger than the actual uncertainty is in the linear part of the model, here the uncertainty can be in the nonlinear part as well as the whole model due to approximate discretization. The paper is organized as follows: In Section 2, an observer design method for a class of nonlinear discrete-time systems is introduced. In Section 3 the practical convergence of the proposed observer via the Euler approximate models is shown. In Section 4, the results of the two previous sections will extend into the $H_\infty$ context followed by an illustrative example showing satisfactory performance of our algorithm.

2. Observer design for nonlinear discrete-time systems

We consider the following system:

\[ x_{k+1} = A_d x_k + F(x_k, u_k), \]
\[ y_k = C_d x_k, \]

(1)

(2)

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $F(x_k, u_k)$ contains nonlinearities of second order or higher. The above system can be either an inherently discrete-time system or the exact discretization of a continuous-time system. We assume that $F(x_k, u_k)$ is locally Lipschitz with respect to $x$ in a region $\mathcal{D}$, uniformly in $u$, i.e. $\forall x_{1k}, x_{2k} \in \mathcal{D}$:

\[ \| F(x_1, u^*) - F(x_2, u^*) \| \leq \gamma_d \| x_1 - x_2 \|, \]

(3)

where $\| \cdot \|$ is the induced 2-norm, $u^*$ is any admissible control sequence and $\gamma_d > 0$ is called the Lipschitz constant. If the nonlinear function $F$ satisfies the Lipschitz continuity condition globally in $\mathbb{R}^n$, then all the results in this and the ensuing sections will be valid globally. The proposed observer is in the following form:

\[ \hat{x}_{k+1} = A_d \hat{x}_k + F(\hat{x}, u) + L(y_k - C_d \hat{x}_k). \]

(4)

Defining the observer error as $e_k = \hat{x}_k - x_k$, we have

\[ e_{k+1} = (A_d - LC_d) e_k + F(\hat{x}_k, u_k) - F(x_k, u_k). \]

(5)

Our goal in this section is two-fold: (i) In the first place, we want to find an observer gain, $L$, such that the observer error dynamics is asymptotically stable. (ii) We want to maximize $\gamma_d$, the allowable Lipschitz constant of the nonlinear system. The following theorem addresses the first goal.

**Theorem 1.** Consider the system (1)–(2) with given Lipschitz constant $\gamma_d$. The observer error dynamics (5) is (globally) asymptotically stable if there exist scalar $\varepsilon > 0$, fixed matrix $Q > 0$ and matrices $P > 0$ and $G$ such that the following set of LMIs has a solution:

\[ \begin{bmatrix} P & Q \\ Q^T & 0 \end{bmatrix} > 0, \]

\[ \begin{bmatrix} A_d & C_d G \\ G^T & P \end{bmatrix} > 0, \]

\[ L(1 + \varepsilon P) + \gamma_d^2 ||e_k||^2 \leq \gamma_d^2 ||e_k||^2. \]

(6)

(7)

(8)

(9)

(10)

where $P, G, \varepsilon$ are the LMI variables and $Q$ is a design parameter to be chosen. Once the problem is solved: $L = P^{-1} G$.

**Proof.** Consider the Lyapunov function candidate as $V_k = e_k^T P e_k$, then

\[ \Delta V = V_{k+1} - V_k = e_k^T (A_d - LC_d)^T P (A_d - LC_d) e_k + 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) + (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k) - e_k^T P e_k, \]

(9)

where we use notations $F_k = F(x_k, u_k)$, $\hat{F}_k = \hat{F}(\hat{x}_k, u_k)$. Suppose $\exists P, Q > 0$ such that the following discrete-time Lyapunov equation has a solution:

\[ (A_d - LC_d)^T P (A_d - LC_d) - P = -Q. \]

(10)

Then (9) becomes

\[ \Delta V = -e_k^T Q e_k + 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) + (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k). \]

(11)

Using Rayleigh and Schwarz inequalities, we have

\[ \| e_k^T Q e_k \| \geq \lambda_{\min}(Q) \| e_k \|^2, \]

(12)

\[ 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) \]

\[ \leq \| 2e_k^T P (F_k - \hat{F}_k) \| \| A_d - LC_d \| \]

\[ = 2\gamma_d^2 \lambda_{\max}(P) \| e_k \|^2 \| A_d - LC_d \|, \]

(13)

\[ \| (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k) \| \leq \lambda_{\max}(P) \| F_k - \hat{F}_k \|^2 \]

\[ \leq 2\gamma_d^2 \lambda_{\max}(P) \| e_k \|^2. \]

(14)

So for $\Delta V < 0$ it is sufficient to have

\[ -\lambda_{\min}(Q) + \lambda_{\max}(P) [2\gamma_d^2 \| A_d - LC_d \| + \gamma_d^2] < 0. \]

(15)

Condition (15) along with (10) are sufficient conditions for asymptotic stability. We now endeavor to convert these
nonlinear inequalities into LMIs. There exists a solution for (10) if there exists \( \varepsilon > 0 \) such that

\[
(\bar{A}_d - LC_d)^T P (\bar{A}_d - LC_d) + P - \bar{Q} - \varepsilon I = (P - \bar{Q} - \varepsilon I) - (\bar{A}_d - LC_d)^T P P^{-1} (\bar{A}_d - LC_d) > 0.
\]

(16)

Using Schur’s complement lemma, defining \( G = PL \) and knowing that \( P^T = P \), the first LMI in Theorem 1 is obtained. The Lyapunov equation in (10) can be rewritten as

\[
C = \bar{A}_d - LC_d + P (\bar{A}_d - LC_d) + Q
\]

and taking into account that \( |\bar{\sigma}(A) - \bar{\sigma}(B)| \leq \bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \), we have

\[
|\bar{\sigma}(\bar{A}_d - LC_d)^T P (\bar{A}_d - LC_d) - \bar{\sigma}(Q)| \leq \bar{\sigma}(Q)
\]

\[
\Rightarrow \bar{\sigma}((\bar{A}_d - LC_d)^T P (\bar{A}_d - LC_d)) \leq \bar{\sigma}(Q) + \bar{\sigma}(P).
\]

(18)

Using Schwarz inequality:

\[
\bar{\sigma}((\bar{A}_d - LC_d)^T P (\bar{A}_d - LC_d)) \leq \bar{\sigma}^2(\bar{A}_d - LC_d) \bar{\sigma}(P).
\]

(19)

Comparing (18) and (19), a sufficient condition for (18) is

\[
\bar{\sigma}^2(\bar{A}_d - LC_d) \leq \frac{\bar{\sigma}(Q)}{\bar{\sigma}(P)}
\]

(20)

Note that since \( P \) and \( Q \) are positive definite their eigenvalues are the same as their singular values. Now, we want to find a sufficient condition for (15). Using (20):

\[
\bar{\sigma}(\bar{A}_d - LC_d) \lambda_{\text{max}}(P) \leq \sqrt{\frac{\bar{\sigma}(Q)}{2}} - \sqrt{\frac{\bar{\sigma}(P)}{2}}.
\]

(21)

For a given \( Q \), define \( g(\bar{\sigma}(P)) = \sqrt{\bar{\sigma}(P)} + \bar{\sigma}(Q) \bar{\sigma}(P) \). Since \( g(\bar{\sigma}(P)) \) is strictly increasing, there is no constant upper limit for this function but we can still bound this nonlinear function with a linear one. We have

\[
g(\bar{\sigma}(P)) < 1 + \frac{\bar{\sigma}(Q)}{\bar{\sigma}(P)} = \bar{\sigma}(P) + \frac{\bar{\sigma}(Q)}{2}.
\]

(22)

Substituting from (21) into (15), a sufficient condition for (15) is

\[
\gamma_d \lambda_{\text{max}}(P) \left( 2 \sqrt{1 + \frac{\lambda_{\text{max}}(Q)}{\lambda_{\text{max}}(P)}} + \gamma_d \right) < \lambda_{\text{min}}(Q).
\]

(23)

For any \( a, b > 0 \), \( a^2 < b^2 \) implies \( a < b \), thus, by squaring the two sides of the above inequality, substituting from (22) and after some algebra, a sufficient condition that satisfies (23) is

\[
(\gamma_d + 2) \lambda_{\text{max}}^2(P) + 2 \lambda_{\text{max}}(Q) \lambda_{\text{max}}(P)
\]

\[
\leq \frac{\lambda_{\text{min}}(Q)}{\gamma_d} \Rightarrow \lambda_{\text{max}}(P) < \Psi_1
\]

or equivalently \( \Psi_1^2 I - PP^T > 0 \) where \( \Psi_1 \) is as is in (8). This, by means of Schur’s complement lemma, is equivalent to the second LMI in Theorem 1.

We now consider the case where the Lipschitz constant of system, \( \gamma_d^* \), is less than 1. This condition is not overly restrictive since the Lipschitz constant can be reduced using a suitable coordinate transformation as discussed in Raghavan and Hedrick (1994). Furthermore, the discretized models of continuous-time systems may also satisfy this condition by appropriate selection of the sampling time. The following theorem shows that under this assumption, the LMI conditions of Theorem 1 are linear in the Lipschitz constant and thus the maximum admissible Lipschitz constant is achievable through an LMI optimization over \( \gamma_d \).

**Theorem 2.** Consider the system (1)–(2). The observer error dynamics (5) is (globally) asymptotically stable with maximum admissible Lipschitz constant \( \gamma_d^* \), if there exist scalars \( \varepsilon > 0 \), \( \xi > 1 \), fixed matrix \( Q > 0 \) and matrices \( P > 0 \) and \( G \) such that the following LMI optimization problem has a solution:

\[
\min(\xi), \quad Z > 0, \quad \begin{bmatrix} \Psi_2 I & P \\ P & \Psi_2 I \end{bmatrix} > 0,
\]

(26)

where \( \Xi \) is defined in (6) and

\[
\Psi_2 = \frac{1}{\xi} [\lambda_{\text{min}}(Q) \xi - \lambda_{\text{max}}(Q)].
\]

(27)

Once the problem is solved: \( L = P^{-1} G, \gamma_d^* = \max(\gamma_d) = 1/\xi \).

**Proof.** Having the same Lyapunov function candidate it follows that \( \Delta V \) is given by (9). Knowing \( \gamma_d < 1 \), (15) reduces to

\[
\gamma_d < \frac{\lambda_{\text{min}}(Q)}{[2\bar{\sigma}(\bar{A}_d - LC_d) + 1] \lambda_{\text{max}}(P)},
\]

(28)

where \( Q \) is the same as before. Based on (21) and (22), it can be written

\[
[2\bar{\sigma}(\bar{A}_d - LC_d) + 1] \lambda_{\text{max}}(P) < 3\bar{\sigma}(P) + \bar{\sigma}(Q).
\]

(29)

From the above, we have

\[
\frac{\lambda_{\text{min}}(Q)}{[2\bar{\sigma}(\bar{A}_d - LC_d) + 1] \lambda_{\text{max}}(P)} > \frac{\lambda_{\text{min}}(Q)}{3\bar{\sigma}(P) + \lambda_{\text{max}}(Q)}.
\]

(30)

Eventually, a sufficient condition for (28) is

\[
\gamma_d < \frac{\lambda_{\text{min}}(Q)}{3\bar{\sigma}(P) + \lambda_{\text{max}}(Q)} \Rightarrow \bar{\sigma}(P) < \frac{\lambda_{\text{min}}(Q)}{3\gamma_d} - \frac{1}{3} \bar{\sigma}(Q).
\]

(31)

which, by means of Schur’s complement lemma, is equivalent to the second LMI in Theorem 2.

**Remark 1.** The purpose of Theorem 2 is two-fold. (i) To find a gain matrix “L” that asymptotically stabilizes the observer error dynamics and (ii) to maximize \( \gamma_d^* \). Dropping the maximization of \( \gamma_d \) still renders a stable observer. In this case the
proposed LMI optimization reduces to an LMI feasibility problem (namely; satisfying the constraints) which is easier. The only parameter to be chosen in both cases is the positive definite matrix \( Q \).

We may also derive a necessary condition for the solvability of (28) for the given \( Q \) and pair \((A_d, C_d)\).

**Proposition 1.** There exists a gain matrix \( L \) which satisfies (28) if there exist scalar \( \varepsilon > 0 \), matrices \( P > 0, Q > 0 \) and \( G \) such that

\[
\begin{gathered}
-Z > 0, \\
\frac{1}{\gamma_d} \lambda_{\min}(Q)
\begin{bmatrix}
PA_d - 2GC_d + P
& \frac{1}{\gamma_d} \lambda_{\min}(Q)
\end{bmatrix}
> 0.
\end{gathered}
\]

**Proof.** The first LMI guarantees the solvability of the discrete-time Lyapunov equation which actually determines the relation between \( P \) and \( Q \).

\[
[2\sigma(A_d - LC_d) + 1] \lambda_{\max}(P)
\geq 2\sigma(PA_d - PLC_d) + \sigma(P)
\geq \sigma(2PA_d - 2PLC_d + P).
\]

Therefore, for (28) to hold, it is necessary that

\[
\gamma_d < \frac{\lambda_{\min}(Q)}{\sigma(2PA_d - 2PLC_d + P)}
\]

which is equivalent to LMI (32). \[ \square \]

**Remark 2 (Nonlinear uncertainty).** The advantage of maximization of \( \gamma_d \) is that if the maximum admissible Lipschitz constant achieved by Theorem 2, \( \gamma_d^* \), is greater than the actual Lipschitz constant of the system, \( \gamma_d \), then it is clear that the proposed observer is robust against any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal to \( \gamma_d^* - \gamma_d \).

3. Observer design for nonlinear sampled-data systems via Euler approximation

Usually, given a continuous-time nonlinear model, a closed-form solution for an exact discretization cannot be found explicitly, thus originating the need of approximate discrete-time models. A framework for nonlinear observer design based on approximate models has been recently proposed in Arcak and Nešić (2004). In this section, our focus will be on Euler approximation which is an important case because it is easy to derive and it preserves the structure of the original nonlinear model. Following the notation in Arcak and Nešić (2004), we consider the following continuous-time system

\[
\dot{x} = Ax + f(x, u), \\
y = Cx,
\]

where \( x \in \mathbb{R}^p, u \in \mathbb{R}^m, y \in \mathbb{R}^p \). We assume that the system has an equilibrium point at the origin and \( f(x, u) \) is locally Lipschitz with the Lipschitz constant \( \gamma_c \). The control input is assumed to be constant during the sampling intervals \([kT, (k + 1)T]\) (zero-order hold assumption). The family of exact discretizations of (35) is

\[
x_{k+1} = A_d^e x_k + F_d^e(x_k, u_k), \\
y_k = C_d x_k.
\]

Index \( T \) means the discretization is dependent to the sampling time. To compute (36) we need a closed-form solution of (35) over the sampling intervals, which is hard to obtain or even impossible. However, it is realistic to assume that a family of approximate discrete-time models is available

\[
x_{k+1}^a = A_d^a x_k^a + F_d^a(x_k^a, u_k), \\
y_k = C_d x_k^a.
\]

Then for the Euler approximation we have

\[
A_d^a = I + AT, \quad C_d = C, \\
F_d^a(x_k^a, u_k) = T f(x_k^a, u_k).
\]

Similar to (4), the proposed observer is

\[
\hat{x}_{k+1}^a = A_d^a \hat{x}_k^a + F_d^a(\hat{x}_k^a, u_k) + L(y_k - C_d \hat{x}_k^a).
\]

Before expressing our result, we recall two properties from Arcak and Nešić (2004), consistency and semiglobal practical convergence.

**Definition 1.** The family \( F_d^a(x, u) \) is said to be (one-step) consistent with \( F_d^a(x, u) \) if for each compact set \( \Omega \subset \mathbb{R}^p \times \mathbb{R}^m \), there exists a class- \( \mathcal{KL} \) function \( \rho(\cdot) \) and a constant \( T_0 > 0 \) such that for all \( (x, u) \in \Omega \) from and all \( T \in (0, T_0] \),

\[
\|F_d^a(x, u) - F_d^a(x, u)\| \leq \rho(T).
\]

It is straightforward that the Euler approximate model of a continuous-time Lipschitz nonlinear system is one-step consistent with the exact discrete-time model.

**Definition 2.** We say that the observer (39) is semiglobal practical in \( T \), if there exists a class- \( \mathcal{KL} \) function \( \beta(\cdot, \cdot) \) such that for any \( d_1 > d_2 > 0 \) and compact sets \( X \subset \mathbb{R}^p, U \subset \mathbb{R}^m \), we can find a \( T^* > 0 \) with the property that for all \( T \in (0, T^*) \),

\[
\|\hat{x}_k^a(0) - x(0)\| \leq d_1 \text{ and } x_k \in X, \ u_k \in U, \ \forall k \geq 0
\]

imply

\[
\|\hat{x}_k^a - x_k\| \leq \beta(\|\hat{x}_k^a(0) - x(0)\|, kT) + d_2.
\]

Based on (38), the Lipschitz constant of the Euler approximation is \( \gamma_d = \gamma_c \). Again, we assume \( \gamma_d < 1 \). This is even less restrictive than in Section 2, because here \( T \) directly multiplies \( \gamma_c \) and can be chosen sufficiently small. The following theorem shows how by the appropriate selection of \( Q \), the algorithm proposed in Theorem 2 can be used to design an observer using Euler approximate discrete-time model (38) guaranteeing the observer practical convergence when applied to the (unknown) exact model (36). In other words, the estimates provided by the
Theorem 3. The observer (39) designed using the Euler approximate model (38) is semiglobal practical in T with the maximum admissible Lipschitz constant \( \gamma_d \), if there exist scalars \( \varepsilon > 0, \xi > 1 \), fixed matrix \( Q > 0 \) and matrices \( P > 0 \) and \( G \) such that the LMI optimization problem (25)–(26) has a solution where \( \lambda_{\text{min}}(Q) = T \).

Proof. Consider the same Lyapunov function used in Theorems 1 and 2, then

\[
\|V_T(e_k) - V_T(e_{k+1})\| = \|e_k^T P e_k - e_{k+1}^T P e_{k+1}\| \\
= \|e_k^T P e_k - e_{k+1}^T P e_{k+1}\| \\
\leq \lambda_{\text{max}}(P)\|e_k - e_{k+1}\| \\
\leq \lambda_{\text{max}}(P)\|e_k\| \leq M. \tag{44}
\]

Thus, from the above:

\[
\|V_T(e_k) - V_T(e_{k+1})\| \leq M\|e_k - e_{k+1}\|. \tag{45}
\]

Similar to what we did in Section 2, we have

\[
V_T(e_{k+1}) - V_T(e_k) \leq [-\lambda_{\text{min}}(Q) + 2\lambda_{\text{max}}(P)\gamma_d + \gamma_d^2\lambda_{\text{max}}(P)]\|e_k\|^2. \tag{46}
\]

Using (28) and (20), it can be written

\[
[-\lambda_{\text{min}}(Q) + 2\lambda_{\text{max}}(P)\gamma_d + \gamma_d^2\lambda_{\text{max}}(P)]\|e_k\|^2 \\
\leq - \frac{\lambda_{\text{min}}(Q)}{2\lambda_{\text{max}}(P)}\|e_k\|^2 \leq - \frac{\lambda_{\text{min}}(Q)}{2\lambda_{\text{max}}(P)}\|e_k\|^2. \tag{47}
\]

Substituting (47) into (46) and knowing that \( \lambda_{\text{min}}(Q) = T \) and \( \gamma_d = \gamma_T \), we have

\[
\frac{V_T(e_{k+1}) - V_T(e_k)}{T} \leq - \frac{\|e_k\|^2}{2(1 + \frac{\lambda_{\text{max}}(Q)}{\lambda_{\text{min}}(Q)}) + 1}. \tag{48}
\]

Now, we define the following functions:

\[
\begin{align*}
\alpha_1(\|e_k\|) & \triangleq \lambda_{\text{min}}(P)\|e_k\|^2, \\
\alpha_2(\|e_k\|) & \triangleq \lambda_{\text{max}}(P)\|e_k\|^2, \\
\gamma_0(\|e_k\|) & \triangleq \frac{\|e_k\|^2}{2(1 + \frac{\lambda_{\text{max}}(Q)}{\lambda_{\text{min}}(Q)}) + 1}, \\
\gamma_1(\|x_k\|) & \triangleq 0, \\
\gamma_2(\|u_k\|) & \triangleq 0.
\end{align*}
\]

Then, the following can be written:

\[
\begin{align*}
\alpha_1(\|e_k\|) & \leq V_T(e_k) \leq \alpha_2(\|e_k\|), \\
\frac{V_T(e_{k+1}) - V_T(e_k)}{T} & \leq - \gamma_0(\|e_k\|) + \rho_T(T)\gamma_1(\|x_k\|) + \gamma_2(\|u_k\|),
\end{align*}
\]

where \( \alpha_1(\cdot) \), \( \alpha_2(\cdot) \), \( \gamma_0(\cdot) \) and \( \rho_T(T) \) are in class-\( \mathcal{K}_\infty \) and \( \gamma_1(\cdot) \), \( \gamma_2(\cdot) \) are nondecreasing functions. The Euler approximate model is consistent with the exact model (36). It follows that all conditions of Theorem 1 in Arcak and Nešić (2004) are satisfied and the proposed observer is semiglobally practically convergent.

Remark 3. Similar to Remark 2, in Section 2, the observer is robust against any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal to \( \gamma_c \). Note that \( Q \) is not necessarily equal to \( T \), nevertheless, it can be figured out from (48) that to have a better convergence rate, \( \lambda_{\text{max}}(Q) \) has to be sufficiently small. Therefore, we can choose \( \lambda_{\text{max}}(Q) = \lambda_{\text{min}}(Q) = T \Leftrightarrow Q = T I \).

If the nonlinear function \( f(x, u) \) is sufficiently smooth, it is possible to pursue the approach of Theorem 3 to guarantee the observer semiglobal practical convergence for higher order approximations provided that the approximate discrete-time models are Lipschitz and consistent. However, the important features of the Euler approximation of being efficiently computed, preserving the nonlinearity structure and scaling the Lipschitz constant will not be valid anymore.

4. Nonlinear \( H_\infty \) observer synthesis

In this section we extend the results of the previous sections by proposing a new nonlinear robust \( H_\infty \) observer design method. Consider the system

\[
x_{k+1} = A_d x_k + F(x_k, u_k) + B_d w_k, \tag{49}
x_k = C_d x_k, \tag{50}
\]

where \( w_k \in \ell_2[0, \infty) \) is an unknown exogenous disturbance. Suppose that

\[
z_k = H e_k \tag{51}
\]

stands for the controlled output for error state where \( H \) is a known matrix. Our purpose is to design the observer parameter \( L \) such that the observer error dynamics is asymptotically stable and the following specified \( H_\infty \) norm upper bound is simultaneously guaranteed:

\[
\|z\| \leq \mu\|w\|. \tag{52}
\]

The following theorem introduces a new method for nonlinear robust \( H_\infty \) observer design.

Theorem 4. Consider Lipschitz nonlinear system (49)–(50) with given Lipschitz constant \( \gamma_d \) along with the observer (4). The observer error dynamics is (globally) asymptotically stable with minimum \( \Sigma_2 \) gain, \( \mu^* \), if there exist scalars \( \varepsilon > 0 \) and

\[
\begin{align*}
\alpha_1(\|e_k\|) & \leq V_T(e_k) \leq \alpha_2(\|e_k\|), \\
\frac{V_T(e_{k+1}) - V_T(e_k)}{T} & \leq - \gamma_0(\|e_k\|) + \rho_T(T)\gamma_1(\|x_k\|) + \gamma_2(\|u_k\|),
\end{align*}
\]
We have 

Thus, a sufficient condition for the following LMI optimization problem has a solution:

\[
\begin{align*}
\min_{\zeta}, \\
\Xi > 0, \\
\begin{bmatrix}
\Psi_1 & P \\
P & \Psi_1
\end{bmatrix} > 0, \\
A_1 \begin{bmatrix} \frac{1}{2}(\gamma_d + 1)\Psi_1 + \lambda_{\max}(Q) \end{bmatrix}I < 0,
\end{align*}
\]

where \( \Xi \) and \( \Psi_1 \) are as in (6) and (8), respectively, and

\[A_1 = H^T H - Q + \gamma_d[3\Psi_1 + \lambda_{\max}(Q)].\]

Once the problem is solved, \( L = P^{-1}G \) and \( \mu = \min(\mu) = \sqrt{\zeta}. \)

**Proof.** Consider the same Lyapunov function candidate as before, thus,

\[
\Delta V = e_k^T(A_d - L C_d)T P (A_d - L C_d) e_k \\
+ 2e_k^T(A_d - L C_d)T P (F_k - \tilde{F}_k) - e_k^T P e_k \\
+ (F_k - \tilde{F}_k)^T P (F_k - \tilde{F}_k) + 2 w_k^T B_d^T P (A_d - L C_d) e_k \\
+ 2 w_k^T B_d^T P (F_k - \tilde{F}_k) e_k + w_k^T B_d^T P B_d w_k,
\]

where the first four terms are the same as those found in the proof of Theorem 1, and the next three terms are due to the disturbance \( w \). If \( w = 0, \Delta V \) is given by (9) so the LMIIs (53) and (54) guarantee the asymptotic stability. If \( w \neq 0 \), we have that

\[
\begin{align*}
w_k^T B_d^T P (A_d - L C_d) e_k &\leq w_k^T B_d^T \bar{\sigma}(P) \tilde{\sigma}(A_d - L C_d) e_k, \\
w_k^T B_d^T P (F_k - \tilde{F}_k) &\leq w_k^T B_d^T \bar{\sigma}(P) \gamma_d e_k.
\end{align*}
\]

From the above and using (13), (14), (21) and (22), it can be written

\[
\Delta V \leq e_k^T[-Q + \gamma_d(3\lambda_{\max}(P) + \lambda_{\max}(Q))]e_k \\
+ w_k^T B_d^T \bar{2}\lambda_{\max}(P)(\gamma_d + 1) + \lambda_{\max}(Q) e_k \\
+ w_k^T B_d^T P B_d w_k.
\]

Now, define \( J \triangleq \sum_{k=0}^{\infty} [z_k^T z_k - \mu^2 w_k^T w_k + \Delta V] \); so,

\[
J < \sum_{k=0}^{\infty} [z_k^T z_k - \mu^2 w_k^T w_k + \Delta V].
\]

Thus, a sufficient condition for \( J \leq 0 \) is that

\[
\forall k \in [0, \infty), \quad z_k^T z - \mu^2 w_k^T w + \Delta V \leq 0.
\]

We have

\[
\begin{align*}
z_k^T z_k - \mu^2 w_k^T w_k + \Delta V &\leq e_k^T[H^T H - Q + \gamma_d(3\lambda_{\max}(P) + \lambda_{\max}(Q))]e_k \\
&+ w_k^T B_d^T \bar{2}\lambda_{\max}(P)(\gamma_d + 1) + \lambda_{\max}(Q) e_k \\
&+ w_k^T (B_d^T P B_d - \mu^2 I) w_k.
\end{align*}
\]

So a sufficient condition for \( J \leq 0 \) is that the right-hand side of the above inequality be negative. Then, \( z_k^T z - \mu^2 w_k^T w \leq 0 \Rightarrow \|z_k\| \leq \mu \|w_k\| \). Substituting \( \lambda_{\max}(P) \) from (24) into (58) and defining \( \zeta = \mu^2 \), the LMI (55) is obtained. \( \square \)

Furthermore, once again we consider a discrete-time system with \( \gamma_d < 1 \) and extend the result of Theorem 2, to the \( H_{\infty} \) context. According to Theorem 3, the case of Euler discretization can always be assumed to accommodate to this case by suitable selection of the sampling period, \( T \).

**Theorem 5.** Consider Lipschitz nonlinear system (49)–(50) with given Lipschitz constant \( \gamma_d < 1 \), along with the observer (4). The observer error dynamics is (globally) asymptotically stable with minimum \( \mathcal{H}_2 \) gain, \( \mu^* \), if there exist scalars \( \varepsilon > 0 \) and \( \zeta > 0 \), fixed matrix \( Q > 0 \) and matrices \( P > 0 \) and \( G \) such that the following LMI optimization problem has a solution:

\[
\min_{\zeta}, \\
\Xi > 0, \\
\begin{bmatrix}
A_2 I & P \\
P & A_2 I
\end{bmatrix} > 0, \\
\begin{bmatrix}
A_4 I & \frac{1}{\gamma_d} A_3 I \\
\frac{1}{\gamma_d} A_3 I & B_d^T P B_d - \zeta I
\end{bmatrix} < 0,
\]

where \( \Xi \) is as in (6) and

\[
A_2 = \frac{\lambda_{\min}(Q)}{3\gamma_d} - \frac{1}{3} \lambda_{\max}(Q), \\
A_3 = \left(1 + \frac{1}{\gamma_d}\right) \lambda_{\min}(Q) + \frac{1}{2}(1 - 2\gamma_d) \lambda_{\max}(Q), \\
A_4 = H^T H - Q + \left[(1 + 2\gamma_d) A_3 + \left(1 + \gamma_d^2 \right) \lambda_{\max}(Q) \right].
\]

Once the problem is solved, \( L = P^{-1}G, \mu^* = \min(\mu) = \sqrt{\zeta}. \)

**Proof.** The proof is similar to the proof of Theorem 4 and thus omitted. \( \square \)

**Remark 4.** We can also first maximize the admissible Lipschitz constant using Theorem 2, and then minimize \( \mu \) for the maximized \( \gamma_d \), using Theorem 5. In this case, according to Remark 2, robustness against nonlinear uncertainty is also guaranteed.

**Example 1.** Consider the Van der Pol oscillator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_1^2 x_2 \end{bmatrix} w, \\
y = [0 \ 1]x,
\]

where \( w \) is an unknown Gaussian disturbance with zero mean and standard deviation 0.05. This system has an unstable equilibrium point at the origin and a stable limit cycle. The system
We assume $H = 0.25I$, and $Q = TI$, and use the Euler approximate model to design observer (39). With $T = 0.1\text{s}$ we get $\gamma_0^2 = 0.7824$, $\mu^4 = 0.8756$ and $L = [-0.8382 \ 1.0938]^T$. With $T = 0.05\text{s}$ we get $\gamma_0^2 = 0.4352$, $\mu^4 = 0.9097$ and $L = [-0.9231 \ 1.0962]^T$. It can be verified using phase plane analysis that for both cases the stable limit cycle falls inside the region $r_c \leq r_c^*$. Therefore, for any initial condition inside the limit cycle, the trajectories will remain inside the region for which the observer asymptotic stability is guaranteed. Fig. 1 shows the state trajectories of the continuous-time system along with their estimations. Simulation is done for 20 s in the presence of disturbance and shows the high performance of the observer with the Euler approximate model.

5. Conclusion

In this paper, a new algorithm for robust $H_\infty$ nonlinear observer design for nonlinear discrete-time systems was proposed. The observer is robust in the sense that it can achieve convergence to the true state, despite nonlinear model uncertainty with optimized exogenous disturbance rejection ratio. When the exact discrete-time model of the system is not available, the same algorithm can still be used for the Euler approximate model.

References


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