Uniform Stability of Discrete Delay Systems and Synchronization of Discrete Delay Dynamical Networks via Razumikhin Technique
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Abstract—In this paper, we consider discrete delay systems and obtain conditions for global uniform exponential stability. Our approach is based on the use of the Razumikhin technique and the Lyapunov function method. In the second part of the paper, we use our stability results to derive exponential synchronization criteria for discrete dynamical networks with coupling time delays. We explicitly consider the case of networks with complex structures, such as networks with chaotic nodes, which have great practical importance in the area of secure communications.

Index Terms—Discrete delay dynamical networks, discrete delay system, global synchronization, impulsive synchronization, Razumikhin technique, stability.

I. INTRODUCTION

Time delays occur frequently in many physical systems and control schemes. Delays occur for a variety of reasons, including finite switching times, hardware speed, and network traffic congestion. It is, therefore, important to analyze systems with time delays. In particular, stability of systems with delay has received considerable attention over the last three decades [1], [2], [13].

Several methods have been proposed to analyze stability of time delay systems, including the Lyapunov functional method, the comparison principle and the Razumikhin technique. The Lyapunov functional method requires a Lyapunov function that decreases in the entire time space. This function is often very difficult to find, making this method applicable to a rather narrow class of systems for which such a function can be found. A similar problem is encountered using the comparison principle. This method requires finding a comparison system with known properties. Stability analysis using this method is based on the fact that, under certain conditions, stability of the comparison system implies stability of the original time-delay system. Finding a suitable comparison system, however, can be difficult, especially in the case of nonlinear time delay systems. The Razumikhin technique [1], [2] requires also the use of a Lyapunov function but, unlike the Lyapunov functional method, this Lyapunov function is not required to decrease in the entire time space.

The Razumikhin technique has been successfully applied to the study of several stability problems for continuous delay systems. See [1]–[3], [5], [6], and the references therein. References [7]–[11] study right-continuous impulsive delay systems using the Razumikhin technique. Razumikhin-type stability theorems for continuous delay systems and right-continuous delay systems are based on the fact that the solution of these type of differential equations is a continuous or right-continuous function. Unlike the case of continuous systems and right-continuous systems, however, the solution of a difference equation is not continuous or right-continuous, thus bringing difficulties in the use of the Razumikhin technique when investigating stability of discrete delay systems.

[4] considers stability of a class of discrete delay systems and reports a Razumikhin-type uniform asymptotic stability result. The main result in this reference, however, is very difficult to use making it very hard to apply in practical applications. To overcome this problem, in this paper, we proceed inspired by reference [11] and investigate Razumikhin-type exponential stability criteria for general discrete delay systems. Our goal is to obtain stability results that can be easily tested. To the best of our knowledge, no Razumikhin-type exponential stability theorem for discrete delay systems has been previously reported.

In the second part of this paper, we consider an application of our results to the synchronization of chaotic systems with time delay. Given the potential application to secure communications, chaotic synchronization has been an active research area for the past 15 years [14]–[22], [38]–[40], [43], [44]. More recently, synchronization of dynamical networks has also received much attention [23]–[36], [41], [42]. A dynamical network consists of coupled nodes that are usually dynamical systems. It has been reported that when a synchronization scheme is applied to a dynamical network, there are several factors that may cause the failure of the synchronization scheme. The main issues are: 1) uncertainties in the network (for example, channel noise); and 2) time delays. In order to deal with these undesirable factors, robust synchronization theory has become a promising research area. In [12], [22], [28], [31], the problem of robust synchronization of uncertain dynamical networks is studied using adaptive control and impulsive
control methods, respectively. Time delays occur commonly in synchronization schemes. [21], and [32]–[34] analyze synchronization of continuous-time delay networks but restrict attention to the case of a single time-delay known a priori. These assumptions are restrictive because time delay present in a typical synchronization scheme is usually not known a priori and, moreover, may by multiple time-varying time delays.

In this paper, we study global synchronization of discrete dynamical networks with coupling time delays. Applying the Razumikhin-type global uniform exponential stability theorem for discrete delay systems established in Section III, we derive several criteria under which global uniform exponential synchronization is achieved. When compared to other previously published results, our solution has the advantage that the synchronization speed with respect to a given admissible error bound can be easily estimated.

The remainder of this paper is organized as follows: In Section II, we introduce our notation and provide several definitions that will be used in later sections. In Section III, we establish a Razumikhin-type global uniform exponential stability theorem for discrete delay systems. In Section IV, the results of Section III are applied to the global synchronization of discrete delay dynamical networks. Finally, in Section V, we present several examples to illustrate our results.

II. PRELIMINARIES

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. Let $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_m = \{0, -1, -2, \ldots\}$, and for some positive integer $m$. Let $\lambda_{\max}(Q) \lambda_{\min}(Q)$ denote the maximum (minimum) eigenvalue of a symmetric matrix $Q$, and $\|A\|$ the norm of $A$ induced by the Euclidean vector norm, i.e., $\|A\| = \left[\lambda_{\max}(A^TA)\right]^{1/2}$. Let $\kappa$ denote the function class: $\kappa = \{c : \mathbb{R}^+ \to \mathbb{R}^+, c \text{ is a strictly increasing function satisfying } c(0) = 0\}$.

For a given positive real number $\rho > 0$, let

$$C([-\rho, 0], \mathbb{R}^n) = \left\{ \psi : [-\rho, 0) \to \mathbb{R}^n, \psi \text{ is continuous} \right\}$$

be a Banach space with topological structure: $\|\psi\| = \sup_{s \in [-\rho, 0)} \|\psi(s)\|$. The restriction of $C([-\rho, 0], \mathbb{R}^n)$ to $\mathbb{N}_m$ is $c([-\rho, 0], \mathbb{R}^n) = \left\{ \varphi : \mathbb{N}_m \to \mathbb{R}^n, \varphi \in \varphi(\mathbb{N}_m) \right\}$, where $\varphi \in C([-\rho, 0], \mathbb{R}^n)$, $\varphi(\mathbb{N}_m)(s) = \varphi(s), s \in \mathbb{N}_m\}$. We will denote $C(\mathbb{N}_m, \mathbb{R}^n)$.

Given a positive integer $m$, we equip the linear space $C(\mathbb{N}_m, \mathbb{R}^n)$ with norm $\|\varphi\| = \max_{s \in \mathbb{N}_m} \|\varphi(s)\|$. Similarly, we define $C(\mathbb{R}^+ \times \mathbb{N}_m, \mathbb{R}^n)$ and its restriction $C(\mathbb{N}_m, \mathbb{R}^n)$.

Consider the discrete delay system of the form

$$\begin{align*}
x(k + 1) &= f(x(k), x_k), \\
x_{k_0} &= \phi,
\end{align*}$$

where $x \in \mathbb{R}^n$, $f \in C(\mathbb{N}_m, \mathbb{R}^n)$, and $\phi \in C(\mathbb{N}_m, \mathbb{R}^n)$. $m \in \mathbb{N}$ represents the delay in the system (1), and $x_{k} \in C(\mathbb{N}_m, \mathbb{R}^n)$ is defined by $x_k(s) = x(k + s)$ for any $s \in \mathbb{N}_m$.

We assume $f(k, 0) \equiv 0, k \in \mathbb{N}_0$, so that (1) admits the trivial solution, and $x = 0$ is an equilibrium point for the system (1). We also assume that system (1) has a unique solution, denoted by $x(k) = x(k, k_0, \phi)$, for any given initial data: $k_0 \in \mathbb{N}$ and $\phi \in C(\mathbb{N}_m, \mathbb{R}^n)$.

**Remark 2.1:** In [4], the function $f(\cdot)$ needs to satisfy:

$$\|f(k, \phi)\| \leq L \|\phi\|, \text{ for some positive constant } L > 0$$

and any $k \in \mathbb{N}_0$. In this paper, this assumption is not necessary. Hence, the discrete delay system considered in this paper is more general than that in [4].

**Definition 2.1:** The equilibrium point $x = 0$ of system (1) is said to be global uniform exponential stable (GUES) if, for any initial condition $k_0 \in \mathbb{N}_0$, $x_{k_0} \equiv \phi$, there exist two positive numbers $\alpha > 0, M > 0$, where both $\alpha$ and $M$ are independent of $k_0$ and $\phi$, such that

$$\|x(k, k_0, \phi)\| \leq M \|\phi\| e^{-\alpha(k - k_0)}, \quad k \geq k_0, \quad k \in \mathbb{N}_0$$

**Remark 2.2:** Obviously, if (2) holds, then the Lyapunov exponent (LE) of system (1) is less than $-\alpha$.

**Remark 2.3:** For simplicity, if the equilibrium point $x = 0$ of the system (1) is GUES, the system (1) is also said to be GUES.

III. RAZUMIKHIN-TYPE THEOREM FOR DISCRETE DELAY SYSTEMS

In this section, we consider the discrete delay system (1) and derive Razumikhin-type stability theorems for global uniform exponential stability.

**Theorem 3.1 (Razumikhin-Type Global Exponential Stability Theorem):** The delay system (1) is GUES with $LE \leq -\alpha(r)$, if there exist a positive definite function $V(k, x)$ and constants $r > 0, p > 1, c_1 > 0, c_2 > 0, 1 > \lambda > 0$, such that:

1) for any $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$

$$c_1 \|x\|^p \leq V(k, x) \leq c_2 \|x\|^p$$

2) for any $s \in \mathbb{N}_m$, if $V(k + s, x(k + s)) \leq pV(k, x(k))$,

$$V(k + 1, x(k + 1)) \leq \lambda V(k, x(k))$$

3) for some $s \in \mathbb{N}_m - \{0\}$, if $V(k + s, x(k + s)) > e^{\alpha} V(k, x(k))$, then

$$V(k + 1, x(k + 1)) \leq \max_{s \in \mathbb{N}_m - \{0\}} \left\{ V(k + s, x(k + s)) \right\} \frac{1}{p}$$

where $\alpha = \min\{\ln(1/\lambda), \ln(p)/(m + 1)\}$.

**Proof:** Let $x(k) = x(k, k_0, \phi)$ be a solution of system (1). Without loss of the generality, let $k_0 = 0$. Define

$$U(k) = \max_{\theta \in \mathbb{N}_m} \{ e^{\theta(k + \theta)} V(k + \theta, x(k + \theta)) \}, \quad k \in \mathbb{N}_0$$

We now prove that

$$U(k + 1) \leq U(k), \quad k \in \mathbb{N}_0$$
For every fixed $k \in \mathbb{N}$, we define

$$
\bar{\theta}_k = \max \{ \theta \in \mathbb{N}_{-m} \cup \mathbb{N} : e^{\alpha(k+\theta)} V(k + \theta, x(k + \theta)) = U(k) \},
$$

which implies that

$$
U(k) = e^{\alpha(k+\bar{\theta}_k)} V(k + \bar{\theta}_k, x(k + \bar{\theta}_k)),
$$

If $\bar{\theta}_k \leq -1$, then for any $\theta \in \mathbb{N}_{-m} - \{0\}$

$$
e^{\alpha(k+\theta)} V(k + 1 + \theta, x(k + 1 + \theta)) \leq e^{\alpha(k+\bar{\theta}_k)} V(k + \bar{\theta}_k, x(k + \bar{\theta}_k))
$$

which implies that

$$
\max_{\theta \in \mathbb{N}_{-m} - \{0\}} \{ e^{\alpha(k+1+\theta)} V(k + 1 + \theta, x(k + 1 + \theta)) \} \leq U(k),
$$

(11)

We claim that

$$
e^{\alpha(k+1)} V(k + 1, x(k + 1)) \leq e^{\alpha(k+\bar{\theta}_k)} V(k + \bar{\theta}_k, x(k + \bar{\theta}_k)),
$$

(12)

By the definition of $\bar{\theta}_k$, we have

$$
e^{\alpha(k+\bar{\theta}_k)} V(k + \bar{\theta}_k, x(k + \bar{\theta}_k)) > e^{\alpha k} V(k, x(k))
$$

which implies that

$$
V(k + \bar{\theta}_k, x(k + \bar{\theta}_k)) > e^{-\alpha \bar{\theta}_k} V(k, x(k)) \geq e^{\alpha} V(k, x(k)),
$$

(13)

From (14) and condition 3) and $\bar{\theta}_k \leq -1$

$$
\max_{s \in \mathbb{N}_{-m} - \{0\}} \left\{ V(k + s, x(k + s)) \right\} \geq p V(k + 1, x(k + 1)) \geq e^{\alpha(m+1)} V(k + 1, x(k + 1)).
$$

Thus, by (15), we have

$$
e^{\alpha(k+1)} V(k + 1, x(k + 1)) \leq e^{-\alpha m} e^{\alpha k} \max_{s \in \mathbb{N}_{-m} - \{0\}} \left\{ V(k + s, x(k + s)) \right\} \leq e^{\alpha(k+m)} V(k + m, x(k + m)) \leq e^{\alpha(k+m)} V(k + m, x(k + m)) = e^{\alpha(k+\bar{\theta}_k)} V(k + \bar{\theta}_k, x(k + \bar{\theta}_k)).
$$

Hence, (12) holds. From (11) and (12)

$$
U(k + 1) \leq U(k), \text{ for } \bar{\theta}_k \leq -1.
$$

(17)

If $\bar{\theta}_k = 0$, then for any $\theta \in \mathbb{N}_{-m}$,

$$
e^{\alpha(k+\theta)} V(k + \theta, x(k + \theta)) \leq e^{\alpha k} V(k, x(k))
$$

which implies that

$$
V(k + \theta, x(k + \theta)) \leq e^{-\alpha \theta} V(k, x(k)) \leq e^{\alpha m} V(k, x(k)) \leq p V(k, x(k)).
$$

(19)

Thus, by condition 2), we have

$$
V(k + 1, x(k + 1)) \leq \lambda V(k, x(k)).
$$

(20)

From (20)

$$
e^{\alpha(k+1)} V(k + 1, x(k + 1)) \leq e^{\alpha k} e^{\alpha} V(k, x(k)) \leq e^{\alpha k} V(k, x(k)) = U(k)
$$

(21)

and from (18) that, for $\theta \in \mathbb{N}_{-m} - \{0\}$

$$
e^{\alpha(k+1+\theta)} V(k + 1 + \theta, x(k + 1 + \theta)) \leq e^{\alpha k} V(k, x(k)) = U(k).
$$

(22)

Hence

$$
U(k + 1) \leq U(k).
$$

(23)

Therefore, (7) holds and it leads to that, for any $k \in \mathbb{N}$

$$
U(k) \leq U(0) \leq \max_{\theta \in \mathbb{N}_{-m}} \left\{ V(\theta, x(\theta)) \right\},
$$

(24)

Thus, by the definition of $U(k)$ and (24), we have that

$$
V(k, x(k)) \leq e^{-\alpha k} U(k) \leq e^{-\alpha k} \max_{\theta \in \mathbb{N}_{-m}} \left\{ V(\theta, x(\theta)) \right\}.
$$

(25)

From (25) and condition 1)

$$
\|x(k)\| \leq \left( \frac{\alpha}{\lambda} \right)^{\frac{1}{r}} e^{-(\alpha/r) k} \|x(0)\|, \quad k \in \mathbb{N}.
$$

(26)

Remark 3.1: An important feature of Theorem 3.1 is that the conditions for global uniform exponential stability can be easily tested. We also emphasize that, for systems with stringent convergence requirements such as synchronization of network controlled systems [12], exponential stability is more significant than stability or asymptotical stability.

We now apply the Razumikhin-type Theorem 3.1 to a class of discrete delay systems. Consider the discrete delay systems of the form

$$
\begin{align*}
\begin{cases}
x(k + 1) = f(k, x(k), x(k - h_1(k)), \ldots, x(k - h_{m_0}(k))) \\
x_{k_0} = \phi
\end{cases}
\end{align*}
$$

(27)

where $f \in C(\mathbb{N} \times \mathbb{R}^{m(m+1)}, \mathbb{R}^n)$, and $h_j(k) \in \{1, 2, \ldots, m\}$, for any $k \in \mathbb{N}$, and $j = 1, 2, \ldots, m_0$.

Corollary 3.1: Assume that condition 1) of Theorem 3.1 holds, while conditions 2)–3) are replaced by the following condition (ii)*:
(i1)* there exists a positive constant 0 < \lambda < 1, such that
V(k + 1, x(k + 1)) \leq \lambda V(k),

where \( \mathcal{P}(k) = \max_{1 \leq j \leq \nu_0} \left\{ V(k - h_j(k), x(k - h_j(k)) \right\} \).

Then, the system (27) is GUES with \( LE \leq (\ln \lambda / r(m + 1)) \).

Proof: We only need to prove that the conditions 2)–3) of Theorem 3.1 still hold. Let \( p = (1 / \lambda)^{m+1/m+2} \), then, from \( \lambda < 1 \), we get that \( 0 < p < (1 / \lambda) \), and \( (\ln p / m + 1) = \ln(1 / \lambda p) \).

Thus, for any \(-h_i(k) \in \mathbb{N}_{-m} \),

\[
V(k - h_i(k), x(k - h_i(k))) \leq p V(k, x(k))
\]

then, by condition (i1)*, we have

\[
V(k + 1, x(k + 1)) \leq \lambda V(k) \leq \lambda p V(k, x(k)).
\]

It follows from the fact \( \lambda^p < 1 \) that the condition 2) of Theorem 3.1 is satisfied.

Let \( \alpha = (\ln p / m + 1) = \ln(1 / \lambda p) \), for some \(-h_i(k) \in \mathbb{N}_{-m} \setminus \{0\} \),

\[
V(k - h_i(k), x(k - h_i(k))) > e^n V(k, x(k))
\]

then, by condition (i1)* that

\[
V(k + 1, x(k + 1)) \leq \lambda V(k)
\]

\[
= \lambda \max_{1 \leq j \leq \nu_0} \left\{ V(k - h_j(k), x(k - h_j(k))) \right\}
\]

\[
\leq \lambda \max_{1 \leq j \leq \nu_0} \left\{ V(k - h_j(k), x(k - h_j(k))) \right\}
\]

\[
< \frac{1}{p} \max_{1 \leq j \leq \nu_0} \left\{ V(k - h_j(k), x(k - h_j(k))) \right\},
\]

Hence, condition 3) of Theorem 3.1 also holds.

Corollary 3.2: Assume that condition 1) of Theorem 3.1 hold, while conditions 2)–3) of Theorem 3.1 are replaced by the following condition (i1)**:

(i1)** There exist positive constants 0 < \lambda < 1, 0 < \lambda_i < 1, \ i = 1, 2, \ldots, \nu_0, \ such that

\[
V(k + 1, x(k + 1)) \leq \lambda V(k, x(k))
\]

\[
+ \sum_{i=1}^{\nu_0} \lambda_i V(k - h_i(k), x(k - h_i(k))), \quad k \in \mathbb{N}.
\]

If \( \lambda + \sum_{i=1}^{\nu_0} \lambda_i < 1 \), then the system (27) is GUES with \( LE \leq (\ln \lambda / r(m + 1)) \), where \( \lambda = \lambda + \sum_{i=1}^{\nu_0} \lambda_i \).

Proof: From (29), we get that for any \( k \in \mathbb{N} \)

\[
V(k + 1, x(k + 1)) \leq \left( \lambda + \sum_{i=1}^{\nu_0} \lambda_i \right) \mathcal{P}(k) = \lambda^2 \mathcal{P}(k).
\]

Since \( \lambda = \lambda + \sum_{i=1}^{\nu_0} \lambda_i < 1 \), the condition (i1)** of Corollary 3.1 holds.

IV. SYNCHRONIZATION OF DISCRETE DELAY
DYNAMICAL NETWORKS

In this section, we consider synchronization of discrete delay dynamical networks. Using the Razumikhin-type stability Theorem 3.1, we derive conditions for global uniform exponential synchronization of these type of networks. Moreover, we also consider a class of dynamical networks with complex structure, such as chaotic nodes, and derive conditions for global uniform synchronization of these networks using the concept of impulsive synchronization and the Razumikhin technique.

Consider a discrete network consisting of \( N \) identical nodes (\( n \)-dimensional discrete systems) and corresponding network coupling delays

\[
x_i(k + 1) = Ax_i(k) + \varphi(k, x_i(k))
\]

\[
+ g_i(x_i(k - \tau_i(k), \ldots, x_N(k - \tau_N(k)))
\]

\[
k \in \mathbb{N}, \quad i = 1, 2, \ldots, N
\]

(31)

where \( A \in \mathbb{R}^{m \times m} \), \( \varphi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) is a smooth non-linear vector function, \( g_i : \mathbb{R}^l \to \mathbb{R}^m \) are smooth but unknown network coupling functions, where \( l = nN \), and for each \( j = 1, 2, \ldots, N \), \( \tau_j(\cdot) \) is the coupling time delay function which represents the delay of the signal transmitted from \( j \)th node to other nodes with \( 0 \leq \tau_j(k) \leq m \) for any \( k \in \mathbb{N} \), \( j = 1, 2, \ldots, N \).

Clearly, each node of the network has the following form

\[
y(k + 1) = Ay(k) + \varphi(k, y(k)), \quad k \in \mathbb{N}
\]

(32)

The solution of (32) may be an equilibrium point, a periodic orbit, or even a chaotic orbit.

Remark 4.1: When the network achieves synchronization, namely, the state \( x_i(k) = x_2(k) = \cdots = x_N(k) = y \), as \( k \to \infty \), the coupling terms should vanish: i.e., \( g_i(y, y, \ldots, y) = 0 \).

Defining the synchronization error as \( e_i(k) = x_i(k) - y(k) \), the error dynamics has the form

\[
e_i(k + 1) = Ae_i(k) + \varphi_i(k, x_i(k), y(k))
\]

\[
+ \dot{g}_i(x_i(k - \tau(k)), y(k))
\]

\[
k \in \mathbb{N}
\]

(33)

where \( \varphi_i(k, x_i, y) = \varphi(k, x_i) - \varphi(k, y), \dot{g}_i(x_i(k - \tau_i(k)), y) = g_i(x_i(k - \tau_i(k)), \ldots, x_N(k - \tau_N(k))) - g_i(y, y, \ldots, y), \) and \( \dot{i} = 1, 2, \ldots, N \).

Definition 4.1: The discrete delay network (31) is said to achieve global uniform synchronization if the error system (33) is global uniform asymptotically stable (GUAS). Moreover, the discrete delay network (31) is said to achieve global uniform exponential synchronization with convergence exponent more than \( \alpha \) if the error system (33) is GUES with \( LE \leq -\alpha \).
Assumption 4.1: There exists a positive constant $L$ with $0 < L < 1$ such that

$$
\|\varphi(k, x_i) - \varphi(k, y)\| \leq L|x_i - y|, 
$$

$i = 1, 2, \ldots, N, k \in \mathbb{N}$. \hfill (34)

Theorem 4.1: Suppose that Assumption 4.1 holds and the coupling is linear, i.e., $g_i(x_i(k - \tau(k)), \ldots, x_N(k - \tau_N(k))) = \sum_{k=1}^{N} B_{ij} x_j(k - \tau_j(k))$. Assume that there exist positive constants $\mu_i$, $i = 1, 2, \ldots, N$ such that

$$
|A| + L + \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\mu_i}{\mu_j} |B_{ij}| < 1. \hfill (35)
$$

Then, the discrete delay network (31) achieves global uniform exponential synchronization with convergence exponent $-\ln p/m + 1$, where $p < 1$ satisfies

$$
p = |A| + L + \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\mu_i}{\mu_j} |B_{ij}|. \hfill (36)
$$

Moreover, for a given admissible error bound $\varepsilon > 0, k$ must satisfy the following condition such that $|\varepsilon_i(k)\| \leq \varepsilon$ for all $i = 1, 2, \ldots, N$

$$
k \geq \left[ \frac{(m + 1) \ln \frac{|A| + \mu_{\max}}{2 \mu_{\min}}}{\ln p} \right]. \hfill (37)
$$

where $X$ stands for the minimal integer not less than $X, \mu_{\max} = \max_{1 \leq i \leq N} \{\mu_i\}, \mu_{\min} = \min_{1 \leq i \leq N} \{\mu_i\},$ and the initial condition with $\phi_i(s) = x_i(s), s \in \mathbb{N}_{m+1}, i = 1, 2, \ldots, N$.

Proof: Let $V(k) = \sum_{i=1}^{N} V_i(k)$, where $V_i(k) = \mu_i|\varepsilon_i(k)|, i = 1, 2, \ldots, N$. Then, by (33), for any $k \in \mathbb{N}$

$$
V_i(k + 1) = \mu_i|\varepsilon_i(k + 1)|
= \mu_i\left[ A\varepsilon_i(k) + \varphi(k, x_i, y) + \gamma_i(x(k - \tau(k)), y) \right]
\leq \mu_i \left[ |A||\varepsilon_i(k)| + L|\varepsilon_i(k)| + \sum_{j=1}^{N} |B_{ij}|||\varepsilon_j(k) - \tau_j(k)|\right]
= (|A| + L)|\varepsilon_i(k)| + \mu_i \sum_{j=1}^{N} |B_{ij}|||\varepsilon_j(k) - \tau_j(k)|\right)
$$

which implies that

$$
V(k + 1) = \sum_{i=1}^{N} V_i(k + 1)
\leq \sum_{i=1}^{N} (|A| + L)V_i(k) + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mu_i}{\mu_j} |B_{ij}|||\varepsilon_j(k) - \tau_j(k)|\right)
$$

where $\lambda = |A| + L, \lambda_j = \sum_{i=1}^{N} (\mu_i/\mu_j)|B_{ij}|, j = 1, 2, \ldots, N$.

By Corollary 3.2, the discrete delay network (31) achieves global uniform exponential synchronization with convergence exponent $-\ln p/m + 1$, where $p = |A| + L + \sum_{j=1}^{N} \sum_{i=1}^{N} (\mu_i/\mu_j)|B_{ij}|$. By (25) and (26), in Theorem 3.1, we have

$$
|\varepsilon_i(k)| \leq \frac{\mu_{\max}}{\mu_{\min}} e^{-\left(\ln p/m + 1\right)k} ||\phi||_m
$$

if (37) holds, then, it is easy to show that $|\varepsilon_i(k)| \leq \varepsilon, i = 1, 2, \ldots, N$, for a given admissible error bound $\varepsilon > 0$.

Theorem 4.2: Suppose that Assumption 4.1 holds and the coupling is linear, i.e., $g_i(x_i(k - \tau(k)), \ldots, x_N(k - \tau_N(k))) = \sum_{k=1}^{N} B_{ij} x_j(k - \tau_j(k))$. Assume that there are positive definite matrices $P_i$ and positive numbers $\mu_i > 0, \mu_i > 0, d_i > 0, \varepsilon_{ij} > 0, such that

$$
\mu_i I \leq P_i \leq \mu_i I, \quad i = 1, 2, \ldots, N
$$

2) the following LMI holds:

$$
\left( \begin{array}{ccc}
\Psi_i - d_i P_i & A^T P_i \tilde{B}_i \\
\tilde{B}_i^T Q A & \tilde{B}_i^T P_i \tilde{B}_i - \tilde{P}_i
\end{array} \right) \leq 0, \quad i = 1, 2, \ldots, N
$$

where

$$
\Psi_i = A^T P_i A + L |A| (\mu_i/\mu_i) P_i + L^2 \mu_i \left[ I + \mu_i \sum_{j=1}^{N} |B_{ij}|^2 \right],
\tilde{P}_i = \text{diag}\{\varepsilon_{i1} P_i, \varepsilon_{i2} P_i, \ldots, \varepsilon_{iN} P_i\},
\tilde{B}_i = (B_{i1}, \ldots, B_{IN}),
$$

and $d_i, \varepsilon_{i1}, \ldots, \varepsilon_{iN}$ satisfying

$$
\max_{1 \leq i \leq N} \sum_{j=1}^{N} d_i \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \varepsilon_{ij} < 1.
$$

The discrete delay network (31) achieves global uniform exponential synchronization with convergence exponent $\ln p/2(m + 1)$, where $p < 1$ satisfies: $p = \max_{1 \leq i \leq N} \{d_i\} + \sum_{i=1}^{N} \sum_{j=1}^{N} \varepsilon_{ij}$. For a given admissible error bound $\varepsilon > 0, |\varepsilon_i(k)| \leq \varepsilon$ for all $i = 1, 2, \ldots, N$, if $k$ satisfies

$$
k \geq \left[ \frac{2(m + 1) \ln \frac{|A| + \mu_{\max}}{2 \mu_{\min}}}{\ln p} \right].
$$

where $\mu_{\max} = \max_{1 \leq i \leq N} \{\mu_i\}, \mu_{\min} = \min_{1 \leq i \leq N} \{\mu_i\},$ and $\phi$ is the initial condition with $\phi_i(s) = x_i(s), -\tau \leq s \leq 0, i = 1, 2, \ldots, N.$
Proof: Let \( V(k) = \sum_{i=1}^{N} V_i(k) \), where \( V_i(k) = e_i(k)^T P_i e_i(k) \), then, we have

\[
V_i(k+1) = e_i(k+1)^T P_i e_i(k+1)
\]

\[
= e_i(k+1)^T P_i e_i(k+1)
\]

\[
= \left[ A e_i(k) + \tilde{\varphi}(k, x_i, y) + \sum_{j=1}^{N} B_{ij} e_j(k - \tau_j(k)) \right] P_i
\]

\[
\cdot \left[ A e_i(k) + \tilde{\varphi}(k, x_i, y) + \sum_{j=1}^{N} B_{ij} e_j(k - \tau_j(k)) \right]^{T}
\]

\[
e_i(k)^T A^T P_i A e_i(k) + 2 e_i(k)^T A^T P_i \tilde{\varphi}(k, x_i, y)
\]

\[
+ 2 e_i(k)^T A^T P_i \sum_{j=1}^{N} B_{ij} e_j(k - \tau_j(k))
\]

\[
+ 2 \tilde{\varphi}_T(k, x_i, y) P_i \tilde{\varphi}(k, x_i, y)
\]

\[
+ \sum_{j=1}^{N} \sum_{l=1}^{N} \epsilon_{ij}^T (k - \tau_j(k)) B_{ij}^T P_i B_{il} e_l(k - \tau_l(k)).
\]

From condition 1) and Assumption 4.1, we obtain

\[
2 \epsilon_i^T(k) A^T P_i \tilde{\varphi}_i(k, x_i, y)
\]

\[
\leq 2 \left\| \epsilon_i^T(k) A^T P_i / \sqrt{\lambda_i} \right\| \cdot \left\| \sqrt{\lambda_i} \tilde{\varphi}_i(k, x_i, y) \right\|
\]

\[
= 2 \sqrt{\epsilon_i^T(k) A^T P_i A e_i(k) \sqrt{\epsilon_i^T(k, x_i, y) P_i \tilde{\varphi}_i(k, x_i, y)}
\]

\[
\leq 2 L \| A \| \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)} \epsilon_i^T(k) P_i e_i(k)
\]

\[
\leq 2 L \| A \| \frac{\mu_i}{\lambda_i} \epsilon_i^T(k) P_i e_i(k)
\]

(45)

and

\[
2 \tilde{\varphi}_i^T(k, x_i, y) P_i \sum_{j=1}^{N} B_{ij} e_j(k - \tau_j(k))
\]

\[
= 2 \sum_{j=1}^{N} \tilde{\varphi}_i^T(k, x_i, y) P_i B_{ij} e_j(k - \tau_j(k))
\]

\[
\leq \sum_{j=1}^{N} \left( L^2 \| P_i B_{ij} \| \epsilon_i^T(k) e_i(k) + \| e_j(k - \tau_j(k)) \| \right)
\]

\[
\leq \sum_{j=1}^{N} \left( L^2 \mu_i \| B_{ij} \| \epsilon_i^T(k) e_i(k) + \| e_j(k - \tau_j(k)) \| \right)
\]

(46)

Substituting (46)–(47) into (45) gives

\[
V_i(k+1) = e_i^T(k+1) P_i e_i(k+1)
\]

\[
= e_i^T(k) A^T P_i A + L \| A \| \frac{\mu_i}{\lambda_i} P_i
\]

\[
+ L^2 \mu_i \sum_{j=1}^{N} \| B_{ij} \|^2 I \epsilon_i^T(k)
\]

\[
+ 2 \sum_{j=1}^{N} \epsilon_i^T(k) A^T P_i B_{ij} e_j(k - \tau_j(k))
\]

\[
+ \sum_{j=1}^{N} \sum_{l=1}^{N} \epsilon_j^T(k - \tau_j(k)) B_{ij}^T P_l B_{il} e_l(k - \tau_l(k))
\]

\[
= (e_i^T(k) \xi_i^T(k - \tau_i(k))) \left( \frac{\Psi_i}{\tilde{\Psi}_i} P_i A \frac{A^T P_i \tilde{B}_i}{\tilde{B}_i^T P_i \tilde{B}_i} \right)
\]

\[
= \left( \frac{\epsilon_i(k)}{\xi_i(k - \tau_i(k))} \right)
\]

(48)

where \( \Psi_i = A^T P_i A + L \| A \| ((\mu_i / \lambda_i) P_i + L^2 \mu_i) I + \mu_i \sum_{j=1}^{N} \| B_{ij} \|^2 I, \tilde{B}_i = (B_{i1}, B_{i2}, ..., B_{iN}) \) and \( \xi_i(k - \tau_i(k)) = (e_i^T(k - \tau_i(k)) \epsilon_i(k)) \).

It follows from (48) and condition 2) that

\[
V_i(k+1) = e_i^T(k+1) P_i e_i(k+1)
\]

\[
\leq d_i \epsilon_i^T(k) P_i e_i(k)
\]

\[
+ \sum_{j=1}^{N} \epsilon_{ij}^T(k - \tau_j(k)) P_j \epsilon_j(k - \tau_j(k))
\]

\[
= d_i V_i(k) + \sum_{j=1}^{N} \epsilon_{ij} V_j(k - \tau_j(k)), \quad k \in \mathbb{N}
\]

(49)

which implies that for any \( k \in \mathbb{N} \)

\[
V(k+1) = \sum_{i=1}^{N} V_i(k+1) \leq \max_{1 \leq i \leq N} \left\{ d_i \right\} V(k) + \sum_{j=1}^{N} \sum_{i=1}^{N} \epsilon_{ij} V_j(k - \tau_j(k))
\]

\[
\leq \max_{1 \leq i \leq N} \left\{ d_i \right\} V(k) + \sum_{j=1}^{N} \sum_{i=1}^{N} \epsilon_{ij} V_j(k - \tau_j(k)).
\]

(50)

Let \( \lambda = \max_{1 \leq i \leq N} \{ d_i \} \) and \( \lambda_j = \sum_{i=1}^{N} \epsilon_{ij} \) \( j = 1, 2, ..., N \). By Corollary 3.2, the discrete delay network (31) achieves global uniform exponential synchronization with convergence exponent \( - \ln p / 2(m + 1) \). Moreover, similar to Theorem 4.1, for a given admissible error bound \( \varepsilon > 0 \), if (44) holds, then \( \| e_i(k) \| \leq \varepsilon \) for all \( i = 1, 2, ..., N \). ■

Theorem 4.2 assumes that the coupling functions in the network 3.1 are linear. Theorem 4.3, given next, generalizes this result to the case nonlinear couplings.

**Theorem 4.3:** Suppose that Assumption 4.1 and the following conditions hold:

1) there exist nonnegative constants \( r_{ij} \geq 0 \) such that for \( i = 1, 2, ..., N, k \in \mathbb{N} \)

\[
\| g_i(x_1, ..., x_N) - g_i(y_1, ..., y_N) \| \leq \sum_{j=1}^{N} r_{ij} \| x_j - y_j \|
\]

(51)

2) there are positive constants \( \mu_i > 0, i = 1, 2, ..., N \), such that

\[
\| A \| + L + \sum_{j=1}^{N} \frac{\mu_i}{\mu_j} r_{ij} < 1.
\]

(52)
Under these conditions, the discrete delay network (31) achieves global uniform exponential synchronization with convergence exponent $-\ln p/m + 1$, where $p < 1$ satisfies

$$
p = (||A|| + L) + \sum_{i=1}^{N} \sum_{j=1}^{N} H_{ij} r_{ij}. \quad (53)
$$

For a given admissible error bound $\varepsilon > 0$, if (37) holds, where $p < 1$ satisfies (53), then $||e_i(k)|| \leq \varepsilon$ for all $i = 1, 2, \ldots, N$.

Proof: The proof follows immediately using an argument analogous to Theorem 4.1 and Corollary 3.1. The details are omitted.

Remark 4.2: It should be noted that condition (35) in Theorem 4.1 and condition 2) in Theorem 4.3 are very stringent. Indeed, in many complex networks, such as the networks in which the discrete chaotic system is used as the nodes, we have that $||A|| \geq 1$ and $||\varphi(k, x) - \varphi(k, y)|| \leq L ||x - y||$ with $L \geq 1$. Hence, the results of Theorems 4.1-4.3 cannot be applied. A natural approach is then to consider the use of linear feedback control $u_i(k) = K_i e_i(k)$ in the $i$th node so to achieve the synchronization. Unfortunately, when $u_i(k) = K_i e_i(k)$, the gain $L$ of $\varphi$ still satisfies $L \geq 1$ even if $||A + K_i|| < 1$. In order to solve this problem, we include an impulsive controller $\{N_j, B_{N_j}\}$ in the synchronization scheme, as shown in Fig. 1. Here, the error system (33) can be rewritten as

$$
e_i(k + 1) = A(k)e_i(k) + \varphi(x_i(k), y_i(k)) + g_i(x_i(k - \tau(k)), y_i(k)), \quad k \neq N_j
$$

$$
\Delta e_i(k + 1) = B_{N_j} e_i(k), \quad k = N_j; \quad i, j \in \mathbb{N}
$$

(54)

where $\Delta e_i(N_j + 1) = e_i(N_j + 1) - e_i(N_j)$. The sequence $\{N_j\}$ satisfies

1. $0 = N_0 < N_1 < N_2 < \cdots$, with $\lim_{j \to \infty} N_j = \infty$
2. For all $j \in \mathbb{N}$, $N_j + 1 - N_j \geq 2$.

The impulsive synchronization scheme subject to network coupling is shown in Fig. 1, where $S_i$ stands for the $i$th node, $Y$ is the isolate node (32) and $g_i$ is the delay network coupling of $i$th node, $i = 1, 2, \ldots, N$.

**Theorem 4.4:** Assume that Assumption 4.1 holds and $||A|| + L \geq 1$. Suppose that there exists a positive constant $q < 1$ such that the following conditions are satisfied:

$$
||I + B_{N_j}|| \leq q, \quad j \in \mathbb{N}_\varepsilon \quad (55)
$$

$$
0 < \sum_{i=1}^{N} r_{ij} < q, \quad j = 1, 2, \ldots, N \quad (56)
$$

$$
\Delta_{sup} \ln \frac{||A|| + L}{1 - q^{-1} \max_{1 \leq j \leq N} \left\{ \sum_{i=1}^{N} r_{ij} \right\}} + \ln q < 0 \quad (57)
$$

where $\Delta_{sup} = \sup_{j \in \mathbb{N}} \left\{ N_{j+1} - N_j \right\}$ satisfying $\Delta_{sup} < \infty$.

The discrete delay network (31) achieves global uniform synchronization under the impulsive controller $\{N_j, B_{N_j}\}$.

Proof: Let $V(k, e) = \sum_{i=1}^{N} V_i(k, e_i)$, where $V_i(k, e_i) = ||e_i(k)||$, then, for any $k \neq N_j$, $j \in \mathbb{N}$, we have, for $i = 1, 2, \ldots, N$

$$
V_i(k + 1, e_i(k + 1)) \leq (||A|| + L)V_i(k, e_i) + \sum_{l=1}^{N} r_{il} V_l(k - \tau(k), e_l(k - \tau(k))), \quad (58)
$$

By (58), it yields that, for any $k \neq N_j$, $j \in \mathbb{N}$

$$
V(k + 1, e(k + 1)) \leq \sum_{i=1}^{N} V_i(k + 1, e_i(k + 1)) \leq (||A|| + L)\sum_{i=1}^{N} V_i(k, e_i) + \sum_{i=1}^{N} \sum_{j=1}^{N} r_{ij} (V_i(k - \tau(k), e_i(k - \tau(k))))
$$

$$
\leq (||A|| + L)\sum_{i=1}^{N} V_i(k, e_i) + \sum_{j=1}^{N} \left( \sum_{i=1}^{N} r_{ij} \right) V_j(k - \tau(k), e_j(k - \tau(k)))
$$

$$
\leq (||A|| + L)\sum_{i=1}^{N} V_i(k, e_i) + \max_{1 \leq j \leq N} \left\{ \sum_{i=1}^{N} r_{ij} \right\} V(k - \tau(k), e(k - \tau(k))), \quad (59)
$$

For any $k \neq N_j$, if for any $s \in \mathbb{N}_\varepsilon$, $qV(k + s, e(k + s)) \leq V(k + 1, e(k + 1))$, it then follows from (59) that

$$
V(k + 1, e(k + 1)) \leq \frac{||A|| + L}{1 - q^{-1} \max_{1 \leq j \leq N} \left\{ \sum_{i=1}^{N} r_{ij} \right\}} V(k, e(k)). \quad (60)
$$

When $k = N_j$, by (55)

$$
V_i(N_j + 1, e_i(N_j + 1)) = ||I + B_{N_j}|| ||e_i(N_j)|| \leq qV_i(N_j + 1, e_i(N_j)), \quad i = 1, 2, \ldots, N \quad (61)
$$
which leads to
\[ V(N_j + 1, e(N_j + 1)) \leq qV(N_j, e(N_j)), \quad (62) \]

Hence, by the Razumikhin-type stability theorem for discrete impulsive hybrid systems with time-delays ([37, Theorem 4.4]), we obtain that the error system (54) is GUAS and, hence, the discrete delay network (31) achieves global uniform synchronization under the impulsive controller \( \{N_j, B_{N_j}\} \).

\[ \square \]

V. EXAMPLES

In this section, we consider several examples that illustrate the results of previous sections.

1) Example 5.1: Consider the nonlinear discrete delay system
\[
\begin{align*}
    x(k+1) &= Ax(k) + F(k, x(k), x(k - h(k))) \\
    x_{k_0} &= \phi
\end{align*}
\]
where \( x(k) \in \mathbb{R}^m, m = 2, h(k) = 1 \), or 2,
\[
A = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{pmatrix}, \quad F(k, x(k), x(k - h(k))) = \frac{1}{4}x_2(k - 1) + \sin^2 x_2(k) + \frac{1}{2}x_2(k - 1)\sin(x_3(k)) + \frac{1}{2}x_2(k - 1)\cos(x_3(k))\end{pmatrix}^T.
\]
Let \( V(k, x) = x^T x \), then
\[
V(k+1, x(k+1)) = x(k+1)^T x(k+1) = x^T A^T x + x^T A^T F + F^T F \leq (||A||^2 + ||A||)||x||^2 + (||A|| + 1)||F||^2 \leq (||A||^2 + ||A||)||V(k, x(k)) + \frac{1}{10}(||A|| + 1)V(k - h(k), x(k - h(k))) + 0.0076V(k - h(k), x(k - h(k))) + 0.00973V(k - h(k), x(k - h(k)))
\]
which implies that the conditions of Corollary 3.2 hold. Thus, by Corollary 3.3, the discrete delay system (63) is GUAS with \( LE \leq -0.0137 \). The numerical simulation of this example is given in Fig. 2.

2) Example 5.2: Consider the discrete delay system, discussed in [4]
\[
x(k+1) = A(k)x(k) + B_1(k)x(k - h_1(k)) + \cdots + B_m(k)x(k - h_m(k)) \quad (64)
\]
where \( x(k) \in \mathbb{R}^n, A(k), B_j(k) \in \mathbb{R}^{n}, j = 1, 2, \ldots, m_0, -h_j(k) \in \mathbb{N}_m \).

Let \( A^* = \sup_{k \in \mathbb{N}} ||A(k)||, B_j^* = \sup_{k \in \mathbb{N}} ||B_j(k)||, j = 1, 2, \ldots, m_0. \)

The system (64) is uniformly exponentially stable if
\[
A^* + B_1^* + B_2^* + \cdots + B_{m_0}^* < 1. \quad (65)
\]

Moreover, \( LE \leq (\ln \lambda/m + 1) \), where \( \lambda = A^* + B_1^* + B_2^* + \cdots + B_{m_0}^* \).

If the Lyapunov function is \( V(k, x(k)) = ||x||^2 \), then
\[
V(k+1, x(k+1)) \leq \left( A^* + B_1^* + B_2^* + \cdots + B_{m_0}^* \right) V(k, x(k)). \quad (66)
\]

Hence, by Corollary 3.1, the system (64) is GUAS with \( LE \leq -0.0137 \).

Remark 5.1: Example 5.2 is discussed in [4], where the objective is to establish uniformly asymptotic stability. Unlike [4], our focus is on global uniform exponential stability. It is immediate, of course, that global uniform exponential stability guarantees uniformly asymptotic stability. Notice, however, the simplicity of our approach when compared to the rather involved method reported in [4].

3) Example 5.3: Consider the entire discrete delay network (31), where \( f(k, x_i) = Ax_i + \varphi(k, x_i) \). The functions \( f, g_j, j = 1, 2, \ldots, N \) satisfy the equation shown at the bottom of the next page for \( j = 1, 2, \ldots, N - 1 \).

\[ \begin{align*}
    g_j(x(k - m)) &= 0.01 \begin{pmatrix} -0.3x_{1j} + 0.1x_{2j} + \sin k \\
    0.1x_{1j} + 0.1x_{2j} - \sin k \end{pmatrix} \\
    f(k, x_i) &= \begin{pmatrix} -0.3x_{1i} + 0.1x_{2i} + \sin^2 k \\
    0.1x_{1i} + 0.1x_{2i} - \cos k \end{pmatrix} + 0.1x_{1i} + 0.1x_{2i} + \sin k \end{align*} \]

It is easy to show that \( ||A|| = 0.3618, ||\varphi(k, e_i)|| \leq 0.1 ||e_i|| \), and \( \sum_{j=1}^{N} \sum_{j=1}^{N} ||B_{ij}|| = 0.4 \). Then
\[
||A|| + L + \sum_{i=1}^{N} \sum_{j=1}^{N} ||B_{ij}|| \leq 0.8618 < 1.
\]

By Corollary 3.1, this discrete delay network achieves global uniform exponential synchronization with convergence exponent \( \ln p/m + 1 = 0.0779 \).

Let \( ||\varphi||_m = 0.7710 \). Figs. 3–5 show that trajectories of the error system for this dynamical network exponentially approach

\[ \quad \]
the origin. Moreover, if the given admissible error bound \( \varepsilon = 0.5 \), then, when

\[
k \geq \left[ \frac{(m+1) \ln \|e_i(k)\|_{\infty} \mu_{\max}}{\ln p} \right] = 7
\]
synchronization errors satisfy: \( \|e_i(k)\| \leq \varepsilon = 0.5 \) for all \( i = 1, 2, \ldots, N \), where \( \mu_{\max} = \mu_{\min} = 1 \).

4) Example 5.4: Consider the discrete delay network (31) and assume that the nodes of the network can be modeled using the chaotic Hénon’s map. A single chaotic Hénon’s map is in form of

\[
y(k+1) = Ay(k) + \varphi(y(k)), \quad k \in \mathbb{N}
\]
where \( y(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} \), \( A = \begin{pmatrix} 0 & 1 \\ 0.3 & 0 \end{pmatrix} \), and \( \varphi(y(k)) = \begin{pmatrix} 1 - 1.4y_1^2(k) \\ 0 \end{pmatrix} \).

The network is

\[
x_i(k+1) = Ax_i(k) + \varphi(x_i(k)) \\
\quad + g_i(x_1(k-1), x_2(k-1), \ldots, x_N(k-1)),
\]
i = 1, 2, \ldots, N,

where \( x_i = (x_{i1}, x_{i2})^T \), and the coupling functions \( g_i, i = 1, 2, \ldots, N \), satisfy

\[
g_i(x) = \begin{pmatrix} -0.01x_{i1} + 0.01x_{i+1,1} \\ 0.01x_{i2} - 0.01x_{i+1,2} \end{pmatrix}, \quad i = 1, 2, \ldots, N - 1
\]
and \( g_N(x_1, x_2, \ldots, x_N) = 0 \).

Let \( \varphi(k, e_i) = \varphi(x_i(k)) - \varphi(y(k)) = \begin{pmatrix} 1.4(y_1^2 - x_{i1}^2) \\ 0 \end{pmatrix} \) and \( x(0) = (0.3, -0.6)^T \), \( y(0) = (0.3, -0.1)^T \). It is easy to show that \( \|A\| = 1.0000 \), \( \|\varphi(k, e_i)\| \leq 2.52 \|e_i\| \), i.e., \( L = 2.52 \), and

\[
\|g_i(x, y)\| = \|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\| \\
\leq 0.01 \left( \|e_i\| + \|e_{i+1}\| \right)
\]
where \( i = 1, 2, \ldots, N - 1 \), and when \( i = N - 1, N \)

\[
\|g_i(x, y)\| = \|g_i(x_1, x_2, \ldots, x_N) - g_i(y, y, \ldots, y)\| = 0.
\]

Let \( N = 10 \). By Theorem 4.4, we can design an impulsive control law \( \{N_j, B_{N_j}, j \in \mathbb{N} \} \) such that the error system is GUAS. If \( N_j = 3j, B_{N_j} = \begin{pmatrix} -0.89 & 0 \\ 0 & -0.89 \end{pmatrix} \), then the conditions of Theorem 4.4 hold. The impulsive controller
{N_j,B_N_j} achieves global uniform synchronization for this discrete dynamical network as illustrated by the simulations shown in Figs. 6 and 7.

VI. CONCLUSION

In this paper, we have derived global uniform exponential stability for discrete delay systems based on the Razumikhin technique and Lyapunov theory. Our conditions can be easily tested and should prove useful in practical applications. The stability results obtained in Section III were employed to derive several criteria for global uniform synchronization of discrete dynamical networks with coupling time delays. In particular, we explicitly considered the case of networks with complex structures, such as networks with chaotic nodes, which have great practical importance in the area of secure communications. We also estimated the convergence speed, a useful parameter that can be used to calculate the synchronization time for a given error bound. Several examples were presented illustrating the efficiency of our results.

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