Quasi-exponential input-to-state stability for discrete-time impulsive hybrid systems

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This paper investigates the problem of input-to-state stability (ISS) for the general discrete-time impulsive hybrid systems. Quasi-exponential input-to-state stability criteria are established for non-linear discrete-time impulsive hybrid systems, based on the Lyapunov function. The results are then specialized to the linear discrete-time impulsive hybrid systems and fairly simple algebraic conditions are derived. Moreover, ISS small-gain properties are also investigated for a class of interconnected discrete-time impulsive systems. Several examples are given to illustrate results.

1. Introduction

Stability analysis of non-linear systems with disturbance inputs is one of a number of very important problems in control theory and engineering. It is well known that external disturbance inputs often lead to the failure of stability for a stable system. ISS analysis aims to investigate how external disturbance inputs affect the system stability. Since the notion of ISS was proposed in Sontag (1989), ISS property analysis of non-linear systems with disturbance inputs has quickly become an active research topic in non-linear feedback analysis and design; see Coron et al. (1995), Teel et al. (1988), Isidori (1999), Jiang and Wang (2001), Jiang et al. (2004), Nešić and Teel (2004) and Laila and Astolfi (2005) and the references therein. Moreover, ISS has been successfully employed in the stability analysis and control synthesis of non-linear systems with disturbance inputs and complex structure; see Teel et al. (1988), Michel (1999), Marquez (2003), Nešić and Teel (2004), Laila and Astolfi (2005) and Huang et al. (2005) and the references therein.

Impulsive hybrid systems represent a type of hybrid system with complex structure. They arise naturally in a wide variety of practical systems and applications. Examples, including evolutionary processes such as biological neural networks (Guan and Chen 1999), ecosystems management (Neuman and Costanza 1990), optimal control models in economics (Bensoussan and Tapiero 1982), and flying object motions (Liu and Willms 1996, Masutani et al. 2001), etc., are characterized by abrupt changes of states at certain times. Those sudden and sharp changes are often of very short duration and are thus assumed to occur instantaneously in the form of impulses. Such impulses may be represented by discrete maps. Systems undergoing abrupt changes may not be well described by using purely continuous or purely discrete models. However, they can be appropriately modelled by impulsive hybrid systems. It is now recognized that the theory of impulsive hybrid systems provides a natural framework for mathematical modelling of many such real world phenomena. The solutions of impulsive hybrid systems are in general discontinuous, which renders most of the standard methods ineffective. Thus new methods and techniques have to be developed to deal with problems arising from impulsive hybrid systems.

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Both the theoretical challenges and the potential applications to real world problems have motivated much of the recent research on impulsive hybrid systems. In recent years, significant progress has been made in the theory of hybrid systems in which the impulses occur in continuous systems at some instances, see Lakshmikantham et al. (1989), Liu (1994), Lakshmikantham and Liu (1998), Ye et al. (1998), Michel (1999) and Li et al. (2000a, b, 2001, 2002), Liu et al. (2003, 2005, 2006) and the references therein. However, the corresponding theory for discrete-time impulsive hybrid systems, in which the impulses occur in a discrete-time system, in some instances, has not yet been developed. Moreover, to the best of our knowledge, no ISS analysis has been previously reported for the discrete-time impulsive hybrid systems. The objective of this paper is to study the ISS problem for discrete-time impulsive hybrid systems. We propose ISS and quasi-exponential ISS notions for discrete-time impulsive hybrid systems and establish quasi-exponential ISS criteria for this kind of system. ISS small-gain theorems are also studied for a class of interconnected discrete-time impulsive hybrid systems. Moreover, we also show that impulses can improve the stability properties of a discrete-time hybrid system, by rendering the system ISS.

The rest of this paper is organized as follows. In §2, we introduce some notations and definitions. In §3–5, we establish quasi-exponential ISS for non-linear discrete-time impulsive hybrid systems. Then, we specialize the results to linear impulsive hybrid systems. Finally, we study the small-gain properties for a class of interconnected discrete-time impulsive hybrid systems. In §6, we discuss some examples to illustrate our results.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real number, $\mathbb{R}^n$ the $n$-dimensional real vector space, $\mathbb{N}$ the set of non-negative integers, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{R}_{+} = [0, +\infty)$. A function $\gamma: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is of class-$K$ ($\gamma \in K$) if it is continuous, zero at zero and strictly increasing. It is of class-$K_{\infty}$ if it is of class-$K$ and unbounded. A continuous function $\beta: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is of class-$K_{\infty}$ if $\beta(t, \cdot)$ is of class-$K$ for each $t \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$. For any $A \in \mathbb{R}^{n \times n}$, $\|A\|$ denotes the norm of a matrix $A$ induced by the Euclidean norm, i.e., $\|A\| = \sqrt{\text{tr}(A^TA)}$. For any function $u: \mathbb{N} \rightarrow \mathbb{R}^n$, we denote $\|u\|_{\infty} = \sup\{|u(k)|, k \in \mathbb{N}\}$. Let $\lambda_{\max}(X)$, (respectively, $\lambda_{\min}(X)$) the maximum (respectively, minimum) eigenvalue of the matrix $X$. For any $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, 2, \ldots, n$.

The discrete-time impulsive hybrid system can be described as

$$
\begin{align*}
\begin{cases}
\dot{x}(n) &= f(x(n), u(n)), &n \neq N_k, \\
\Delta x(n) &= I_k(x(n), u_d(n)), &n = N_k,
\end{cases}
\end{align*}
$$

where $x(n) \in \mathbb{R}^n$ are the states; $f \in C([\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n], \mathbb{R}^n)$, are known continuous functions; $u_k(n) \in \mathbb{R}^n$ and $u_d(n) \in \mathbb{R}^n$ are external disturbance inputs, and $\Delta x(n+1) = x(n+1) - x(n)$, $n = N_k$, and the sequence $\{N_k\}$ satisfies: $N_k \in \mathbb{N}$ and $0 \leq N_0 < N_1 < \cdots < N_k < \cdots$, with $\lim_{k \to \infty} N_k = \infty$, and $\Delta_{k+1} \neq N_{k+1} - N_k > 1, k \in \mathbb{N}$. Let $x(n) \Delta x(n, x_0, u_c, u_d)$ be the solution of system (1) with initial condition $x(N_0) = x_0$.

Remark 1: If there exists a positive integer $k_0$ such that $\Delta_{k_0+1} = +\infty$, then system (1) becomes a normal discrete-time system with the initial point $(N_0 = N_{k_0}, x_0)$, for which ISS has been previously discussed in the literatures (see Jiang and Wang (2001) and Jiang et al. (2004)). In this paper, we study the ISS problem for discrete-time impulsive hybrid system under the following assumption.

Assumption 1:

$$
\Delta_{\sup} \triangleq \sup_{k \in \mathbb{N}} \{\Delta_k\} < \infty.
$$

Definition 1: The system (1) is said to be globally ISS if there exist a $K\mathcal{L}$-function $\beta: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ and $K$-functions $\gamma_c$ and $\gamma_d$ such that, for every input $(u_c, u_d) \in U$ and each $\xi \in \mathbb{R}^n$,

$$
\|x(n, \xi, u_c, u_d)\| \leq \beta(\|\xi\|, n - N_0) + \gamma_c(\|u_c\|_\infty) + \gamma_d(\|u_d\|_\infty),
$$

$$
n \in \mathbb{N}.
$$

Remark 2: Property (3) of above definition is equivalent to the following property: there exist a $K\mathcal{L}$-function $\beta: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ and $K$-functions $\gamma_c$ and $\gamma_d$ such that, for every input $(u_c, u_d) \in U$ and each $\xi \in \mathbb{R}^n$,

$$
\|x(n, \xi, u_c, u_d)\| \leq \max_{n \geq N_0, n \in \mathbb{N}} \left\{ \beta(\|\xi\|, n - N_0), \gamma_c(\|u_c\|_\infty), \gamma_d(\|u_d\|_\infty) \right\}.
$$

Condition (4) is frequently used in the literatures (see for example Jiang and Wang (2001) and Jiang et al. (2004)) instead of (3) in the definition of ISS.
Definition 2: The system (1) is said to be globally quasi-exponential ISS if there exist constants \( \alpha > 0, K > 0, \) and \( K \)-functions \( \gamma_c, \) and \( \gamma_d, \) such that, for any \( n \in (N_k, N_{k+1}), k \in \mathbb{N}, \) and every input \((u_c, u_d) \in U,\)

\[
\|x(n, z, u_c, u_d)\| \leq Ke^{-\alpha_k}\|x(N_0)\| + \gamma_c(\|u_c\|_\infty) + \gamma_d(\|u_d\|_\infty).
\]

(5)

Remark 3: In inequality (5), if \( k \) is replaced by \( n, \) then the system is said to be globally exponential ISS. From the fact that \( k \to \infty \) if and only if \( n \to \infty, \) the Definition 2 is well defined.

Lemma 1: For any integer \( l \geq 1, \) and any \( \xi(k), n(k), z(k) \in \mathbb{R}^l, X_k, Y_k, W_k \in \mathbb{R}^{p \times l}, \) if

\[
\xi(k + 1) \leq X_k\xi(k) + Y_kn(k) + W_kz(k), \quad k \in \mathbb{N},
\]

then, the following inequality holds:

\[
\xi(k + 1) \leq \sum_{j=k}^{N_k} G_k(X_j, Y_j)\eta(j)
\]

(7)

where, \( \prod_{i=k}^{N_k} X_j = X_kX_{k-1} \cdots X_1X_0, \) and let \( P_j = \{Y_j, W_j\}, \) and

\[
G_k(X, P) = \begin{cases}
    P_k, & j = k; \\
    \prod_{i=k}^{j-1} X_iP_i, & 0 \leq j \leq k - 1.
\end{cases}
\]

(8)

Proof: It can be derived by using the induction principle. The details are omitted.

Lemma 2: Let constants \( p > 0, a > 0, b > 0, \) then

\[
(a + b)^p \leq \max\{2^{p-1}, 1\}(a^p + b^p).
\]

(9)

Proof: When \( p = 1, \) equality (9) obviously holds. When \( p > 1, \) we prove \( (a + b)^p \leq 2^{p-1}(a^p + b^p). \) Denote \( q = 1/p. \) By H"older’s inequality, for any \( \rho > 0, \) we have

\[
(a + b)^p \leq \left(1 + a + \frac{b^p}{\rho}\right)^p \leq \left(1 + a + \frac{b^p}{\rho}\right)^{1/p} \left(1 + a + \frac{b^p}{\rho}\right)^{p-1} \leq (1 + a + \frac{b^p}{\rho})^{p-1} \cdot \left(1 + \frac{b^p}{\rho}\right).
\]

(10)

Let \( \rho = 1, \) then (10) implies that \( (a + b)^p \leq 2^{p-1}(a^p + b^p) \) for any \( p > 1. \) When \( p < 1, \) we prove \( (a + b)^p \leq a^p + b^p. \)

For a fixed \( b > 0, \) let function \( F(s) \) be \( F(s) = s^p + b^p - (s + b)^p, \) \( s \in \mathbb{R}_+. \) Since \( p < 1, \) then \( F(s) = p(s^{p-1} - (s + b)^{p-1}) \geq 0. \) Thus, function \( F(s) \) is non-decreasing and hence \( F(a) \geq F(0) = 0, \) which implies \( (a + b)^p \leq a^p + b^p \) for \( p < 1. \) Hence, inequality (9) holds and the proof is complete.

3. Quasi-exponential ISS for non-linear discrete-time impulsive hybrid systems

The Lyapunov function method is an important approach to the analysis of stability of dynamical systems. In this section, by using the Lyapunov function method, we establish ISS criteria for non-linear discrete-time impulsive system (1).

Theorem 1: Suppose Assumption 1 holds and furthermore assume that there exists a Lyapunov function \( V(x) \) such that the following conditions are satisfied:

(i) there exist constants \( c_1 > 0, c_2 > 0 \) and \( p > 0 \) such that for any \( x \in \mathbb{R}^n, \)

\[
c_1\|x\|^p \leq V(x) \leq c_2\|x\|^p;
\]

(11)

(ii) for every \( n \in (N_k, N_{k+1}), k \in \mathbb{N}, \) there exist \( K \)-function \( \phi_{1k} \) and \( K \)-function \( \gamma_1 \) such that

\[
V(x(n + 1)) - V(x(n)) \leq -\phi_{1k}(V(x(n))) + \gamma_1(\|u_i(n)\|),
\]

(12)

where \( \phi_{1k} \) satisfies \( \phi_{1k}(s) \leq cs \) for some constant \( \leq c < 1 \) and any \( s \in \mathbb{R}_+;\)

(iii) for every \( n = N_k, k \in \mathbb{N}, \) there exist \( K \)-functions \( \phi_{2k} \) and \( \gamma_2 \) such that

\[
V(x(N_k + 1)) \leq \phi_{2k}(V(x(N_k))) + \gamma_2(\|u_i(N_k)\|); \quad (13)
\]

(iv) there exist constants \( \sigma_k > 0 \) such that, for every \( k \in \mathbb{N}, \)

\[
\phi_{2k}(V(x(N_k))) \leq \sigma_k V(x(N_k)) + \sum_{i=1}^{\Delta x_k+1} \phi_{1k}(V(x(N_k + i))),
\]

(14)

where \( \sigma_k, (k \in \mathbb{N}) \) satisfying \( \prod_{i=k}^{x_k} \sigma_i = e^{-\alpha k + \theta_k}, \) with \( \theta_k \) satisfying

\[
\lim_{k \to \infty} \frac{\theta_k}{k} < \alpha, \quad \sup_{k \in \mathbb{N}} \{\theta_k - \theta_{k-1}\} < \infty,
\]

(15)

where \( \theta_{k-1} = \min_{0 \leq i \leq k-1} \{\theta_i\}, \) for any \( k \in \mathbb{N}. \)
Then, the discrete-time impulsive hybrid system (1) is globally quasi-exponential ISS.

**Proof:** By condition (ii), we get that, for any $n \in (N_k, N_{k+1})$,

\[
V(x(n + 1)) \leq V(x(n)) - \phi_{ik}(V(x(n))) + \gamma_1(||u_c(n)||)
\]
\[
\leq V(x(n - 1)) - \phi_{ik}(V(n - 1)) - \phi_{ik}(V(n)) + \gamma_1(||u_c(n - 1)||) + \gamma_1(||u_c(n)||)
\]
\[
\leq \cdots
\]
\[
\leq V(x(N_k + 1)) - \sum_{i=N_k+1}^{n} \phi_{ik}(V(x(i)))
\]
\[
+ \sum_{i=N_k+1}^{n} \gamma_1(||u_c(i)||). \tag{16}
\]

It follows from (16) and conditions (iii) and (iv) that

\[
V(x(N_{k+1})) \leq V(x(N_k + 1)) - \sum_{i=N_k+1}^{N_{k+1}-1} \phi_{ik}(V(x(i)))
\]
\[
+ \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||)
\]
\[
\leq \phi_{2k}(V(x(N_k))) + \gamma_2(||u_d(N_k)||)
\]
\[
- \sum_{i=1}^{\Delta_{k+1}-1} \phi_{ik}(V(x(N_k + i)))
\]
\[
+ \sum_{i=1}^{\Delta_{k+1}-1} \gamma_1(||u_c(N_k + i)||)
\]
\[
\leq \sigma_k V(x(N_k)) + \gamma_2(||u_d(N_k)||)
\]
\[
+ \sum_{i=1}^{\Delta_{k+1}-1} \gamma_1(||u_c(N_k + i)||), \quad k \in \mathbb{N}. \tag{17}
\]

Denote $\sigma! \Delta \prod_{i=k}^{\Delta_{k+1}-1} \sigma_i$. By Lemma 1 and condition (iv), we get that

\[
V(x(N_{k+1})) \leq \sigma_k! V(x(N_k)) + \sum_{j=0}^{k} \frac{\sigma_j!}{\sigma_j^j} \gamma_2(||u_d(N_j)||)
\]
\[
+ \sum_{j=0}^{\Delta_{k+1}-1} \frac{\sigma_j!}{\sigma_j^j} \gamma_1(||u_c(N_j + i)||)
\]
\[
\leq e^{-\Delta_{k+1}} V(x(N_k))
\]
\[
+ \left[ 1 + e^{\theta_{k+1} - \theta_{k+1}} \frac{1 - e^{-\theta_k}}{\alpha - 1} \right] \gamma_2(||u_d||_{\infty})
\]
\[
+ (\Delta_{\sup} - 1) \left[ 1 + e^{\theta_{k+1} - \theta_{k+1}} \frac{1 - e^{-\theta_k}}{\alpha - 1} \right] \gamma_1(||u_c||_{\infty}). \tag{18}
\]

By (16) and conditions (iii) and (iv), we have

\[
V(x(N_{k+1} - 1)) \leq V(x(N_k + 1)) - \sum_{i=N_k+1}^{N_{k+1}-2} \phi_{ik}(V(x(i)))
\]
\[
+ \sum_{i=N_k+1}^{N_{k+1}-2} \gamma_1(||u_c(i)||)
\]
\[
\leq \phi_{2k}(V(x(N_k))) + \gamma_2(||u_d(N_k)||)
\]
\[
- \sum_{i=1}^{N_{k+1}-1} \phi_{ik}(V(x(i))) + \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||)
\]
\[
\leq \sigma_k V(x(N_k)) + \gamma_2(||u_d(N_k)||)
\]
\[
+ \phi_{ik}(V(x(N_k + 1))) + \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||). \tag{19}
\]

It follows from (19) and $\phi_{ik}(s) \leq cs$ that

\[
V(x(N_{k+1} - 1)) \leq \frac{\sigma_k}{1 - c} V(x(N_k)) + \frac{1}{1 - c} \gamma_2(||u_d(N_k)||)
\]
\[
+ \frac{1}{1 - c} \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||) \tag{20}
\]

For any $1 \leq i \leq \Delta_{k+1} - 1$, by the induction, we have

\[
V(x(N_{k+1} - i)) \leq V(x(N_k + 1)) - \sum_{i=N_k+1}^{N_{k+1}-i-1} \phi_{ik}(V(x(i)))
\]
\[
+ \sum_{i=N_k+1}^{N_{k+1}-i-1} \gamma_1(||u_c(i)||)
\]
\[
\leq \sigma_k V(x(N_k)) + \gamma_2(||u_d(N_k)||)
\]
\[
+ \sum_{i=N_k+1}^{N_{k+1}-1} \phi_{ik}(V(x(i))) + \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||)
\]
\[
\leq \sigma_k V(x(N_k)) + \frac{1}{1 - c} \gamma_2(||u_d(N_k)||)
\]
\[
+ \frac{1}{1 - c} \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||)
\]
\[
\leq \lambda_c(c) \left[ \sigma_k V(x(N_k)) + \gamma_2(||u_d(N_k)||) + \sum_{i=N_k+1}^{N_{k+1}-1} \gamma_1(||u_c(i)||) \right], \tag{21}
\]

where $\lambda_c(c) = (1/(1 + c))(1 + c/(1 - c))^{i-1}$, for $1 \leq i \leq \Delta_{k+1} - 1$. 

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Hence, for any \( n \in (N_k, N_{k+1}], k \in \mathbb{N}, \) from (18), (21), and \( 1 < \lambda_i(c) < \lambda_{\Delta n-1}(c), \) we get that

\[
V(n) \leq \lambda_{\Delta n-1}(c) \left[ \sum_{i=N_k+1}^{n} \gamma_1(\|u_\ell(i)\|) + \sum_{j=0}^{k} \frac{\sigma_j}{\sigma_i} \gamma_2(\|u_d(N_j)\|) + \sum_{i=1}^{\Delta_{\Delta n-1}} \gamma_1(\|u_c(N_j + \ell)\|) \right] \\
\leq \lambda_{\Delta n-1}(c) \left\{ \sum_{i=N_k+1}^{n} 1 + \sum_{j=0}^{k} \frac{\sigma_j}{\sigma_i} \gamma_2(\|u_d(N_j)\|) + \sum_{i=1}^{\Delta_{\Delta n-1}} \gamma_1(\|u_c(N_j + \ell)\|) \right\} \\
\leq \lambda_{\Delta n-1}(c) \left\{ e^{-\alpha k + \theta_k} V(n_0) + \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_2(\|u_d\|_{\infty}) + (\Delta_{\Delta n-1} - 1) \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_1(\|u_c\|_{\infty}) \right\}.
\]

(22)

Thus, by condition (i), (22) and Lemma 2, for any \( n \in (N_k, N_{k+1}], k \in \mathbb{N}, \) we obtain that

\[
\|x(n)\| \leq \left( \frac{c 2 \lambda_{\Delta n-1}(c)}{\epsilon_1} \right)^{1/p} \max \left\{ \gamma_1(\|u_d\|_{\infty}), \gamma_2(\|u_d\|_{\infty}) \right\} \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_2(\|u_d\|_{\infty}) + (\Delta_{\Delta n-1} - 1) \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_1(\|u_c\|_{\infty}),
\]

(23)

where \( \hat{\theta} \Delta \lim_{k \to \infty} \theta_k/k < \alpha, \hat{\theta} \Delta \sup_{k \in \mathbb{N}} \theta_k - \theta_{k-1} < \infty. \)

Hence, the system (1) is globally quasi-exponential ISS with gain \((\gamma_\epsilon, \gamma_d), \) where

\[
\gamma_\epsilon(s) = \lambda_{\Delta n-1}(c) (\Delta_{\Delta n-1} - 1)^{1/p} \left[ \max \left\{ \gamma_1, \gamma_2 \right\} \right]^2 \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_1(\|u_c\|_{\infty}),
\]

and

\[
\gamma_d(s) = \lambda_{\Delta n-1}(c) \left[ \max \left\{ \gamma_1, \gamma_2 \right\} \right]^2 \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_2(\|u_c\|_{\infty}).
\]

The proof is complete. □

**Theorem 2:** Suppose Assumption 1 holds and furthermore assume that there exists a Lyapunov function \( V(x) \) such that the condition (i) and (iii) of Theorem 1 hold while conditions (ii) and (iv) are replaced by

\[(ii^*) \text{ for every } n \in (N_k, N_{k+1}], k \in \mathbb{N}, \text{ there exist } K\text{-functions } \phi_{1k} \text{ and } \gamma_1 \text{ such that} \]

\[
V(x(n+1)) - V(x(n)) \leq \phi_{1k}(V(x(n))) + \gamma_1(\|u_c(n)\|); \tag{24}
\]

\[(iv^*) \text{ there exist constants } \sigma_k > 0 \text{ such that, for every } k \in \mathbb{N}, \]

\[
\phi_{2k}(V(x(N_k))) \leq \sigma_k V(x(N_k)) - \sum_{i=1}^{\Delta_{\Delta n-1}} \phi_{1k}(V(x(N_k + i)));
\]

(25)

where \( \sigma_k, (k \in \mathbb{N}) \) satisfies: \( \prod_{i=0}^{\Delta_{\Delta n-1}} \sigma_i = e^{-\alpha k + \theta_k}, \) with \( \theta_k \) satisfying (15).

Under these conditions, system (1) is globally quasi-exponential ISS.

**Proof:** By similar proof of Theorem 1, we obtain that, for any \( n \in (N_k, N_{k+1}], k \in \mathbb{N}, \)

\[
\|x(n)\| \leq \left( \frac{c 2 \lambda_{\Delta n-1}(c)}{\epsilon_1} \right)^{1/p} \max \left\{ \gamma_1(\|u_d\|_{\infty}), \gamma_2(\|u_d\|_{\infty}) \right\} \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_2(\|u_d\|_{\infty}) + (\Delta_{\Delta n-1} - 1)^{1/p} \left[ \max \left\{ \gamma_1(\|u_d\|_{\infty}), \gamma_2(\|u_d\|_{\infty}) \right\} \right]^2 \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_1(\|u_c\|_{\infty}),
\]

(26)

where \( \hat{\theta} \Delta \lim_{k \to \infty} \theta_k/k < \alpha, \hat{\theta} \Delta \sup_{k \in \mathbb{N}} \theta_k - \theta_{k-1} < \infty. \)

Hence, system (1) is globally quasi-exponential ISS with gain \((\gamma_\epsilon, \gamma_d), \) where \( \gamma_\epsilon(s) = (\Delta_{\Delta n-1} - 1)^{1/p} \left[ \max \left\{ \gamma_1(\|u_d\|_{\infty}), \gamma_2(\|u_d\|_{\infty}) \right\} \right]^2 \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_1(\|u_c\|_{\infty}), \) and

\[
\gamma_d(s) = (\Delta_{\Delta n-1} - 1)^{1/p} \left[ \max \left\{ \gamma_1(\|u_d\|_{\infty}), \gamma_2(\|u_d\|_{\infty}) \right\} \right]^2 \\
\times \left[ 1 + e^{\theta_k - \theta_{k-1}} \frac{1 - e^{-\alpha k}}{e^{\alpha k} - 1} \right] \gamma_2(\|u_c\|_{\infty}).
\]

The proof is complete. □

**Theorem 3:** Suppose condition (i) of Theorem 1 holds and furthermore assume that there exist constants \( 0 < \epsilon_k \leq a, 0 < b_k \leq b, 0 < c_k \leq c, 0 < d_k \leq d, \) \( k \in \mathbb{N}, \) with such that the following conditions hold:

\[(ii) \text{ for every } n \in (N_k, N_{k+1}], k \in \mathbb{N}, \]

\[
V(x(n+1)) \leq a_k V(x(n)) + b_k \|u_c(n)\|; \tag{27}
\]

\[(iii) \text{ for every } n = N_k, k \in \mathbb{N}, \]

\[
V(x(N_k + 1)) \leq c_k V(x(N_k)) + d_k \|u_d(N_k)\|; \tag{28}
\]
(iv) there exist constants $\sigma > 0$ and $\theta_k (k \in \mathbb{N})$ such that
\[
\prod_{i=k}^{1} c_i a_{\gamma_{i}}^{N_{\gamma_{i}}-1} = e^{-\sigma \gamma + \theta_k},
\]
where $\theta_k$ satisfying (15).

Then, the system (1) is globally quasi-exponential ISS.

**Proof:** By condition (ii) and Lemma 1, it is easy to get that for any $n \in (N_k, N_{k+1})$,
\[
V(x(n+1)) \leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} V(x(N_k)) + h_{N_{\gamma_{k}}-1} a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k \|u_d\|_\infty,
\]
where $h_m(a_k) = a^\alpha_k + a^\beta_k + \cdots + a_k + 1$.

Hence, for any $k \in \mathbb{N}$, by condition (iii) and (30), we get
\[
V(x(N_{k+1})) \leq a_{\gamma_{k+1}}^{N_{\gamma_{k+1}}-1} c_k V(x(N_k)) + a_{\gamma_{k+1}}^{N_{\gamma_{k+1}}-1} d_k \|u_d\|_\infty
\]
\[
+ h_{N_{\gamma_{k+1}}-1} a_{\gamma_{k+1}}^{N_{\gamma_{k+1}}-1} d_k \|u_c\|_\infty
\]
\[
\Delta = a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k + \|u_c\|_\infty + \gamma_k \|u_c\|_\infty,
\]
where $\alpha_k = a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k$, $\beta_k = a_{\gamma_{k+1}}^{N_{\gamma_{k+1}}-1} d_k$, $\gamma_k = h_{N_{\gamma_{k}}-1} a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k$.

By Lemma 1, we have
\[
V(x(N_{k+1})) \leq \prod_{j=k}^{0} \alpha_j V(x(N_0)) + \sum_{j=0}^{k} g_k(\alpha, \beta_j) \|u_d\|_\infty
\]
\[
+ \sum_{j=0}^{k} g_k(\alpha, \gamma_j) \|u_c\|_\infty,
\]
where, let $c_k! = \prod_{i=k}^{0} c_i$, and
\[
g_k(\alpha, \beta_j) = \left\{ \begin{array}{ll}
\beta_j, & j = k \\
\prod_{i=j+1}^{k} \alpha_i \beta_j, & 0 \leq j \leq k - 1
\end{array} \right.
\]
\[
= c_k! \prod_{i=j}^{0} a_{\gamma_{i}}^{N_{\gamma_{i}}-1} a_{\alpha_{i}}^{N_{\alpha_{i}}-1} d_{j},
\]
\[
g_k(\alpha, \gamma_j) = \left\{ \begin{array}{ll}
\gamma_j, & j = k \\
\prod_{i=j+1}^{k} \alpha_i \gamma_j, & 0 \leq j \leq k - 1
\end{array} \right.
\]
\[
= c_k! \prod_{i=j}^{0} a_{\gamma_{i}}^{N_{\gamma_{i}}-1} a_{\gamma_{i}}^{N_{\gamma_{i}}-1} h_{N_{\gamma_{i}}-N_{\gamma_{i}}-2}(a_j) b_j.
\]

Noting that $a_j \leq a$, we have
\[
h_{N_{\gamma_{j+1}}-N_{\gamma_{j}}}(a_j) = a_{\gamma_{j+1}}^{N_{\gamma_{j+1}}-1} + \cdots + a_j + 1
\]
\[
\leq H(a) \Delta \left\{ \begin{array}{ll}
1, & a < 1; \\
\Delta_{\sup} - 1, & a \geq 1.
\end{array} \right.
\]

Thus, by (30)–(33), we get that for any $n \in (N_k, N_{k+1})$,
\[
V(x(n+1)) \leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} V(x(N_k)) + h_{N_{\gamma_{k}}-1} a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k \|u_c\|_\infty
\]
\[
\leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} V(x(N_k)) + H(a) \|u_c\|_\infty
\]
\[
\leq c_k! a_{\gamma_{k}}^{N_{\gamma_{k}}-1} \prod_{j=0}^{k-1} a_{\gamma_{j}}^{N_{\gamma_{j}}-1} V(x(N_0)) + r_1(n) \|u_d\|_\infty
\]
\[
+ r_2(n) \|u_c\|_\infty,
\]
where $r_1(n) = a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} g_{k-1}(a_j, \beta_j)$,
\[
r_2(n) = H(a) b_j + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} g_{k-1}(a_j, \gamma_j).
\]

It follows from condition (iv) and (27), $b_j \leq b, d_j \leq d$ that
\[
r_1(n) = a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} g_{k-1}(a_j, \beta_j)
\]
\[
= a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} c_k! \prod_{i=j}^{0} a_{\gamma_{i}}^{N_{\gamma_{i}}-1} a_{\alpha_{i}}^{N_{\alpha_{i}}-1} d_j
\]
\[
\leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} g_{k-1}(a_j, \gamma_j)
\]
\[
\leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} d_k + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} \Delta_{\sup} - 1 \|u_c\|_\infty.
\]

and
\[
r_2(n) = H(a) b_j + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} g_{k-1}(a_j, \beta_j, \gamma_j)
\]
\[
= H(a) b_j + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} c_k \sum_{j=0}^{k-1} c_k! \prod_{i=j}^{0} a_{\gamma_{i}}^{N_{\gamma_{i}}-1} a_{\alpha_{i}}^{N_{\alpha_{i}}-1} d_j
\]
\[
\times \left( h_{N_{\gamma_{j+1}}-N_{\gamma_{j}}}(a_j) - 1 \right) b_j
\]
\[
\leq H(a) b_j + a_{\gamma_{k}}^{N_{\gamma_{k}}-1} \Delta_{\sup} - 1 \|u_c\|_\infty.
\]

Moreover, for any $n \in (N_k, N_{k+1})$, we have
\[
a_{\gamma_{k}}^{N_{\gamma_{k}}-1} \leq a_{\gamma_{k}}^{N_{\gamma_{k}}-1} \leq M(a) \Delta \left\{ \begin{array}{ll}
1, & a < 1; \\
\Delta_{\sup} - 1, & a \geq 1.
\end{array} \right.
\]

Denote $\hat{\theta} = \lim_{k \to \infty} \theta_k/k$, and $\hat{\theta} = \sup_{k \in \mathbb{N}} (\theta_k - \theta_{k-1})$, then there exists some $K_0 \in \mathbb{N}$ such that $\theta_k/k \leq \hat{\theta}$ when
By Lemma 2, condition (i) and (37), we obtain that, for any $n \in (N_k, N_{k+1}], k \in \mathbb{N},$
\[
V(x(n)) \leq d_k^{n-N_k-1} c_k! \sum_{i=k}^{n} a_i^{n-i-1} V(x(N_0)) + r_1(n-1)\|u_d\|_\infty \\
+ M(a) \cdot \max_{k \leq K_0} \{e^{\delta k}\} e^{-(\sigma - \delta)k} V(x(N_0)) \\
+ M(a)d \left[ 1 + \frac{e^{\delta}}{e^{\sigma} - 1} \right] \|u_d\|_\infty \\
+ \left[ H(a)b + M(a)(H(a^{-1}) - 1)\frac{e^{\delta}}{e^{\sigma} - 1} \right] \|u_c\|_\infty
\]
(37)

By Lemma 2, condition (i) and (37), we obtain that, for any $n \in (N_k, N_{k+1}], k \in \mathbb{N},$
\[
\|x(n)\| \leq \left( \frac{c_2}{c_1} M(a) \max_{k \leq K_0} \{e^{\delta k}\} \right)^{1/p} \\
\times \max \{ s^{(1/p-1)} \} \left[ e^{-(\sigma - \delta)k} \|x(N_0)\| \right] \\
+ \gamma_{\epsilon}(\|u_d\|_\infty) + \gamma_{\delta}(\|u_c\|_\infty)
\]
(38)

where
\[
\gamma_{\epsilon}(s) = \max \{ s^{(1/p-1)} \} \left[ e^{-(\sigma - \delta)k} \|x(N_0)\| \right] \\
\gamma_{\delta}(s) = \max \{ s^{(1/p-1)} \} \left[ e^{-(\sigma - \delta)k} \|x(N_0)\| \right]
\]
for any $s \in \mathbb{R}_+.$

Hence, the system (1) is ISS with gain $(\gamma_{\epsilon}, \gamma_{\delta})$ and the proof is complete. □

**Remark 4:**

(i) When the system is free from inputs, i.e., $u_c = u_d = 0,$ then Theorems 1–3 imply that the discrete-time impulsive hybrid system (1) is globally quasi-exponential stable.

(ii) From the conditions of Theorems 1–3, it is not difficult to see that Theorem 3 is not a consequence of Theorems 1 and 2.

(iii) When $u_c = u_d = 0,$ from Theorems 1–3, $V(x(n))$ is not a strictly decreasing function with respect to time $n.$ Hence, the Lyapunov function $V(x(n))$ needs not to be strictly decreasing with respect to time $n$ to achieve ISS for a discrete-time impulsive hybrid systems, while it is required for a discrete-time system, see Jiang and Wang (2001). Here, the impulses in the system may do help to achieve the ISS property for a discrete-time impulsive hybrid system.

(iv) By Jiang and Wang (2001) and Jiang et al. (2004) a discrete-time system is ISS if and only if it has a ISS-Lyapunov function $V(x(n))$ which is strictly decreasing with respect to time $n$ when the system is free from inputs. However, from Theorems 1–3, it is no longer true for a discrete-time impulsive hybrid system.

4. **Quasi-exponential ISS for linear discrete-time impulsive hybrid systems**

In this section, we consider linear discrete-time impulsive hybrid systems and state several theorems, based on the results of §3.

Consider the following linear discrete-time impulsive hybrid system
\[
\begin{align*}
\begin{cases}
\dot{x}(n+1) = Ax(n) + K_c u_c(n), \quad n \neq N_k, \\
\Delta x(n+1) = B_n x(n) + K_d u_d(n), \quad n = N_k, \\
x(N_0) = x_0,
\end{cases}
\end{align*}
\]
(39)

where $x(n) \in \mathbb{R}^n,$ $\Delta x(n+1) = x(n+1) - x(n),$ $A \in \mathbb{R}^{n \times n}, K_c, K_d \in \mathbb{R}^{n \times m},$ are constant matrices, and for every $k \in \mathbb{N}, B_k \in \mathbb{R}^{n \times n}$ with $B_{N_k} = 0,$ $u_c \in \mathbb{R}^n$ and $u_d \in \mathbb{R}^d$ are disturbance inputs or control inputs. Denote $C_{N_k} = I, C_{N_{k+1}} = I + B_{N_k},$ where $I$ is the identity matrix.

**Theorem 4:** Suppose Assumption 1 holds and furthermore assume that there exist constants $\alpha > 0$ and $\theta_k (k \in \mathbb{N})$ such that
\[
\|A\|^N_k - N_{k-1} \prod_{i=k}^{N_k} \|C_{N_i}\| = e^{-\alpha k + \theta_k},
\]
(40)

where $\theta_k$ satisfying (15).

Then, the linear discrete-time impulsive hybrid system (39) is globally quasi-exponential ISS.

**Proof:** It is a direct consequence of Theorem 3 with $V(x(n)) = \|x(n)\|, a_k = \|A\|, b_k = \|K_c\|, c_k = \|C_{N_k}\|, d_k = \|K_d\|.$ □

**Theorem 5:** Assume that Assumption 1 holds and furthermore suppose that the following conditions are satisfied:

(i) $\|A\| < 1;$

(ii) for any $j \in \mathbb{N},$ the series $\sum_{j=1}^{\infty} \ln \|C_{N_j}\|$ is uniformly bounded from above.

Then, the system (39) is globally quasi-exponential ISS.

**Proof:** Since the series $\sum_{j=1}^{\infty} \ln \|C_{N_j}\|$ is uniformly bounded from above, there exists a positive constant $M > 0$ such that for any $j \in \mathbb{N}$ and $k \geq j,$
\[
-\infty \leq \sum_{j=1}^{k} \ln \|C_{N_j}\| \leq M < \infty.
\]

Noting the fact $N_k - N_0 \geq 2k,$ which implies that
\[
k \leq N_k - N_0 - k = \sum_{j=1}^{k} \Delta_j - k \leq k(\Delta_{\sup} - 1),
\]
(42)
we can set $N_k - N_0 - k = k + t_k$, where $t_k$ is integer with satisfying: $0 \leq t_k \leq k(\Delta_{\sup} - 2)$. Thus, we have

$$N_{k+1} - N_k = 2 + (t_{k+1} - t_k), \quad k \in \mathbb{N}. \quad (43)$$

It follows from (43) and the fact that $N_{k+1} - N_k \geq 2$ that the sequence $\{t_k\}$ is non-decreasing.

Let $\alpha = -\ln \|A\|$, then, $\alpha > 0$. Denote $\theta_k = -\alpha t_k + \sum_{j=0}^{k} \ln \|C_{N_j}\|$, we test that all the conditions of Theorem 4 hold:

Obviously, we have $\|A\|^{N_k - N_0 - k} \cdot \prod_{i=j+1}^{k} \|C_{N_i}\| = e^{-\alpha k + \theta_k}$, and $\theta_k$ satisfying, for any $k \in \mathbb{N},$

$$\lim_{k \to \infty} \frac{\theta_k}{k} \leq 0 < \alpha,$$

and by the fact that the series $\sum_{j=0}^{\infty} \ln \|C_{N_j}\|$ is uniformly bounded from above, we get

$$\sup_{k \in \mathbb{N}} \{\max_{0 \leq j \leq k-1} \{\theta_k - \theta_j\}\} = \sup_{k \in \mathbb{N}} \left\{ \max_{0 \leq j \leq k-1} \left\{ -\alpha(t_k - t_j) + \sum_{i=j+1}^{k} \ln \|C_{N_i}\| \right\} \right\}
\leq \sup_{k \in \mathbb{N}} \{\max_{0 \leq j \leq k-1} \left\{ \sum_{i=j+1}^{k} \ln \|C_{N_i}\| \right\}\} < \infty. \quad (45)$$

Hence, all conditions of Theorem 4 hold. The proof is complete. \hfill $\Box$

It is well known that the discrete-time system $x(n+1) = Ax(n)$ is exponential stable if and only if matrix $A$ is a Schur matrix, i.e., $\rho(A) < 1$. Hence, if $A$ is a Schur matrix, then the matrix equation

$$A^T P A - P = -I \quad (46)$$

has an unique solution $P$, where $P$ is a positive definite matrix. In the following, we derive the globally quasi-exponential ISS properties for system (39) by using the Schur matrix.

**Theorem 6:** Assume that Assumption 1 holds and furthermore suppose that the following conditions are satisfied:

(i) $\rho(A) < 1$; \quad (47)

(ii) for any $j \in \mathbb{N}$, the series $\sum_{i=j+1}^{\infty} \ln[\max (P^{-1} C_{N_i}^T P C_{N_i})]$ is uniformly bounded from above, where $P$ is the unique solution to the matrix equation (46).

Then, the system (39) is globally quasi-exponential ISS.

**Proof:** Let Lyapunov function $V(x)$ be: $V(x) = x^T P x$, then by (39) and (46), for any $n \in (N_k, N_{k+1}), k \in \mathbb{N}$, we have

$$V(x(n+1)) - V(x(n)) = -x^T(n)x(n) + 2x^T(n) A^T P K_{c} u_c(n) + u_c(n)^T K_c^T P K_c u_c(n)
\leq -\frac{1}{2 \lambda_{\max}(P)} V(x(n))
\leq (2 \|A^T P K_c\|^2 + \|K_c^T P K_c\|) \|u_c(n)\|^2,$$

and for $n = N_k$, by Li et al. (200a, Lemma 2.1),

$$V(x(N_k + 1)) = x^T(N_k) C_{N_k}^T P C_{N_k} x(N_k)
\leq 2x^T(N_k) C_{N_k}^T P K_d u_d(N_k) + u_d(n)^T K_d^T P K_d u_d(n)
\leq (1 + \epsilon_k) \lambda_{\max}(P^{-1} C_{N_k}^T P C_{N_k}) V(x(N_k))
\leq (1 + \epsilon_k)^{-1} \max \{ P^{-1} C_{N_k}^T P C_{N_k} \}, \quad k \in \mathbb{N}.$$ \quad (49)

Therefore, we have

$$\lambda_{\min}(P) \|x\|^2 \leq V(x) \leq \lambda_{\max}(P) \|x\|^2.$$ \quad (50)

Second,

$$\prod_{i=k}^{1} e^{\alpha(t_i - t_{i-1}/2)} = \left(1 - \frac{1}{2 \lambda_{\max}(P)} \right)^{N_k - N_0 - k - 1} \prod_{i=k}^{N_k - N_0 - k - 1} (1 + \epsilon_i) \lambda_{\max}(P^{-1} C_{N_i}^T P C_{N_i})
\leq e^{-\epsilon_k (t + k)} \prod_{i=k}^{N_k - N_0 - k - 1} e^{\alpha(t_i - t_{i-1}/2)} \lambda_{\max}(P^{-1} C_{N_i}^T P C_{N_i})
\leq e^{-\alpha t_k} e^{-\alpha k + \theta_k}.$$ \quad (51)
Moreover, by condition (ii), for any \( k \in \mathbb{N} \), we get
\[
\lim_{k \to \infty} \frac{\theta_k}{k} = \lim_{k \to \infty} \frac{\sum_{j=1}^{k} \ln \left[ \lambda_{\max} \left( P^{-1} C_{N_j}^T P C_{N_j} \right) \right]}{k} = 0 < \frac{\alpha}{2}, \tag{52}
\]
and by the fact that the series \( \sum_{j=1}^{\infty} \ln \left[ \lambda_{\max} \left( P^{-1} C_{N_j}^T P C_{N_j} \right) \right] \) is uniformly bounded from above, we get
\[
\sup_{k \in \mathbb{N}} \left\{ \frac{\theta_k - \theta_{k-1}}{k} \right\} = \sup_{k \in \mathbb{N}} \left\{ \max_{0 \leq j \leq k-1} \left\{ \theta_k - \theta_j \right\} \right\} = \sup_{k \in \mathbb{N}} \left\{ \max_{0 \leq j \leq k-1} \left\{ \sum_{i=j+1}^{k} \ln \left[ \lambda_{\max} \left( P^{-1} C_{N_i}^T P C_{N_i} \right) \right] \right\} \right\} < \infty. \tag{53}
\]
Hence, all conditions of Theorem 3 hold. The proof is complete. \( \square \)

5. Small-gain properties for interconnected discrete-time impulsive hybrid systems

In this section, we discuss the small-gain property for a class of interconnected non-linear discrete-time impulsive hybrid systems. This class of systems can be described as:
\[
\begin{aligned}
  x_1(n+1) &= f_1(x_1(n), u_{1}(n)), & n \neq N_k, \\
  \Delta x_1(n+1) &= I_{1,n}(x_1(n), v_1(n), u_{1,n}(n)), & n = N_k, \\
  x_1(N_0) &= x_{1,0},
\end{aligned}
\tag{54}
\]
and
\[
\begin{aligned}
  x_2(n+1) &= f_2(x_2(n), u_{2}(n)), & n \neq N_k, \\
  \Delta x_2(n+1) &= I_{2,n}(x_2(n), v_2(n), u_{2,n}(n)), & n = N_k, \\
  x_2(N_0) &= x_{2,0},
\end{aligned}
\tag{55}
\]
subject to the interconnection constraints
\[
v_1(n) = x_2(n), \quad v_2(n) = x_1(n), \tag{56}
\]
where for \( i = 1, 2 \) and for each \( n \in \mathbb{N} \), \( x_i(n) \in \mathbb{R}^{n_i}, u_{ic} \in \mathbb{R}^{n_i}, u_{id} \in \mathbb{R}^d, \) and \( f_i, I_{i,n} \) are continuous.

Here, without loss of generality, we assume that the two impulsive hybrid systems have the same impulsive time instant sequence \( \{0 \leq N_0 < N_1 < \cdots < N_k < \cdots \} \) because otherwise we can put the two impulsive time instants of the two impulsive hybrid systems together and then reorder them to form a unified impulsive time sequence.

**Theorem 7:** Suppose Assumption 1 holds and furthermore assume that there exist Lyapunov functions \( V(x_i) \) (\( i = 1, 2 \)) such that the following conditions hold:

(i) there exist constants \( c_{1i} > 0, c_{2i} > 0 \) and \( P_i > 0 \) such that for any \( x \in \mathbb{R}^n \),
\[
   c_{1i} \|x\|_p \leq V_i(x) \leq c_{2i} \|x\|_p; \tag{57}
\]

(ii) for every \( n \in (N_k, N_{k+1}), k \in \mathbb{N}, \) and \( i = 1, 2, \) there exist \( \mathcal{K}_\infty \)-function \( \phi_i \) and \( \mathcal{K} \)-function \( \gamma_i \) such that
\[
   V_i(x_i(n+1)) - V_i(x_i(n)) \leq \phi_i(V_i(x_i(n))) + \gamma_i(\|u_i(n)\|), \tag{58}
\]
where \( \phi_i \) satisfying \( \phi(s) \leq cs \) for some constant \( 0 < c < 1 \) and any \( s \in \mathbb{R}^+ \);

(iii) there exist \( \mathcal{K} \)-functions \( \psi_i \) and \( \gamma_{i,\ell} \) and constants \( 0 < \sigma_i < 1, \mu_i > 0 \) such that, for \( i = 1, 2; \) \( j = 2, 1, \) i.e., \( (i, j) = \{(1, 2), (2, 1)\} \), and every \( k \in \mathbb{N}, \)
\[
   V_i(x_i(n_k+1)) \leq \psi_i(V_i(x_i(N_k))) + \mu_i V_i(v_i(N_k)) + \gamma_i(\|u_i(N_k)\|), \tag{59}
\]
and
\[
   \psi_i(V_i(x_i(N_k))) \leq \sigma_i V_i(x_i(N_k)) + \sum_{j=1}^{\Delta_{k+1} - 1} \phi_j(V_i(x_i(N_k + l))); \tag{60}
\]

(iv) \( \mu_1 \mu_2 < \left(1 - \max_{i=1,2} \sigma_i \right)^2. \tag{61} \)

Then, the interconnected system (54)-(56) is globally quasi-exponential ISS with \( (u_c, u_d) \) as the input.

**Proof:** For \( i = 1, 2, \) by condition (ii), we get that, for any \( n \in (N_k, N_{k+1}), \)
\[
   V_i(x_i(n+1)) \leq V_i(x_i(n)) - \phi_i(V_i(x_i(n))) + \gamma_i(\|u_i(n)\|)
   \leq V_i(x_i(N_k + 1)) - \sum_{l=N_k+1}^{n} \phi_l(V_i(x_i(l)))
   + \sum_{l=N_k+1}^{n} \gamma_l(\|u_l(l)\|). \tag{62}
\]
Thus, by (62) and condition (iii), we get that
\[
V_i(x_i(N_{k+1} + l)) \leq \sigma_i V_i(x_i(N_k)) + \mu_i V_i(v_i(N_k)) + \sum_{s=N_k}^{N_{k+1}-1} \gamma_s(\|u_s(K_s)\|). \tag{63}
\]
where \(i = 1, 2; \ j = 2, 1.\)

Denote:
\[
W(k) = \begin{pmatrix} V_1(x_1(k)) \\ V_2(x_2(k)) \end{pmatrix}, \quad \gamma(x(\|u(\|))) = \begin{pmatrix} \gamma_1(\|u_1(k)\|) \\ \gamma_2(\|u_2(k)\|) \end{pmatrix},
\]
\[
\gamma_d(\|u_d(k)\|) = \begin{pmatrix} \gamma_d(\|u_{d1}(k)\|) \\ \gamma_d(\|u_{d2}(k)\|) \end{pmatrix}.
\]

Then, (63) implies that
\[
W(N_{k+1}) \leq A_{\sigma,\mu} W(N_k) + \gamma_d(\|u_d(N_k)\|)
\]
\[
+ \sum_{s=N_k}^{N_{k+1}-1} \gamma_s(\|u_s(K_s)\|), \quad k \in \mathbb{N}, \tag{64}
\]
where
\[
A_{\sigma,\mu} = \begin{pmatrix} \sigma_1 & \mu_1 \\ \mu_2 & \sigma_2 \end{pmatrix}.
\]

By Lemma 1, we get that
\[
W(N_{k+1}) \leq A_{\sigma,\mu}^{k+1} W(N_0) + \sum_{j=0}^{k} A_{\sigma,\mu}^{k-j} \gamma_d(\|u_d(N_j)\|)
\]
\[
+ \sum_{j=0}^{k} \sum_{i=N_j+1}^{N_{k+1}-1} A_{\sigma,\mu}^{k-j} \gamma_s(\|u_s(i)\|)
\]
\[
= S_{\sigma,\mu}^{k+1} S^{-1} W(N_0) + \sum_{j=0}^{k} S_{\sigma,\mu}^{k-j} S^{-1} \gamma_d(\|u_d(N_j)\|) + \sum_{j=0}^{k} \sum_{i=N_j+1}^{N_{k+1}-1} A_{\sigma,\mu}^{k-j} \gamma_s(\|u_s(i)\|), \tag{65}
\]

where \(A = S_{\sigma,\mu}^{k+1} S^{-1}\) and \(J_{\sigma,\mu}^{k-j}\) is the Jordan matrix of \(A_{\sigma,\mu}.\)

On the other hand, for any \(1 \leq l \leq \Delta_{k+1} - 1,\) denote \(\lambda(c) = (1/(1-c))(1+c/(1-c)^{l-1}),\) then by the induction and similar proof of Theorem 1, we have
\[
V_i(x_i(N_{k+1} + l)) \leq \lambda_l(c) \left[ \sigma_i V_i(x_i(N_k)) + \mu_i V_i(v_i(N_k)) + \gamma_d(\|u_d(N_k)\|) + \sum_{s=N_k}^{N_{k+1}-1} \gamma_s(\|u_s(S)\|) \right], \tag{66}
\]
which implies that, for any \(1 \leq l \leq \Delta_{k+1} - 1,\)
\[
W(N_{k+1} + l) \leq \lambda_l(c) \left[ A_{\sigma,\mu} W(N_k) + \gamma_d(\|u_d(N_k)\|) + \sum_{i=N_k}^{N_{k+1}-1} \gamma_s(\|u_s(i)\|) \right], \quad k \in \mathbb{N}. \tag{67}
\]

Hence, for any \(n \in (N_k, N_{k+1}), k \in \mathbb{N},\) letting \(n = N_{k+1} - l \) with \(1 \leq l \leq \Delta_{k+1} - 1,\) from (65) and (67), we get that
\[
W(n) = W(N_{k+1} + l) \leq \lambda_{N_{k+1} - n}(c) \left[ A_{\sigma,\mu} W(N_k) + \gamma_d(\|u_d(N_k)\|) + \sum_{i=N_k}^{N_{k+1}-1} \gamma_s(\|u_s(i)\|) \right]
\]
\[
+ \sum_{j=0}^{k} \sum_{i=N_j+1}^{N_{k+1}-1} A_{\sigma,\mu}^{k-j} \gamma_s(\|u_s(i)\|) + \frac{\gamma_d(\|u_d(N_0)\|)}{\lambda_{N_{k+1} - n}(c)}
\]
\[
\leq \frac{\gamma_d(\|u_d(N_0)\|)}{\lambda_{N_{k+1} - n}(c)} + \frac{\gamma_d(\|u_d(N_0)\|)}{\lambda_{N_{k+1} - n}(c)} \left( S_{\sigma,\mu}^{k+1} S^{-1} W(N_0) + \sum_{j=0}^{k} A_{\sigma,\mu}^{k-j} \gamma_d(\|u_d(N_j)\|) + \sum_{j=0}^{k} \sum_{i=N_j+1}^{N_{k+1}-1} A_{\sigma,\mu}^{k-j} \gamma_s(\|u_s(i)\|) \right). \tag{68}
\]

Thus, by (65), (68), and \(1 < \lambda(c) < \lambda_{\lambda_{\Delta} - 1}(c),\) for any \(n \in (N_k, N_{k+1}), k \in \mathbb{N},\) we get
\[
\left\| W(n) \right\| \leq \frac{\gamma_d(\|u_d(N_0)\|)}{\lambda_{\lambda_{\Delta} - 1}(c)} \left\{ \frac{\left\| S \right\|}{\left\| S^{-1} \right\|} \left\| J_{\sigma,\mu}^{k+1} \right\| \left\| W(x(N_0)) \right\| + \left\| S \right\| \left\| S^{-1} \right\| \left\| (I - A_{\sigma,\mu})^{-1} \right\| \right.
\]
\[
\times \left( \left\| (I - J_{\sigma,\mu}^{k+1}) \right\| \left\| \gamma_d(\|u_d(\|) \right\| \right)
\]
\[
\left. + \left\| (\lambda_{\lambda_{\Delta} - 1}(c)) \right\| \gamma_s(\|u_s(\|) \right\| \right) \right\}, \tag{69}
\]
where
\[
\gamma_s(\|u_s(\|) \right\| = \begin{pmatrix} \gamma_1(\|u_1(\|) \right\| \right) \gamma_2(\|u_2(\|) \right\| \right), \gamma_d(\|u_d(\|) \right\| = \begin{pmatrix} \gamma_1(\|u_1(\|) \right\| \right) \gamma_2(\|u_2(\|) \right\| \right).
\]

By condition (iv), it is not difficult to obtain that
\[
\rho(A_{\sigma,\mu}) < 1. \tag{70}
\]

Hence, \(\rho(J_{\sigma,\mu}) = \rho(A_{\sigma,\mu}) < 1\) and hence there exist constants \(K_1 > 0, K_2 > 0, \alpha > 0,\) which are determined by \(\sigma_i\) and \(\mu_i, i = 1, 2,\) such that \(\|J_{\sigma,\mu}^k\| \leq K_1 e^{-\alpha t}\) and \(\|I - J_{\sigma,\mu}^k\| \leq K_2\) for all \(k \in \mathbb{N}.\)
Hence, for any \( n \in (N_k, N_{k+1}), k \in \mathbb{N} \), from (69), we get

\[
\| W(n) \| \leq \frac{\lambda_{\Delta_{\text{sup}}(c)}}{c_1} \| S \| \| S^{-1} \| \left\{ K_1 e^{-\alpha k} \| W(x(N_0)) \| + K_2 \|(I - A_{\alpha, \mu})^{-1} \cdot \| \gamma_d(\| u_d \|_\infty) \| + (\Delta_{\text{sup}} - 1) \| \gamma_c(\| u_c \|_\infty) \| \right\}.
\]

Thus, by condition (i) and (71) and Lemma 2, for any \( n \in (N_k, N_{k+1}), k \in \mathbb{N} \), we obtain that

\[
\| x_i(n) \| \leq \left( \frac{\lambda_{\Delta_{\text{sup}}(c)}}{c_1} \| S \| \| S^{-1} \| \right)^{1/p_i} \max \left\{ 2 \left( \frac{1}{p_i} \right) - 1, 1 \right\} \times \left[ d_i \sup_{k=1}^{\infty} \| x(N_0) \| + \max \left\{ 2 \left( \frac{1}{p_i} \right) - 1, 1 \right\} \times K_2^{1/p_i} \|(I - A_{\alpha, \mu})^{-1} \cdot \| \gamma_d(\| u_d \|_\infty) \|^{1/p_i} + (\Delta_{\text{sup}} - 1)^{1/p_i} \| \gamma_c(\| u_c \|_\infty) \|^{1/p_i} \right]\},
\]

where

\[
X(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix},
\]

and for \( i = 1, 2 \), and some \( d_i > 0 \), \( \| W(N_0) \|^{1/p_i} \leq d_i \| x(N_0) \| \).

Thus, for any \( n \in (N_k, N_{k+1}), k \in \mathbb{N} \), there exist constants \( M_1 > 0 \), and \( \gamma_c, \gamma_d \) such that

\[
\| X(n) \| \leq M_1 e^{-\alpha k} \| X(N_0) \| + \gamma_c(\| u_c \|_\infty) + \gamma_d(\| u_d \|_\infty),
\]

where

\[
p = \max\{p_1, p_2\} \quad \text{and} \quad \gamma_c(s) = 3 \sqrt{v_1^2 (\Delta_{\text{sup}} - 1)^{2/p} \| \gamma_c(s) \|^{2/p} + v_2^2 (\Delta_{\text{sup}} - 1)^{2/p} \| \gamma_c(s) \|^{2/p}},
\]

\[
\gamma_d(s) = 3 \sqrt{v_1^2 \| \gamma_d(s) \|^{2/p} + v_2^2 \| \gamma_d(s) \|^{2/p}},
\]

Thus, the interconnected system (54)–(56) is globally quasi-exponential ISS with gain \((\gamma_c, \gamma_d)\).

**Theorem 8:** Suppose all conditions of Theorem 7 hold except conditions (ii) and (iii), which are replaced by the following conditions:

(ii\*): for every \( n \in (N_k, N_{k+1}), k \in \mathbb{N} \), and \( i = 1, 2 \), there exist \( \gamma_c \) and \( \gamma_d \) such that

\[
V_i(x_i(n + 1)) - V_i(x_i(n)) \leq \rho_i(\| V_i(x_i(n)) \|)
\]

(iii\*): there exist \( \gamma_c \) and \( \gamma_d \) such that

\[
V_i(x_i(n + 1)) \leq \sigma_i \| V_i(x_i(n)) \| + \gamma_i(\| u_i \|_\infty)
\]

Then, interconnected system (54)–(56) is globally quasi-exponential ISS with \((u_c, u_d)\) as the input.

**Proof:** By the similar proof of Theorem 7, we obtain the result of this theorem. The details are omitted here.

**Remark 5:** In the condition (ii) of Theorem 7 or Theorem 8, it requires that \( \phi_i(s) < cs \) for every \( s \in \mathbb{R}_+ \) and some \( 0 < c < 1 \). This means that \( \text{id} - \phi_i \in \mathcal{K} \) and hence guaranteeing the positive definite function of \( V(x) \). In the condition (iv): \( \mu_1 \mu_2 < (1 - \max_{i=1,2} \| \sigma_i \|)^2 \), it can be seen as the small-gain condition for the interconnected system, although it is more stronger than \( \mu_1 \mu_2 < 1 \). Moreover, from the proof of Theorem 7, this condition can be replaced by:

(iv\*): denote \( A_{\alpha, \mu} \triangleq \left( \frac{\sigma_1 \mu_1}{\sigma_2} \right) \), then \( \rho(A_{\alpha, \mu}) < 1 \).

**6. Examples**

In this section, we consider several examples to illustrate the results of previous sections.

**Example 1:** Consider system in the form of system (39), where

\[
A = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.2 & 0.1 \\ 0.2 & 0 & 0.5 \end{pmatrix}, \quad B_{N_k} = \begin{pmatrix} 0.14 & 0 & 0 \\ 0 & 0.14 & 0 \\ 0 & 0 & 0.14 \end{pmatrix},
\]

\( N_k = 3k + N_0, \quad (N_0, k \in \mathbb{N}). \)

Obviously, \( \rho(A) = 0.9 < \| A \| = 0.9309 < 1, \| C_{N_k} \| = \| I + B_{N_k} \| = 1.14 \). Thus, for any \( j \in \mathbb{N} \), the series
Moreover, for any positive definite matrix $P$ and any $j \in \mathbb{N}$, the series $\sum_{i=j}^{\infty} \ln\left[\ln(P^{-1}C_{N_j}^T P C_{N_j})\right] = +\infty$. Hence, the conditions of Theorems 5 and 6 are not satisfied. However, we have

$$\|A\|N_{i-N_0-k} \prod_{i=1}^{k} \|C_{N_i}\| = 0.9879^k = e^{-0.0122k}.$$ 

Hence, by Theorem 4, we conclude that this system is globally quasi-exponential ISS. Moreover, by Theorem 4 and (38), for any $n \in (N_k, N_{k+1}], k \in \mathbb{N}$, we obtain

$$\|x(n)\| \leq e^{-0.0122k}\|x(N_0)\| + 82.4682\|K_d\|\|u_d\|_{\infty} + 184.3248\|K_c\|\|u_c\|_{\infty}.$$ 

Figure 1 shows a computer simulation, assuming first that $x_0 = (0.3, -0.6, 0.8)^T$, and $u_c = 0, u_d = 0$. The quasi-exponential stability property of system is shown in figure 1. Then, we set the external disturbance inputs as: $K_c = 0.1 I, K_d = 0.1 I, u_c(k) = (\sin(k), \sin(k), \sin(k))^T, u_d(k) = (\cos(k), \cos(k), \cos(k))^T$. Figures 2–4 show the system response, clearly illustrating the ISS property.

**Example 2:** Consider system in the form of system (39), where

$$A = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.2 & 1 \\ 2 & 0 & 0.5 \end{pmatrix},$$

$$B_{N_k} = \begin{pmatrix} -0.5560 & 0 & 0 \\ 0 & -0.8 & 0.1 \\ 0.1 & 0 & -0.8 \end{pmatrix},$$

$$N_k = N_0 + 3k, \ k \in \mathbb{N}.$$
Obviously, \( \| A \| = 2.2521 > 1, \) \( \| C_{N_k} \| = \| I + B_{N_k} \| = 0.4579, \) and

\[
\| A \|^{N_0-N_0-k} \prod_{i=1}^{k} \| C_{N_i} \| = 2.3224^k = e^{0.8426k}.
\]

Hence, the conditions of Theorem 4 are not satisfied. But \( \rho(A) = 0.9 < 1, \) thus \( A \) is a Schur matrix. By Matlab toolbox on Riccati equation, the matrix equation (46) has a unique solution

\[
P = \begin{pmatrix} 188.8249 & 0.1088 & 6.3885 \\ 0.1088 & 1.0400 & 0.2231 \\ 6.3885 & 0.2231 & 2.9695 \end{pmatrix}.
\]

Hence, we get that \( \lambda_{\text{max}}(P^{-1}C_{N_k}^TPC_{N_k}) = 0.2055, \) which means that for any \( j \in \mathbb{N}, \) the series \( \sum_{j=0}^{\infty} \ln(\lambda_{\text{max}}(P^{-1}C_{N_k}^TPC_{N_k})) = -\infty < 0 < \infty. \) Hence, by Theorem 6, we conclude that this system is globally quasi-exponential ISS. Moreover, by Theorem 6 and (38), for any \( n \in (N_k, N_{k+1}), k \in \mathbb{N}, \) we obtain

\[
\| x(n) \| \leq 6.3065e^{-0.7924k} \| x(N_0) \| + 19.4445 \| K_d^TPK_c \|^{1/2} \| u_d \|_\infty + 12.7076 \left( 2 \| A^TPK_c \|^2 + \| K_d^TPK_c \| \right)^{1/2} \| u_c \|_\infty.
\]

Figure 5 shows a computer simulation, assuming first that \( x_0 = (1.0, -0.6, 0.2)^T, \) and \( u_c = 0, u_d = 0. \) Figure 5 shows that the system is globally quasi-exponential stable. Then, we set the disturbance inputs as: \( K_c = 0.05I, K_d = 0.1I, \) \( u_c(n) = (\text{rand}(1), \text{rand}(1), \text{rand}(1))^T, \) \( u_d(n) = (\cos(n), \cos(n), \cos(n))^T, \) where \( \text{rand}(1) \) is a random number satisfying: \( 0 \leq \text{rand}(1) \leq 1. \) Hence, \( u_c \) can be seen as a random disturbance of the system. Figures 6–8 show the system response, clearly illustrating the ISS property.

**Example 3:** Consider system in the form of system (39), where \( N_k = N_0 + 3k, N_0, k \in \mathbb{N}, \) and
$A = 1.1I, B_{N_k} = -0.3I,$ where $I$ is $3 \times 3$ identity matrix. Then, $\|A\| = \rho(A) = 1.1 > 1$. Hence, the conditions of Theorem 5 and Theorem 6 are not satisfied. But from $\|C_{N_k}\| = \|I + B_{N_k}\| = 0.7$, for any $k \in \mathbb{N}$, we derive

$$\|A\|^N_{N_k - N_0} \sum_{i=1}^k \|C_{N_i}\| = 0.889^k = e^{-0.1177k}.$$ 

Thus, by Theorem 4 and (38), for any $n \in (N_k, N_{k+1}], k \in \mathbb{N}$, we have

$$\|x(n)\| \leq 1.21e^{-0.1177k}\|x(N_0)\| + 10.8972\|K_d\|\|u_d\|_\infty + 98.9724\|K_c\|\|u_c\|_\infty.$$ 

Therefore, this system is globally quasi-exponential ISS.

**Remark 6:** For any linear discrete-time systems without impulses: $x(n+1) = Ax(n) + Bu(n)$, by Jiang and Wang (2001), this system is ISS if and only if $A$ is Schur matrix, i.e., $\rho(A) < 1$. However, in Example 3, $\rho(A) = 1.1 > 1$. Hence, in here, impulses in the system do contributions to a discrete-time impulsive hybrid system to achieve ISS property.

**Example 4:** Consider the following non-linear discrete system

$$
\begin{align*}
x(n+1) &= Ax(n) + F(n, x(n)) + u_c(n), \
\Delta x(n) &= Bx(n) + u_d(n), \
x(0) &= x_0,
\end{align*}
$$

where

$$A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0.1 & 0.5 \end{pmatrix},$$

$$B = -2.1I.$$ 

$u_c, u_d$ are the external inputs satisfying $\|u_c\|_\infty < \infty$, $\|u_d\|_\infty < \infty$, and $F(n, x(n)) = 1/4(x_1(n)/(1 + \sin^2 n + \|x(n)\|^2)x_2(n)\sin(x_3(n))x_3(n)\cos(x_3(n)))^T$. Let $V(x) = \|x\|$, then, by (76), we obtain that

$$V(x(n+1)) \leq 0.9105V(x(n)) + \|u_c\|_\infty, \quad n \neq N_k, k \in \mathbb{N},$$

and

$$V(x(N_k+1)) \leq 1.1V(x(N_k)) + \|u_d\|_\infty, \quad k \in \mathbb{N}.$$ 

Thus, by Theorem 3 and (38), for any $n \in (N_k, N_{k+1}], k \in \mathbb{N}$, we have

$$\|x(n)\| \leq e^{-0.0922k}\|x_0\| + 14.4778\|u_c\|_\infty + 11.3537\|u_d\|_\infty.$$ 

Therefore, this system is globally quasi-exponential ISS.

![Figure 9. Quasi-exponential stability of system (76) without external inputs.](image)

The numerical simulations of this example are given as follows: firstly, let $u_c = 0, u_d = 0$, i.e., the system is free from external disturbance inputs, then the quasi-exponential stability of the system is shown in figure 9, where the initial condition is given as: $x_0 = (0.8, -0.4, 0.2)^T$. Secondly, under the same initial condition, we set random inputs: $u_c(n) = 0.1(\text{rand}(1, \text{rand}(1), \text{rand}(1)))^T$, $u_d(n) = 0.05(\text{rand}(1, \text{rand}(1), \text{rand}(1)))^T$, where $\text{rand}(1)$ is a random number satisfying: $0 \leq \text{rand}(1) \leq 1$, then, the quasi-exponential ISS property is shown in figure 10.

![Figure 10. Quasi-exponential ISS property of system (76) with random inputs.](image)

7. Conclusions

In this paper, we proposed ISS and quasi-exponential ISS notions for discrete-time impulsive hybrid systems. By using Lyapunov function, we established quasi-exponential ISS criteria for non-linear
discrete-time impulsive hybrid systems. The obtained ISS criteria were used to derive the ISS properties for linear discrete-time impulsive hybrid systems. In addition, ISS small-gain theorems have also been obtained for a class of interconnected discrete-time impulsive systems. Finally, some representative examples have been solved so as to illustrate the theoretical results obtained in this paper. Moreover, the theorems and examples obtained in this paper emphasize the fact that impulses may help a discrete-time system without the ISS property to achieve ISS property, and hence the ISS property for a discrete-time impulsive hybrid system is different from that of discrete-time systems, in which the system is ISS if and only if it has an ISS-Lyapunov function.

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References


