stable regardless of the initial points $u_i(0)$ or $x_i(0)$. Repeating the same argument for all flows $i$, we establish their individual convergence.

The necessity of condition $0 < \beta < 2$ directly follows from Corollary 3.

Combining EMKC’s Lyapunov and global quasi-asymptotic stability, we have:

**Corollary 4:** EMKC is globally asymptotically stable under constant feedback delay $D$ if and only if $0 < \beta < 2$.

VI. CONCLUSION

This paper offered a comprehensive stability analysis of a new congestion controller called MKC, which is proven to be locally stable with arbitrary (heterogeneous) feedback delays under easily verifiable conditions. This property makes MKC a highly appealing platform for congestion control in future high-speed networks with heterogeneous users. Moreover, we proposed a negative packet-loss feedback function to be used in conjunction with MKC and called the resulting controller EMKC. We proved that EMKC achieves both RTT-independent stability and fairness and converges to link utilization exponentially fast. Our investigation of global stability shows that all EMKC flows converge to their unique stationary points regardless of the initial point in which the system is started. We proved this fact for constant delays $D$ and our future work is to extend the analysis to heterogenous delays.

REFERENCES


**Controllability and Observability for a Class of Controlled Switching Impulsive Systems**

Bin Liu and Horacio J. Marquez

Abstract—In this note, we study the controllability and observability problem for a class of controlled switching impulsive systems. By posing several formulas of variation of parameters for this type of time-varying systems and employing the characteristic polynomial theory of matrix, we establish necessary and sufficient conditions for controllability and controlled observability with respect to a given switching time sequence. Specializing the obtained results to the case of time-invariant linear switching impulsive systems, we derive some simple algebraic criteria, which include the results reported in the literature for time-invariant linear switching systems, linear impulsive systems and classical linear systems. One example is worked out for illustration.

Index Terms—Controllability, controlled observability, hybrid systems, switching impulsive systems, variation of parameters.

I. INTRODUCTION

Motivated by the fact that hybrid systems provide a natural framework for mathematical modeling of many physical phenomena, their study has received considerable attention for the last two decades [1]–[3]. Most of the work encountered in the literature has focused on two types of hybrid systems, namely; switching and impulsive systems. See [4]–[10] and the references cited therein for recent work on these two classes of systems. It is, however, worth noticing that switching and impulsive systems do not include some important hybrid systems existing in some applications characterized by switches of the states and abrupt changes at the switching instants.

Indeed, in many natural phenomena in systems such as evolutionary processes, biological neural networks and bursting rhythmic models in pathology, when certain quantities accumulate, the nature of the reaction undergoes an abrupt change. In this case, one needs to switch to a new system of differential equations taking into consideration a momentary perturbation of an impulsive nature. This class of systems exhibit simultaneously continuous-time dynamic switching and impulsive jump phenomena. A general description of these systems is called switching impulsive system. Examples include evolutionary processes and biological systems, as well as frequency-modulated signal processing systems, networked control systems, optimal control models in economics, and flying object motions. See [22]–[28].

The controllability and observability problem for hybrid systems has recently received considerable attention. See [11]–[18], [20], [21]. In [11], controllability and observability of periodic switching linear systems was studied. In [18] and [31], geometric criteria for controllability of switching systems was established. In [30], controllability of
switching bilinear systems was investigated using Lie algebraic techniques. In [12], [15] and [21], the controllability problem was studied for linear impulsive hybrid systems. Most of the literature on the subject, however, deals with time-invariant systems almost exclusively. Moreover, so far very few results for time-variant switching impulsive systems in which both switching and impulse are simultaneously considered have been reported.

In this note, we study the controllability and observability problem for a class of time-variant controller-switched impulsive hybrid systems, where the impulses occur at the switching instants. We derive controllability and controlled observability criteria expressed in the form of a matrix rank condition that is much easier to check than the geometric criteria previously reported for time-invariant switched systems ([17], [18]). Moreover, when specializing to the case of time-invariant linear hybrid systems, our results coincide with those previously reported in the literature (see e.g., [11], [12], [15], [18], [20], [21]) for time-invariant linear switching systems, impulsive linear systems and classical linear systems. The extension considered here is, however, nontrivial, as we deal with a much broader class of systems.

The rest of this note is organized as follows. In Section II, we present preliminaries and propose the results of variation of parameters for the system. In Sections III–IV, we establish necessary and sufficient conditions for controllability and controlled observability, respectively. For illustration, one representative example is given in Section V. In Section VI we provide our conclusions and final remarks.

II. PRELIMINARIES

In the sequel, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathbb{R}_+ = [0, +\infty)$. For any positive integers $j$, $k$ ($j \leq k$), denote $\prod_{i=j}^{k} a_i = a_j a_{j+1} \cdots a_k$, where $a_i \in \mathbb{R}$. Let $I$ denote the $n \times n$ identity matrix.

Consider the time-variant controller-switching hybrid system of the form

$$
\dot{x}(t) = A_q(t)(x(t))x(t) + B_q(t)(x(t))u_q(t), \quad t_s < t \leq t_{k+1},
$$

$$
\Delta x = C_q(t)x(t) + D_q(t)v_q(t), \quad t = t_k
$$

$$
y(t) = F_q(t)x(t)
$$

where $A_q(t) \in \mathbb{R}^{n \times n}$, $B_q(t) \in \mathbb{R}^{n \times m}$, $C_q(t) \in \mathbb{R}^{p \times n}$, $D_q(t) \in \mathbb{R}^{p \times m}$, $F_q(t) \in \mathbb{R}^{q \times n}$, with all the entries in these matrices being continuous scalar functions of $t \in \mathbb{R}_+$; $0 \leq t_s < t_k < \cdots < t_{j+1} = t_j$ is the switching time sequence, at which the impulses occur, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^q$ is the output control, $u_q(t) \in U_q$, $v_q(t) \in U_q$ is a continuous input, $a_{i-1} > 0$, $i \in \mathbb{R}^n$ is a discrete input signal and $g : [t_0, t_f] \rightarrow Q = \{1, 2, \ldots, r\}$ is the switching control signal of the system. Since $Q$ is finite, the switching control $g(t)$ is necessarily piecewise constant with values in $Q$. We normalize the switching control $g(t)$ to be left continuous on the time interval $[t_0, t_f]$, i.e., $q(t) = g(t^-) = 1 \text{ mod } r$ at $g(t)$. Let $x(t)$ denote the solution at time $t$ of system (1) starting from $x(t_0) = x_0$.

Remark 2.1:

1) In engineering applications, the cost of controllers is often important. To reduce unnecessary costs, the controllers $u_q(t)$ and $v_q(t)$ usually accommodate to one of the following two cases:

(i) $D_q(t) = 0$, i.e., the controllers are added only to the continuous part; or

(ii) $B_q(t) = 0$, i.e., the controllers are added only to the discrete part.

2) In system (1), the impulses $\Delta x$ and the mode changes $q$ occur at the same time. Otherwise, if there exists some switching time $t_i$ at which impulse does not occur, then, we set $\Delta x = 0$ at $t_i$.

On the other hand, if there is some impulse instance $t_j$ at which the mode change does not occur, i.e., $q(t_j) = q(t_{j+1})$, then, we look this case as there is a mode change between the same subsystem. Hence, for convenience, we put the impulse instances and switching times as the same.

Definition 2.1: For the given switching time sequence $\Sigma = \{\{t_i\}_{i=0}^{\infty}\}$ with $t_{k+1} = t_f$, function $q : [t_0, t_f] \rightarrow Q$ is said to be an admissible switching control if

\begin{enumerate}
\item $q(t) \equiv q(t_0) \equiv q(t_1)$, for all $t \in [t_0, t_1]$;
\item $q(t) \equiv q(t_{i+1})$, for $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, M$.
\end{enumerate}

Remark 2.2: It should be noticed that for any switching system, the switching control $q(t)$ includes two parts (see e.g., [23], [27]): (i) when should the switching control be applied? i.e., $t_k = ?$ ($k = 1, 2, \ldots, M$); and (ii) which system should be activated at the switching times $t_k$, $k = 1, 2, \ldots$? i.e., $q(t_k) = ?$ ($k = 1, 2, \ldots$). Hence, if the switching time sequence $\Sigma = \{\{t_i\}_{i=0}^{\infty}\}$ with $t_{k+1} = t_f$ is given, then there remains to consider the second part of $q(t)$. In this note, for the given switching time sequence $\Sigma = \{\{t_i\}_{i=0}^{\infty}\}$ with $t_{k+1} = t_f$, we consider an admissible switching control.

Corresponding to system (1), for every $q(t) \in Q$, consider linear time-varying system:

$$
\dot{x}(t) = A_{q(t)}(t)x(t).
$$

Suppose that $X_{q(t)}(t)$ is the fundamental solution matrix of (2). Then, for any switching-times $\{\{t_i\}_{i=0}^{\infty}\}$ and $s \in (t_k, t_{k+1})$, the transition matrix associated with $A_{q(t)}(t)$ is

$$
X_{q(t)}(t, s) = X_{q(t)}(t_k, t)\prod_{j=k}^{l-1} X_{q(t)}(t_j, t_{j+1}).
$$

It is easy to see that, for any $t, s, t' \in \mathbb{R}_+$ and $s, t, \tau \in (t_k, t_{k+1})$, $X_{q(t)}(t', \tau)X_{q(t)}(\tau, s) = X_{q(t)}(t, s)$, $X_{q(t)}(t, s') = I_q$, where $I_q$ is the identity matrix of order $n$, $X_{q(t)}(t, s)$ is $(s \in Q)$ (see [19]).

Lemma 2.1: Let $X_{q(t)}(t) = X_{q(t)}(t) x(t) + D_{q(t)}(t) v_{q(t)}(t)$. For the switching time sequence $\Sigma = \{\{t_i\}_{i=0}^{\infty}\}$ and $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, M$, the solution of the system (1) satisfies (3), shown at the bottom of the page.

Proof: By using variation of parameters of ordinary differential system [19] and an induction argument, we can derive the result of this lemma. The details are omitted.

Lemma 2.2: In system (1), assume that $D_{q(t)}(t) = 0$, and $C_{q(t)}(t_0) = 0$, then, for the switching time sequence $\Sigma = \{\{t_i\}_{i=0}^{\infty}\}$ and $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, M$, the solution of system (1) satisfies (4), shown at the bottom of the next page.

Proof: The proof follows an argument analogous to that of Lemma 2.1 and it is omitted.

Lemma 2.3: For any matrix $X \in \mathbb{R}^{n \times n}$, there exist linearly independent functions $f_i(t), i = 0, 1, \ldots, n - 1$, such that the following equation holds:

$$
e^{xt} = \sum_{i=0}^{n-1} f_i(t) X^i.
$$

(5)
Proof: A direct consequence of the Cayley–Hamilton theorem and Lemma 2 in [12].

III. CONTROLLABILITY

In this section, for a given switching time sequence (i.e., impulse instance sequence) \( \Sigma = \{t_{j}\}_{j=1}^{M+1} \), we introduce the concept of controllability of system (1) and investigate the conditions under which the system is controllable.

Definition 3.1: The switching impulsive system (1) is said to be controllable on \([t_0, t_f]\) \((t_f > t_0)\) with respect to the switching time sequence \( \Sigma = \{t_{j}\}_{j=1}^{M+1} \) with \( t_0 < t_{j+1} = t_j \) if for all \( x_0 \in \mathbb{R}^n \), \( x_f \in \mathbb{R}^n \), there exist an admissible switching control input \( q(t) \) and continuous control inputs \( u_{i}(t) \) \((i = 1, \ldots, k)\) such that \( (1) \) has a solution \( x(t) \) satisfying \( x(t_0) = x_0 \) and \( x(t_f) = x_f \).

Definition 3.2: The switching impulsive system (1) is said to be controllable to the origin on \([t_0, t_f]\) \((t_f > t_0)\) with respect to the switching time sequence \( \Sigma = \{t_{j}\}_{j=1}^{M+1} \) with \( t_0 < t_{j+1} = t_j \) if for all \( x_0 \in \mathbb{R}^n \) there exist an admissible switching control input \( q(t) \) and continuous control inputs \( u_{i}(t) \) \((i = 1, \ldots, k)\) such that \( (1) \) has a solution \( x(t) \) satisfying \( x(t_0) = x_0 \) and \( x(t_f) = 0 \).

Remark 3.1: It was shown in [14] that controllability of an impulsive systems is not equivalent to controllability to the origin.

A. Only Continuous Control Signals are Input: \( D_{0}(q(t)) = 0 \)

Without loss of generality, let \( C_{a(t_0)}(t_0) = 0 \). Denote \( E_{M+1} = I \), and for \( j = 0, 1, \ldots, M \),

\[
E_j = \prod_{i=M+1}^{j+1} X_{a(t_j)}(t_i, t_{i-1})(I + C_{a(t_{i-1})}(t_{i-1})).
\]

Then, it follows from Lemma 2.2 that

\[
x(t_j) = E_{0}x_0 + \sum_{j=1}^{M+1} \int_{t_{j-1}}^{t_j} X_{a(t_j)}(t, s)B_{a(t_j)}(s)u_{a(t_j)}(s)ds.
\]

(7)

Theorem 3.1: Let \( D_{0}(q(t)) = 0 \), \( C_{a(t_0)}(t_0) = 0 \) and assume that every matrix \( I + C_{a(t_j)}(t_j) \) is non-singular. Then, the switching impulsive system (1) is controllable on \([t_0, t_f]\) with respect to the switching time sequence \( \Sigma \) if and only if there exists an admissible switching control input \( q(t) \) such that \( \rho_{a}(\Phi) = n \)

\[
\Phi = (\Phi_{1}, \Phi_{2}, \ldots, \Phi_{M+1}), \text{ and for } j = 1, 2, \ldots, M + 1
\]

Proof: Denote \( \Phi = \sum_{j=1}^{M+1} \Phi_j \). It is easy to see that \( \rho_{a}(\Phi) = \rho_{a}(\Phi) \).

Sufficiency: If there exists an admissible switching control input \( q(t) \) such that \( \rho_{a}(\Phi) = n \), we show system (1) is controllable on

\[
x(t) = X_{a(t_{k+1})}^{-1}(t_{k+1}, t)\left\{ \prod_{i=k+1}^{1} X_{a(t_j)}(t_i, t_{i-1})(I + C_{a(t_{i-1})}(t_{i-1}))x_0 \
+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} X_{a(t_j)}(t, s)B_{a(t_j)}(s)u_{a(t_j)}(s)ds \right. \\
+ \int_{t_k}^{t_{k+1}} X_{a(t_{k+1})}(t, s)B_{a(t_{k+1})}(s)u_{a(t_{k+1})}(s)ds \right\}
\]

(4)
where the $n \times n$ matrix $\Omega$ satisfies
\begin{equation}
\Omega = \left( E_1 \hat{C}_1, \ldots, E_{\Delta T} \hat{C}_M, \hat{C}_{M+1} \right) 
\end{equation}
(17)
where
\begin{align*}
E_j &= e^{A(t_{M+1})T^{j+1}}(I + C(t_{M+1}))e^{A(t_M)T^j}(I + C(t_{M-1})) \\
&\quad \cdots e^{A(t_{j+1})T^j}(I + C(t_{j})) \\
\delta_i &= t_i - t_{i-1}, \text{ and } \hat{C}_j \text{ is the controllable matrix of matrices } A_q(t_j) \text{ and } B_q(t_j)
\end{align*}
and
\begin{equation*}
\hat{C}_j = \left( B_q(t_j), A_q(t_j)B_q(t_j), A_q(t_j)^2B_q(t_j), \ldots, A_q(t_j)^{n-1}B_q(t_j) \right).
\end{equation*}

Proof: Obviously, when $A_q(t_j)(t) \equiv A_q(t)$, $B_q(t_j)(t) \equiv B_q(t)$, from (6), one can obtain that
\begin{equation*}
E_j = e^{A(t_{M+1})T^{j+1}}(I + C(t_{M+1}))e^{A(t_M)T^j}(I + C(t_{M-1})) \\
&\quad \cdots e^{A(t_{j+1})T^j}(I + C(t_{j})).
\end{equation*}
(18)

By Lemma 2.3, there exist functions $f_{i}(t_{j})(t)$, $f_{i}(t_{j})(t)$, which are linearly independent on $[t_{j}, t_{j+1}]$, and hence, $f_{i}(t_{j})(t)$ is non-singular, the matrix $\Theta_j$ is nonsingular, the matrix $\Omega_j$ is invertible. Moreover, it is easy to see that
\begin{equation*}
rank(\Omega) \leq rank(\Omega) \leq n.
\end{equation*}
(24)

Thus, $rank(\Omega) = rank(\Omega) = n$. By Corollary 3.1, the result of this corollary is true.

Remark 3.2: If system (1) is time-invariant (i.e., $A_i(t) = A_i$, $B_i(t) = B_i$, $i = 1, 2, \ldots, r$) and no impulses happened in it (i.e., $C_0 = 0$, $D_i(t) = 0$, $i = 1, 2, \ldots, r$), results in Corollaries 3.1–3.2 include those results for switching systems in the literature, (see e.g., [11, 15, 18, 20]).

B. Only the Discrete Control Signals are Input: $B_q(t_j)(t) = 0$

For convenience, let $D_q(t_0)(t_0) = 0$ and denote
\begin{equation*}
W_j = \sum_{i=1}^{j+1} A_q(t_j)(t_j), \quad j = 0, 1, \ldots, M.
\end{equation*}
(25)

Then, it follows from Lemma 2.1 and (3) and (25) that
\begin{equation*}
x(t_j) = W_0x_0 + \sum_{j=1}^{M} W_j C_q(t_j) + x_0(t_j) x(t_j)
\end{equation*}
+ \sum_{j=1}^{M} W_j D_q(t_j)v_q(t_j).
(26)

Theorem 3.2: The switching impulsive system (1) with $B_q(t_j)(t) = 0$, $C_q(t_0)(t_0) = 0$ and $D_q(t_0)(t_0) = 0$ is controllable on $[t_0, t]$ with respect to the switching time sequence $\Sigma = \{t_i\}_{i=0}^{M+1}$ if and only if there exists an admissible switching control input $q(t)$ such that
\begin{equation*}
\Psi = \left( W_1 D_q(t_1)(t_1), W_2 D_q(t_2)(t_2), \ldots, W_M D_q(t_M)(t_M) \right)
\end{equation*}
(27)
has full rank.

Proof: By Definition 3.1 and Lemma 2.1, for the given $\Sigma = \{t_i\}_{i=0}^{M+1}$ with $t_i \neq t_j$, and every $x_0, x_j \in R^n$, system (1) is controllable on $[t_0, t]$ if and only if there exist an admissible switching control input $q(t)$ and discrete control inputs $v_q(t_j)$ such that
\begin{equation*}
x(t_j) = x(t_j) + W_0x_0 + \sum_{j=1}^{M} W_j C_q(t_j)x_q(t_j) + x_0(t_j) x(t_j)
+ \sum_{j=1}^{M} W_j D_q(t_j)v_q(t_j).
\end{equation*}
(28)
Denote \( Y = (v^0_{a(t_1)}(t_1), v^T_{a(t_2)}(t_2), \ldots, v^T_{a(t_M)}(t_M))^T \) and

\[
\beta = x_f - W_0 x_0 - \sum_{j=1}^M W_j C_{a(t_j)}(t_j) x_{a(t_j)}(t_j).
\]

Then, it follows from (28) that

\[
\Psi Y = \beta,
\]

which implies that \( \beta \in \text{Span} \{ \Psi \} \). Thus, for any \( x_0, x_f \in R^n \), (29) has a solution \( Y \) if and only if \( \text{Span} \{ \Psi \} = R^n \). That is, if and only if matrix \( \Psi \) has full rank.

**Corollary 3.3:** For the \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) with \( t_{M+1} = t_f \), denote \( \delta_i = t_i - t_{i-1} \), assume that \( A_{a(t)}(t) \equiv A_{a(t)} \), \( D_{a(t)}(t) \equiv D_{a(t)} \), where \( A_{a(t)} \) and \( D_{a(t)} \) are constant matrices, then the switching impulsive system (1) with \( B_{a(t)}(t) = 0 \), \( C_{a(t)}(t_0)(t_0) = 0 \) and \( D_{a(t_0)}(t_0) = 0 \) is controllable on \([0, t_f] \) with respect to \( \Sigma \) if and only if there exists a switching input \( q(t) \) such that

\[
\Psi = \begin{pmatrix}
    e^{A_{a(t_0)}(t_0)^T} & \cdots & e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} & \cdots & e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} e^{A_{a(t_M)}(t_M)^T} \\
    e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} & \cdots & e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} e^{A_{a(t_M)}(t_M)^T} & \cdots & e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} e^{A_{a(t_M)}(t_M)^T} & \cdots & e^{A_{a(t_0)}(t_0)^T} e^{A_{a(t_2)}(t_2)^T} e^{A_{a(t_M)}(t_M)^T}
\end{pmatrix}
\]

(30)

has full rank.

**Proof:** If \( A_{a(t)}(t) \equiv A_{a(t)} \), where \( A_{a(t)} \) are constant matrices, then

\[
X_{a(t_j)}(t_i, t_i-1) = e^{A_{a(t_i)} t_j} I, \quad i = 1, 2, \ldots, M.
\]

(31)
The result is the direct consequence of Theorem 3.2.

**Corollary 3.4:** For the \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) with \( t_{M+1} = t_f \), denote \( \delta_i = t_i - t_{i-1} \), assume that \( A_{a(t)}(t) \equiv A \), \( D_{a(t)}(t) \equiv D \), where \( A \) and \( D \) are constant matrices, then system (1) is controllable on \([0, t_f] \) with respect to \( \Sigma \) if and only if

\[
\text{rank}(D, AD, AD^2, \ldots, A^{n-1}D) = n.
\]

(32)

**Proof:** When \( A_{a(t)}(t) \equiv A \), \( D_{a(t)}(t) \equiv D \), it follows that \( W_j = e^{A_{a(t)} \delta_i} I \), where \( \gamma_j = t_{j+1} - t_j, \quad j = 1, 2, \ldots, M \). By Lemma 2.3, there exist linearly independent scalar functions \( f_i(t) \), \( i = 0, 1, \ldots, n-1 \), such that \( e^{A_{a(t)} \delta_i} = \sum_{i=0}^{n-1} f_i(t) A^i \). Thus, we get

\[
W_j D = e^{A_{a(t)} \delta_i} D = \sum_{i=0}^{n-1} f_i(\gamma_j) A^i D
\]

(33)

which implies that \( W_j D \in \text{Span} \{ D, AD, AD^2, \ldots, A^{n-1}D \} \), \( j = 1, 2, \ldots, M \).

Hence, \( \text{Span} \{ \Psi \} \subseteq \text{Span} \{ D, AD, AD^2, \ldots, A^{n-1}D \} \). That is

\[
\text{rank}(\Psi) \leq \text{rank}(D, AD, AD^2, \ldots, A^{n-1}D) \leq n.
\]

(34)

On other hand, by Theorem 3.2, system (1) is controllable with respect to \( \Sigma \) if and only if the matrix \( \Psi \) has full rank. Therefore, by (34), system (1) is controllable with respect to \( \Sigma \) if and only if (32) holds.

**Remark 3.3:** If system (1) is time-invariant (i.e., \( A_{a(t)}(t) = A_{a(t)} \), \( B_{a(t)}(t) = B_{a(t)} \), \( C_{a(t)}(t) = C_{a(t)} \), \( D_{a(t)}(t) = D_{a(t)} \), \( i = 1, 2, \ldots, r \)), results in Corollaries 3.3–3.4 include those results for impulsive hybrid systems in the literature, (see e.g., [12], [21]).

**IV. CONTROLLABILITY**

In this section, we first define the concept of controllability for time-varying switching impulsive systems. Then we establish the criteria of controllable observability for this class of systems.

**Definition 4.1:** The switching impulsive systems (1) is called to be controllable observable on \([t_0, t_f] \) if \( t_f > t_0 \) with respect to the switching time sequence \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) with \( t_{M+1} = t_f \). If there exists an admissible switching control input \( q(t) \) such that any initial state \( x_0 \in R^n \) is uniquely determined by the corresponding system inputs \( u_{a(t)}(t) \) (or \( v_{a(t)}(t) \)) and system output \( q(t) \) for \( t \in [0, t_f] \).

**Theorem 4.1:** The switching impulsive system (1) with \( D_{a(t)}(t) = 0 \), \( C_{a(t)}(t_0)(t_0) = 0 \) is controllable observable with respect to the \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) with \( t_{M+1} = t_f \) and if only if there exists an admissible switching control input \( q(t) \) such that the \( n \times n \) matrix \( \Gamma \) satisfies

\[
\text{rank}(\Gamma) = n,
\]

for \( k = 0, 1, \ldots, M \), \( t \in (t_k, t_{k+1}] \),

\[
\Gamma_{k+1} = \int_{t_k}^{t_{k+1}} \Gamma_{k+1}(s, t_0) F_{a(t_k)}(s) y(s) y^T(s) ds + \Gamma_{k+1}(t_0, t_0) x_0 \quad t \in (t_k, t_{k+1}]
\]

(35)

**Proof:**

**Sufficiency:** Letting the control inputs \( u_{a(t)}(t) = 0 \), it follows from Lemma 2.2 that for any \( t \in (t_k, t_{k+1}] \), \( k = 0, 1, \ldots, M \),

\[
y(t) = F_{a(t_k+1)}(t) x(t) = F_{a(t_k+1)}(t) \Gamma_{k+1}(t_0, t_0) x_0.
\]

(37)

Multiplying (37) by \( \Gamma_{k+1}(t, t_0)^T F_{a(t_k+1)}(t) \) and integrating from \( t_0 \) to \( t_f \), we have

\[
\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} \Gamma_{k+1}(s, t_0) F_{a(t_k)}(s) y(s) ds \leq \sum_{k=0}^{M} \Gamma_{k+1} x_0 \leq 0.
\]

(38)

Hence, \( x_0 \) is uniquely determined by \( y(t) \) and therefore the system is controllable observable.

**Necessity:** Suppose \( \text{rank}(\Gamma) = n \), then the symmetry matrix \( \tilde{\Gamma} \) is singular and there exists a nonzero vector \( \alpha \in R^n \) such that

\[
\alpha^T \tilde{\Gamma} = 0.
\]

(40)

Since every matrix \( \Gamma_{k+1}(t, t_0)^T F_{a(t_k+1)}(t) y(t) \) is nonnegative definite and its entries are all continuous, for \( t \in (t_k, t_{k+1}] \), \( k = 1, 2, \ldots, M + 1 \), we get

\[
F_{a(t_k)}(t) \Gamma_{k+1}(t, t_0) \alpha = 0.
\]

(41)

For \( t \in (t_k, t_{k+1}] \), define \( \tilde{y}(t) \) as: See equation (42) at the bottom of the next page. From Definition 4.1, it is easy to see that system (1)
is controlled observable is equivalent to \( x_0 \) being uniquely determined by \( \tilde{y}(t) \). Let \( x_0 = \alpha \), by Lemma 2.2 and (41)-(42), we have
\[
\tilde{y}(t) = F_{0(t_{k+1})}(t)X^{-1}_{0(t_{k+1})}(t_{k+1}, t)
\cdot \prod_{\ell=k+1}^{M} X_{0(t_{\ell})}(t_\ell, t_{\ell-1}) \cdot (I + C_{0(t_{\ell-1})}(t_{\ell-1})) x_0
\]
\[
= F_{0(t_{k+1})}(t) \Gamma_{k+1}(t, t_0)x_0.
\]
(43)
Thus, for \( t \in (t_k, t_{k+1}] \), \( k = 0, 1, \ldots, M \), we obtain
\[
0 = \tilde{y}(t) = F_{0(t_{k+1})}(t) \Gamma_{k+1}(t, t_0)x_0
\]
which yields that system (1) is not controlled observable on \([t_0, t_j] \). This is a contradiction and therefore the matrix \( \Gamma \) satisfies \( \text{rank} k(\Gamma) = n \).

**Theorem 4.2:** The switching impulsive system (1) with \( B_{0(t)}(t) = 0 \), \( C_{0(t)}(t) = 0 \), is controlled observable with respect to \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) if and only if there exists an admissible switching control input \( q(t) \) such that the \( n \times n \) matrix
\[
\Upsilon = \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} Y_{k+1}(s, t_0) F_{0(t_{k+1})}(s) F_{0(t_{k+1})}(s) Y_{k+1}(s, t_0) ds
\]
is non-singular, where for \( t \in (t_k, t_{k+1}] \), \( k = 0, 1, \ldots, M \)
\[
Y_{k+1}(t, t_0) = X^{-1}_{0(t_{k+1})}(t_{k+1}, t) \prod_{\ell=k+1}^{M} X_{0(t_{\ell})}(t_\ell, t_{\ell-1}).
\]
(45)

**Proof:** The proof follows an argument analogous to that of Theorem 4.1 and is omitted.

**Corollary 4.1:** For the given \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) with \( t_{M+1} = t_f \), denote \( \delta_i = t_i - t_{i-1} \). Assume that \( A_{0(t_i)}(t) = A_{0(t_{i-1})}(t) \neq F_{0(t_i)}(t) \), where \( A_{0(t_i)}(t), F_{0(t_i)}(t) \) are constant matrices, and furthermore suppose that every matrix \( I + C_{0(t_{i-1})}(t_{i-1}) \) is non-singular. Then, system (1) with \( B_{0(t)}(t) = 0, C_{0(t)}(t) = 0 \), or with \( B_{0(t)}(t) = 0, C_{0(t)}(t) = 0 \), is controlled observable on \([t_0, t_f] \) with respect to \( \Sigma = \{ t_i \}_{i=1}^{M+1} \) if and only if there exists an admissible switching control input \( q(t) \) such that the matrix \( S = \left\{ S_{0(t_1)}, S_{0(t_2)}, \ldots, S_{0(t_{M+1})} \right\} \) satisfies \( \text{rank} k(S) = n \), where \( S_{0(t_j)} \) is the observable matrix of matrices \( A_{0(t_j)}(t) \) and \( F_{0(t_j)}(t) \)
\[
S_{0(t_j)} = \begin{pmatrix}
F_{0(t_j)} & F_{0(t_j)} A_{0(t_j)} & \cdots & A_{0(t_j)}^{n-1} F_{0(t_j)}
\end{pmatrix}, \quad j = 1, 2, \ldots, M + 1.
\]
(47)

**Proof:** The proof is similar to that for Corollary 3.1. The details are omitted.

**Remark 4.1:**
1) In [29], the observer-based stabilization problem for switching single input and single output linear systems is studied. The results derived in this note are less conservative than those in [29]. Hence, by using the results obtained in this note and the methods in [29], for the given switching time sequence, we can study the design of observer-based controllers for the stabilization of time-variant switching impulsive systems. This will be a topic of further research.

2) The results obtained in Theorems 3.1–3.2 and Theorems 4.1–4.2 can be used to design the switching modes for switching systems to achieve the controllability and controlled observability, respectively.

**V. Example**
In this section, we give an example to illustrate the obtained results.

**Example 5.1:** Consider the time-variant system with two subsystems in form of (1). Here, the matrices are given as: \( C_1(t) = 0, C_2(t) = 0 \), and \( A_1(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \), \( A_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( \Phi_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \Phi_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( \Phi_3(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). By calculation, we get the transition matrices \( A_1(t) \) and \( A_2(t) \), respectively
\[
X_1(t, s) = \begin{pmatrix} e^{-(t-s)} & -e^{-(t-s)+3s} + e^{-2(t-s)+4s} \\ 0 & e^{-2(t-s)} \end{pmatrix},
\]
\[
X_2(t, s) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix}.
\]
Thus, we further get that
\[
\Phi_1 = \begin{pmatrix} 5.7171 & -0.0129 \\ -0.0129 & 0.0000 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 1.6922 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 327.0940 & -4.4064 \\ -4.4064 & 0.1206 \end{pmatrix}.
\]
It follows from \( \text{rank} k(\Phi_1, \Phi_2, \Phi_3) = 2 \) and Theorem 3.1 that the system is controllable on \([t_0, t_f] \) with respect to the switching time sequence.

In fact, let \( x_0 = 0 \), and for some constants \( c_1, c_2 \in R \), let \( u_1(t) = c_1 \), \( u_2(t) = c_2 \), \( u_3(t) = 1 \), \( u_4(t) = 2 \), then, by Lemma 2.2, we obtain that \( x(t_f) = x(2.5) = \begin{pmatrix} 17.6233 \\ -2.2055 \end{pmatrix} \), \( c_1 + (1 - 0 \times c_2) \).
Obviously, $x(t_f)$ can be driven anywhere by selecting proper $c_1$ and $c_2$. Thus, the system is controllable on $[0, t_f]$ with respect to the switching time sequence.

Moreover, by calculation, we get that

$$
\begin{align*}
\Gamma_1 &= \begin{pmatrix} 0.4323 & 0.0831 \\ 0.0831 & 0.2994 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0.2767 & -0.1776 \\ -0.1776 & 0.0056 \end{pmatrix}, \\
\Gamma_3 &= \begin{pmatrix} 0.2913 & -0.1908 \\ -0.1908 & 0.1251 \end{pmatrix}.
\end{align*}
$$

Thus, rank $[\Gamma_1, \Gamma_2, \Gamma_3] = 2$ and hence, by Theorem 4.1, we obtain that the system is controllable observable. In fact, let $u_1(t) = u_2(t) = 0$, we get that

$$
y(t) = \begin{pmatrix} e^{-t} \\ 0 \\ e^{2t} + e^{-3t} \\ e^{-4t} \end{pmatrix} x_0, \quad t \in (0, t_1],
$$

$$
y(t) = 0.8 e^{3 t/2} \begin{pmatrix} 1 \\ -1 + e^{-1} - e^{3 t} \\ e^{3 t} \end{pmatrix} x_0, \quad t \in (t_1, t_2]
$$

and for $t \in (t_2, t_3]$,

$$
y(t) = 0.96 e^{-3 t/2} \begin{pmatrix} 1 \\ -e^{-2} + e^{-1} - 1 + e^{-t} - e^{-3 t} \\ e^{-3 t} \end{pmatrix} x_0.
$$

It yields that $x_0 \in R^m$ is uniquely determined by $y(t)$ for any $t \in [0, t_f]$.

VI. CONCLUSION

In this note, we have studied the controllability and controlled observability problems for a class of time-varying controlled switching impulsive systems. Necessary and sufficient conditions for controllability and controlled observability were derived and expressed in the form of matrix rank conditions that can be easily tested. Restricting attention to time-invariant linear switching impulsive systems, corresponding algebraic criteria were also derived. These criteria include those results for classical linear systems and time-variant linear switching systems previously reported in the literature.

ACKNOWLEDGMENT

The authors would like to thank the editor and the referees for their helpful suggestions and comments.

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