Dynamic Observer-Based Sensor Fault Diagnosis for LTI Systems

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Abstract

Most observer-based methods applied in fault detection and diagnosis (FDD) schemes use the classical two-degrees of freedom observer structure in which a constant matrix is used to stabilize the observer error dynamics while a post filter helps to achieve some desired properties for the residual signal. In this paper, we consider the use of a more general framework which is the dynamic observer structure in which an observer gain is seen as a filter designed so that the error dynamics has some desirable frequency domain characteristics. This structure offers extra degrees of freedom and we show how this freedom can be used for the sensor fault diagnosis problem with the objective to make the residual converge to the faults vector achieving detection and estimation at the same time. The use of appropriate weightings to transform this problem into a standard $H_{\infty}$ optimal control problem is also demonstrated. The introduced strategies are applied to a real-time example of an inverted pendulum system.

1 Introduction

Owing to the growing demand for higher reliability in control systems, the fault detection and diagnosis (FDD) problem is gaining increasing consideration in both theory and application. The FDD problem is defined as the synthesis of a monitoring system that detects faults and specifies their location and significance in the control system [1]. Model-based approaches, represented by the structure shown in Figure 1, have been a useful tool to solve this problem specially for linear time invariant (LTI) systems. This structure consists of: (i) A residual generation module that generates a fault indicating signal (residual) based on a mathematical model of the monitored system that includes the faults effect and that can also include any disturbances and/or model uncertainties affecting the system, (ii) A decision making phase where the residuals are examined and a decision rule is applied to determine if a fault has occurred.

A residual generator can be synthesized in the frequency domain via factorization of the transfer matrix of the monitored system. This is one of the most famous techniques used for residual generation. A very comprehensive study of this method was made by Ding and Frank in 1990 [3] (see also [6, 8, 9, 10, 7] for good surveys on this subject). This approach was also studied by other investigators such as Marquez and Diduch in 1992 [15], Yao et. al. in 1994 [21], and Kinnaert and Peng in 1995 [11]. The recent developments include the optimally robust detection filter (ORDF) developed in [2] where the optimal filter consists of a bandpass filter and a linear system which is obtained by solving a general eigenvalue problem. A method was also proposed to design a bank of ORDFs that have different sensitivities to a different set of faults for the purpose of fault isolation. This techniques was further considered in [4] focusing on the relationship between the minimum size of the detectable faults and the design parameters as well as the frequency window, reducing the problem to an optimization problem that is solvable using frequency domain optimization techniques. Within the same context, the tradeoff between the false alarm rate and the missed detection rate was studied in [5, 23] using different performance indices such as the $H_2$, $H_{\infty}$.

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$H_{\infty/\infty}$ and $H_{\infty/-}$ optimization approaches. The $H_-$ index was fully developed and characterized in terms of LMIs in [12] as a measure for the least effect of faults on the residuals. Recently, to avoid the time window problem introduced in the $H_2$ based design, an integrated design method using the absolute value of the residual signal was proposed in [22].

In this paper, we reconsider the frequency domain approach within a different context, in which the problem formulation is frequency based while the design is state space based. In our formulation the residual generator structure is also different, where in lieu of the classical observer structure in which a constant observer gain and a post filter help to achieve the FDD objectives, we consider a more general framework by making use of the dynamic observer structure introduced in [14, 16, 17, 18]. Using the dynamical structure, we show that the sensor fault isolation problem can be solved using a single observer without the need for using a bank of observers as in the ORDF approach. Also, by adding fault identification to the list of objectives (where the objective of estimating the faults magnitudes is considered in addition to detection and isolation), we show that the classical “static gain” observer structure can not solve the problem in this case. We then demonstrate that these objectives are achievable using a single dynamical observer by minimizing the faults effect in a narrow frequency band on the observer’s estimation error, and without the need for the number of faults to be known a priori. Different frequency patterns for the faults are also considered and the use of weightings to model the problem as a standard $H_\infty$ optimal control problem is illustrated. Unlike the results in [12] where the $H_-$ index was used, we here show that for the low and high frequency ranges the $H_\infty$ approach can still be used to maximize the effect of faults on the residuals. For simplicity, this paper considers systems with no model uncertainties. The robustness problem will be addressed in a separate paper.

The introduced techniques are demonstrated through experiments on a real-time example of the QUANSER$^{TM}$ rotary inverted pendulum available at the Advanced Control Systems Laboratory, University of Alberta. Different fault scenarios are considered including an important sensor fault introduced by the pendulum encoder. The dynamical observer structure is compared to the classical structure in real-time, with emphasis on both the fault diagnosis performance and the observer-based control response. The observer structure used in this paper has been further considered by the authors in [19] to study systems with Lipschitz nonlinearities. Although the observer structures are similar, the underlying approaches in both articles are very different where an analytical design with observer parameterization is adopted in this paper while a numerical approach based on LMIs is considered in [19].

Figure 1: Structure of the model-based FDD system.
The rest of the paper is organized as follows: section 2 introduces some mathematical background and notations used throughout the paper. In section 3, the problem of diagnosing sensor faults in a narrow frequency band is considered. In section 4, we consider the two cases of low frequency and high frequency ranges, formulating these problems as weighted $H_{\infty}$ optimal control problems. Real-time experiments on an inverted pendulum system are presented in section 5 and some conclusions are drawn in section 6.

2 Preliminaries and notation

The linear fault detection and diagnosis (FDD) problem considers the general class of LTI-MIMO systems affected by faults that can be modeled as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + R_1f(t)$$
$$y(t) = Cx(t) + Du(t) + R_2f(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $f(t) \in \mathbb{R}^o$, and where the matrices $A, B, C, D, R_1$ and $R_2$ are known matrices of appropriate dimensions. Here $f(t)$ is the fault vector, and can represent the different types of system faults shown in Figure 1 (i.e sensor, actuator and component faults).

As mentioned in section 1, the most famous technique used for residual generation is the observer-based approach that uses the following Luenberger observer structure:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t) + Du(t)$$

in addition to a transfer matrix (weighting) $Q(s)$ to generate the residual $r$ as:

$$r = Q(s)(y - \hat{y}); \quad r(t) \in \mathbb{R}^q$$

The residual obtained from (5) is therefore the weighted output estimation error of the observer, and the residual generator (3)-(5) has two degrees of freedom, namely, the constant observer gain $L$ and the post filter $Q(s)$. This freedom can be used to achieve different specifications of the FDD problem. The different tasks of a residual generator [1] can be summarized in: (i) Fault detection, (ii) Fault isolation, (iii) Fault identification. The residual generator achieves fault detection when the residual is nonzero if and only if a fault is affecting the system, helping to make a binary decision either that a fault occurred or not. Fault isolation is achieved if in addition to the detection objective, the residual can distinguish between different faults. This can be accomplished by dedicating each element of the residual vector $r$ to a specific element in the fault vector $f$. In fault identification, the residual must also monitor the size of the fault if possible. This is particularly important when a reconfiguration control is involved. The relative importance of the three tasks is subjective and depends on the application, however it is important to note that fault identification implies isolation and that fault isolation implies detection (but not the opposite).

For the sensor faults diagnosis problem, which is our focus in this paper, the system (1)-(2) is the special case where $R_1 = 0$, $R_2 = I_p$ (the identity matrix of order $p$) and $f(t) = f_s(t) \in \mathbb{R}^p$. The effect of using the classical residual generator in (3)-(5) is clear by noting that the observer error dynamics is given from (6)-(7) (where $e = x - \hat{x}$, $\tilde{y} = y - \hat{y}$) which can also be represented by the system in Figure 2.

$$\dot{e}(t) = (A - LC)e(t) - Lf_s(t)$$
$$\tilde{y}(t) = Ce(t) + f_s(t)$$
Figure 2: Error dynamics of the classical observer structure.

The fault vector $f_s$ has direct effect on the output estimation error $\hat{y}$, and hence on the residual. Therefore sensor fault detection by definition is achievable by this structure [1]. Sensor fault isolation can also be achieved by using the dedicated observer scheme, where a bank of observers (3)-(4) is used to differentiate between different faults. This technique was considered in [20].

In this paper we consider the multiple sensor faults identification problem using a novel approach. Our methodology is based on the extension of the Luenberger structure in (3)-(4) to a more general dynamic framework. We tackle the case when the sensor faults $f_s$ are in a narrow frequency band by showing that the sensor fault identification problem is equivalent to an output zeroing problem which is solvable only with a dynamic observer, proving that the classical structure can not solve the problem in this case. We further consider the cases of low and high frequency ranges showing that the problem can be modeled as a weighted $H_\infty$ optimal control problem solvable using commercially available software. The extra design freedom offered by the dynamic formulation is used to solve the proposed problems. We also show that the number of sensor faults does not need to be known a priori, and that our approach is applicable to the case of multiple sensor faults without any additional restrictive observability assumptions.

The following definitions and notation will be used throughout the paper:

For a system $H : \mathcal{L}_2 \to \mathcal{L}_2$, we will represent by $\gamma(H)$ the $\mathcal{L}_2$ gain of $H$ defined by $\gamma(H) = \sup_{\omega} \| H(j\omega) \|_{\mathcal{L}_2}$. It is well known that, for a linear system $H : \mathcal{L}_2 \to \mathcal{L}_2$ (with a transfer matrix $\hat{H}(s)$), $\gamma(H)$ is equivalent to the H-infinity norm of $\hat{H}(s)$ defined as follows:

$$\gamma(H) \equiv \| \hat{H}(s) \|_{\infty} \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}}(\hat{H}(j\omega))$$

where $\sigma_{\text{max}}$ represents the maximum singular value of $\hat{H}(j\omega)$. The matrices $I_n, 0_n$ and $0_{nm}$ represent the identity matrix of order $n$, the zero square matrix of order $n$ and the zero $n$ by $m$ matrix respectively. $\text{Diag}_r(a)$ represents the diagonal square matrix of order $r$ with $[a \ a \ \cdots \ a]_{1 \times r}$ as its diagonal vector, while $\text{diag}(a_1, a_2, \cdots, a_r)$ represents the diagonal square matrix of order $r$ with $[a_1 \ a_2 \ \cdots \ a_r]$ as its diagonal vector. The symbol $\hat{T}_{yu}$ represents the transfer matrix from input $u$ to output $y$. The symbol $RH_\infty$ denotes the space of all proper and real rational stable transfer matrices. The partitioned matrix $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (when used as an operator from $u$ to $y$, i.e, $y = Gu$) represents the state space representation ($\dot{\xi} = A\xi + Bu, \ y = C\xi + Du$), and in that case the transfer matrix is $\hat{G}(s) = C(sI - A)^{-1}B + D$. We will also make use of the following property on the rank of $\hat{G}(s)$ [24]:

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{rank} \left( \hat{G}(s) \right)$$

if $s$ is not an eigenvalue of $A$ and where $n$ is the dimension of the matrix $A$. 

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3 Narrow frequency band sensor fault diagnosis

In most observer-based FDD designs, maximizing the faults effect on the observer’s estimation error is considered as an optimal objective. However, for the sensor faults case (shown in Figure 2) the opposite is true. By minimizing \( e \), the output estimation error \( \hat{y} \) converges to \( f_s \) which guarantees fault identification in this case. In this section, we consider the solution of this design problem when \( f_s \) is in a narrow frequency band around a nominal frequency \( \omega_o \). This problem has many applications such as the case of sensor bias and the case of faults of known harmonics that frequently appear in power systems. We apply a new dynamical observer structure, showing that the problem is not tractable for the static gain structure in (3)-(4).

3.1 Dynamic generalization of the classical observer structure

Throughout this paper, following the approach in [14, 16, 17], we will make use of dynamical observers of the form:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + \eta(t) \\
\dot{y}(t) &= Cx(t) + Du(t)
\end{align*}
\]

where \( \eta(t) \) is obtained by applying a dynamical compensator on the output estimation error \( (y - \hat{y}) \). In other words \( \eta(t) \) is given from

\[
\begin{align*}
\dot{\xi} &= A_L \xi + B_L (y - \hat{y}), \quad A_L \in \mathbb{R}^{k \times k}, \quad B_L \in \mathbb{R}^{k \times p} \\
\eta &= C_L \xi + D_L (y - \hat{y}), \quad C_L \in \mathbb{R}^{n \times k}, \quad D_L \in \mathbb{R}^{n \times p}
\end{align*}
\]

We will also write

\[
K = \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}
\]

(13)

to represent the compensator in (11)-(12). It is straightforward to see that this observer structure reduces to the usual observer in (3)-(4) in the special case where the gain \( K \) is the constant gain given by \( K = \begin{bmatrix} 0_n & 0_{np} \\ 0_n & L \end{bmatrix} \). The additional dynamics provided by this observer brings additional degrees of freedom in the design, something that will be exploited in the problem of sensor fault estimation.

First, note that the observer error dynamics in (6) is now given by \( \dot{e} = Ae - \eta \) which can also be represented by the following so-called standard form:

\[
\begin{align*}
\dot{z} &= [A] z + [0_{np} - I_n] \begin{bmatrix} \tau \\ \nu \end{bmatrix} \\
\begin{bmatrix} \xi \\ \varphi \end{bmatrix} &= [I_n \ C] z + [0_{np} \ 0_n] \begin{bmatrix} \tau \\ \nu \end{bmatrix}
\end{align*}
\]

(14)

(15)

where

\[
\begin{align*}
\tau &= f_s \\
\nu &= \eta = K(y - \hat{y}) \\
\zeta &= e = x - \hat{x} \\
\varphi &= y - \hat{y}
\end{align*}
\]

(16)
which can also be represented by Figure 3 where the plant $G$ has the state space representation in (17) with the matrices in (14)-(15) and where the controller $K$ is given in (13).

$$\hat{G}(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}$$  \hfill (17)

![Figure 3: Standard setup.](image)

Therefore, all possible observer gains for the observer in (9)-(13) (including the static case in (3)-(4)) can be parameterized by the set of all stabilizing controllers for the setup in Figure 3. This is a standard result in control theory [24] and, for the observer problem considered in this paper, it can be represented by the following theorem (as a special case of Theorem 11.4 in [24]):

**Theorem 1** Let $F$ and $L$ be such that $A+LC$ and $A-F$ are stable; then all possible observer gains $K$ for the observer (9)-(13) can be parameterized as the transfer matrix from $\varphi$ to $\nu$ in Figure 4 with any $Q(s) \in RH_{\infty}$.

$$\hat{J}(s) = \begin{bmatrix}
A - F + LC & -L & -I_n \\
F & 0_{np} & I_n \\
-C & I_p & 0_{pn}
\end{bmatrix}$$

![Figure 4: Parameterization of all observer gains.](image)

### 3.2 State and sensor fault estimation

As mentioned earlier, our objective is to minimize (in some sense) the effect of sensor faults (in a narrow frequency band around a nominal frequency $\omega_o$) on the state estimation error in order to achieve sensor faults estimation.

In this section, we consider the solution of this minimization problem in an $L_2$ sense when $f_s$ is in a narrow frequency band around a nominal frequency $\omega_o$. From the special cases of interest is the case of sensor bias and the case of faults of known harmonics that frequently appear in power systems. We apply the dynamic observer structure introduced in section 3.1, showing that the problem is not tractable for the classical structure in (3)-(4). Towards that goal, we will first assume that the Fourier transform of the sensor fault $F_s(j\omega)$ have a frequency pattern restricted to the narrow band $\omega_o \pm \Delta \omega$ as described by equation (18):

$$|F_s(j\omega)| \leq \begin{cases}
\mu : |\omega - \omega_o| < \Delta \omega \\
\delta : \text{otherwise}
\end{cases}$$  \hfill (18)
where $\delta$ is a small neglected number for the frequency magnitudes outside the region of interest, and where $\mu$ is a positive upper bound on these magnitudes inside the considered domain. We will then define an observer gain $K$ as optimal if $\|e\|_{L_2}$ can be made arbitrarily small for all possible sensor faults satisfying (18). But by studying the gain of the error dynamics in Figure 3, we have: $\|e\|_{L_2} \leq \|\hat{T}_{ef}(s)\|_{\infty} \|f_s\|_{L_2}$. And since (as $\Delta \omega \to 0$), $\hat{T}_{ef}(j\omega) \to T_{ef}(j\omega_0)$ then we have $\|e\|_{L_2} \leq \sigma_{\max}(\hat{T}_{ef}(j\omega_0)) \|f_s\|_{L_2}$, and it directly follows that an optimal gain $K$ is one that satisfies $\hat{T}_{ef}(j\omega_0) = 0$.

The following lemma shows that a static observer gain can never be an optimal observer gain according to the previous discussion.

**Lemma 1** A static observer gain (such as the constant matrix $L$ in (3)-(4)) can never satisfy $\hat{T}_{ef}(j\omega_0) = 0$ for any nominal frequency $\omega_0$.

**Proof:** The proof follows by noting that the transfer matrix from $f_s$ to $e$ (as seen in (6) and in Figure 2) is $\hat{T}_{ef}(s) = \begin{bmatrix} A - LC & -L \\ I_n & 0_{np} \end{bmatrix}$. And since the gain $L$ is chosen to stabilize $(A - LC)$, then $(\forall \omega_0)$, $j\omega_0$ is not an eigenvalue of $(A - LC)$. Therefore, by using (8), we have $\text{rank}(\hat{T}_{ef}(j\omega_0)) = \text{rank}\left(\begin{bmatrix} A - LC - j\omega_0 I_n & -L \\ I_n & 0_{np} \end{bmatrix} - n\right)$. But rank $\begin{bmatrix} A - LC - j\omega_0 I_n & -L \\ I_n & 0_{np} \end{bmatrix} = \text{rank}\left(\begin{bmatrix} L & 0_n \\ 0_{np} & I_n \end{bmatrix} \right) = n + \text{rank}(L)$. Therefore, rank $\left(\hat{T}_{ef}(j\omega_0)\right) \neq 0$ unless $L = 0^1$. This implies that no gain $L$ can satisfy $\hat{T}_{ef}(j\omega_0) = 0$, and therefore a static observer gain can never be an optimal gain, for cancelling the effect of sensor faults in the narrow band $\omega_0 \pm \Delta \omega$.

We now consider the case of the dynamic observer introduced in (9)-(13). As a result of the gain parametrization presented in theorem 1, the transfer matrix from $f_s$ to $e$, achievable by an internally stabilizing gain $K$, is equal to the Linear Fractional Transformation (LFT) between $T$ and $Q$ as follows [24]:

$$
\hat{T}_{ef}(s) \equiv LFT(T, Q) = \hat{T}_{11}(s) + \hat{T}_{12}(s)\hat{Q}(s)\hat{T}_{21}(s)
$$

where $\hat{Q}(s) \in RH_{\infty}$ and where $T$ is given from

$$
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} = \begin{bmatrix}
A - F & F \\
0_n & A + LC \\
I_n & 0_n \\
0_{pm} & C
\end{bmatrix}
\begin{bmatrix}
0_{np} & -I_n \\
I_n & 0_{np} \\
0_n & C
\end{bmatrix}
$$

(20)

We will denote $\hat{T}_{11}(s)$, $\hat{T}_{12}(s)$ and $\hat{T}_{21}(s)$ by $\hat{T}_1(s)$, $\hat{T}_2(s)$ and $\hat{T}_3(s)$ respectively. The following lemma presents a result on the invertibility of the transfer matrices $\hat{T}_2(s)$ and $\hat{T}_3(s)$ at a frequency $\omega_0$ (i.e., at $s = j\omega_0$).

**Lemma 2** The $(n \times n)$ and $(p \times p)$ matrices $\hat{T}_2(j\omega_0)$ and $\hat{T}_3(j\omega_0)$ are invertible if $j\omega_0$ is not an eigenvalue of $A$.

**Proof:** By using (20), $\hat{T}_2(s) = \begin{bmatrix} A - F & F \\
0_n & A + LC \\
I_n & 0_n \\
0_{pm} & C
\end{bmatrix} = \begin{bmatrix} A - F & -I_n \\
I_n & 0_n \\
0_n & C
\end{bmatrix}$. Similarly, we have $\hat{T}_3(s) = \begin{bmatrix} A + LC & L \\
C & I_p
\end{bmatrix}$. Therefore, using (8) the following two rank properties are satisfied:

\[\text{The case of } L = 0 \text{ is the special case where } A \text{ is Hurwitz. It is important to note that no observer is needed for state estimation in this case. In this paper we are interested in the more general case.}\]
1. \( \text{rank}(\hat{T}_2(j\omega_o)) = \text{rank} \begin{bmatrix} A - F - j\omega_o I_n & -I_n \\ I_n & 0_n \end{bmatrix} - n \).

2. \( \text{rank}(\hat{T}_3(j\omega_o)) = \text{rank} \begin{bmatrix} A + LC - j\omega_o I_n & L \\ C & I_p \end{bmatrix} - n \).

But \( \text{rank} \begin{bmatrix} A - F - j\omega_o I_n & -I_n \\ I_n & 0_n \end{bmatrix} = 2n, \forall \omega_o \). We also have, \( \text{rank} \begin{bmatrix} A + LC - j\omega_o I_n & L \\ C & I_p \end{bmatrix} = \text{rank} \begin{bmatrix} A - j\omega_o I_n & L \\ 0_{pm} & I_p \end{bmatrix} = n + p \); if \( j\omega_o \) is not an eigenvalue of \( A \). Therefore, \( \text{rank}(\hat{T}_2(j\omega_o)) = n \) and \( \text{rank}(\hat{T}_3(j\omega_o)) = p \) (full ranks) if \( j\omega_o \) is not eigenvalue of \( A \), and the proof is completed. \( \triangle \)

Based on the results in lemma 2, it can be proven that, for \( \hat{T}_{ef}(s) \) in (19), there exists a transfer matrix \( \hat{Q}(s) \in RH_\infty \) that satisfies \( \hat{T}_{ef}(j\omega_o) = 0 \) (see Appendix A for details about the computation of \( \hat{Q}(s) \)). Therefore, for the dynamic observer in (9)-(13), an optimal gain can be found (unlike the static case as was shown in Lemma 1). This shows the advantage of using the dynamic observer in this case. To summarize, based on the previous results, we will define an optimal residual generator for the narrow frequency band in the form of the following theorem:

**Theorem 2** An observer of the form (9)-(13) along with \( r = y - \hat{y} \) is an optimal residual generator for the sensor fault identification problem, with faults in a narrow frequency band around \( \omega_o \), only if the observer gain \( K \) is dynamic and if the dynamic gain \( K \) is chosen as the Linear Fractional Transformation \( \text{LFT}(J, Q) \) in Figure 4 where \( \hat{Q}(s) \in RH_\infty \) solves the problem \( \hat{T}_{ef}(j\omega_o) = 0 \) for the transfer matrix \( \hat{T}_{ef}(s) \) in (19). Moreover, assuming \( \hat{Q}(s) \) has the state space matrices \( A_q, B_q, C_q \) and \( D_q \), the dynamic gain parameters are given from:

\[
A_L = \begin{bmatrix} A - F + LC + D_q C & -C_q \\ -B_q C & A_q \end{bmatrix}, \quad B_L = \begin{bmatrix} -L - D_q \\ B_q \end{bmatrix}, \quad C_L = \begin{bmatrix} F - D_q C & C_q \end{bmatrix}, \quad D_L = [D_q].
\]

**Proof:** A direct result of Lemma 1, Lemma 2 and the results in Appendix A. \( \triangle \)

**Remarks**

- According to the previous theorem, an optimal residual generator guarantees sensor faults estimation and at the same time state estimation (with minimum energy for the estimation errors). This constitutes another advantage of our dynamical approach: an advantage of having state estimation in presence of sensor faults is the possibility to use the observer in fault tolerant output feedback control (i.e, if a reconfiguration control action is involved).

- From the special cases of interest is the case of sensor bias, where the previous approach can be used to get an exact estimation of all sensor biases at the same time. A sufficient condition is that the matrix \( A \) has no eigenvalues at the origin.

4 \( H_\infty \) sensor fault diagnosis

In this section, we consider two different cases: the low frequency range and the high frequency range. For the first case, we assume the system to be affected by sensor faults of low frequencies determined by a cutoff frequency \( \omega_1 \), i.e the frequency pattern for \( f_s(t) \) is confined to the region \([0, \omega_1]\). On the other hand, in the high frequency case, we assume these faults to have very high
frequencies above a minimum frequency \( \omega_b \), i.e. the frequencies are confined to the region \([\omega_b, \infty)\).

As mentioned in section 3, by using the dynamic observer in (9)-(13), the error dynamics (due to general sensor faults) can be represented by Figure 3 where the plant \( G \) has the state space representation shown in (17) with the matrices defined in (14)-(15) and where the controller \( K \) is given in (13). Therefore, the two previous problems can be solved by adding weightings to the standard setup in Figure 3 that emphasize the frequency range under consideration, and by solving these problems as weighted \( H_\infty \) control problems.

However, before introducing weightings, it is important to note that the standard form in (14)-(15) does not satisfy all of the regularity assumptions in the \( H_\infty \) framework (Notice that \( D_{12}^T D_{12} \) is singular), and hence observer synthesis can not be carried out directly using the standard \( H_\infty \) solution. Fortunately, regularization can be done by extending the external output \( \zeta \) in Figure 3 to include the “scaled” vector \( \beta \nu \); with \( \beta > 0 \). It can be seen that the standard form in (14)-(15) has now the following form:

\[
\begin{align*}
\dot{z} &= [A] z + [0_{np} \ -I_n] \begin{bmatrix} \tau \\ \nu \end{bmatrix} \\
\begin{bmatrix} e \\
\beta \nu \end{bmatrix} &= \begin{bmatrix} I_n \\ 0_n \end{bmatrix} z + \begin{bmatrix} 0_{np} \\ 0_{np} \end{bmatrix} \begin{bmatrix} 0_n \\ \beta I_n \end{bmatrix} \begin{bmatrix} \tau \\ \nu \end{bmatrix}
\end{align*}
\]

which can also be represented by the standard setup shown in Figure 3 with the same variables in (16), except for redefining the matrices of \( G(s) \) in (17) and defining \( \zeta \triangleq \begin{bmatrix} e \\ \beta \nu \end{bmatrix}^T \). All the regularity assumptions of the so called “general \( H_\infty \) problem” (see [24], Chapter 14, for more details) are now satisfied iff \( A \) has no eigenvalues on the imaginary axis.

The following lemma demonstrates a certain equivalence relationships between the standard form in (14)-(15) and the regularized one in (21)-(22) (see Appendix B for the detailed proof. The proof is similar to that of Lemma 2 in [17] and is included for clarification). This result is the basis to find a parameterization of all possible observers similar to the one obtained in section 3.

**Lemma 3** Let \( R_1 \) be the setup in Figure 3 associated with (14)-(15), \( R_2 \) be the one associated with (21)-(22) and consider a stabilizing controller \( K \) for both setups. Then \( \| \hat{R}_1 \|_\infty < \gamma \) if and only if \( \exists \beta > 0 \) such that \( \| \hat{R}_2 \|_\infty < \gamma \).

### 4.1 The low frequency range case

We now consider sensor faults of low frequencies determined by a cutoff frequency \( \omega_l \). The SISO weighting \( \hat{w}_l(s) = \frac{a s + b}{s^2} \) [24], emphasizes this low frequency range with “b” selected as \( \omega_l \) and “a” as an arbitrary small number for the magnitude of \( \hat{w}_l(j \omega) \) as \( \omega \to \infty \). Therefore, with a diagonal transfer matrix \( \hat{W}(s) \) that consists of these SISO weightings, the observer problem in Figure 3 can be modified to the weighted version in Figure 5.

It can be seen that the augmented plant \( \hat{G} \) (consisting of the weighting \( W \) cascaded with \( G \) in (21)-(22)) is given by:

\[
\hat{G}(s) = \begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} = \begin{bmatrix} A_w & 0_{pn} \\ 0_{np} & A \end{bmatrix} \begin{bmatrix} I_p \\ 0_{np} \end{bmatrix} \begin{bmatrix} 0_{pn} \\ \beta I_n \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{B}_1 \\ \hat{B}_2 \\ \hat{C}_1 \\ \hat{D}_{11} \\ \hat{D}_{12} \\ \hat{C}_2 \\ \hat{D}_{21} \\ \hat{D}_{22} \end{bmatrix}
\]
Figure 5: Weighted standard setup.

where \( A_w = 0_p \), \( C_w = \text{diag}_p(b) \) and \( D_w = \text{diag}_p(a) \). However, this standard form violates assumptions 1 and 3 of the regularity assumptions summarized earlier; since \((A, B_2)\) is not stabilizable and the rank conditions are not satisfied at \( \omega = 0 \). Therefore, we introduce the modified weighting \( \tilde{w}_{\text{mod}}(s) = \frac{as + b}{\tilde{c}} \) with arbitrary small positive “c”. It is easy to see that, with this modification, the augmented plant \( \tilde{G} \) is the same as (23) except for \( A_w \) which is now given by the stable matrix \( \text{diag}_p(-c) \) and \( C_w \) given by \( \text{diag}_p(b - ac) \). Similar to the non weighted case, all the regularity assumptions are satisfied iff \( A \) has no eigenvalues on the imaginary axis. To this end, we define the regular \( H_\infty \) problem associated with the low frequency range as follows:

**Definition 1** (Low frequency \( H_\infty \)) Given \( \beta > 0 \), find \( S \), the set of admissible controllers \( K \) satisfying \( \| T_{\tilde{c}} \|_\infty < \gamma \) for the setup in Figure 5 where \( \tilde{G} \) has the state space representation (23) with \( A_w = \text{diag}_p(-c) \), \( C_w = \text{diag}_p(b - ac) \) and \( D_w = \text{diag}_p(a) \).

Based on the previous results, we now present the main result of this section in the form of the following theorem for an optimal residual generator in \( L_2 \) sense:

**Theorem 3** An observer of the form (9)-(13) along with \( r = y - \hat{y} \) is an optimal residual generator for the sensor fault identification problem, with faults of low frequencies below the cutoff frequency \( \omega_1 \), if the dynamic gain \( K \in S^* \), where \( S^* \) is the set of controllers solving the \( H_\infty \) optimal control problem in Definition 1 with the minimum possible \( \gamma \).

**Proof**: A direct result of the previous discussion.

**Comments**

- A residual generator that is optimal in the sense of Theorem 3 can be found by using an iterative binary search algorithm over the constant \( \beta \) (in order to achieve the minimum possible \( \gamma \) for the problem in Definition 1 which has \( \beta \) as one of its parameters). Existing software packages can be used to solve this regular \( H_\infty \) problem for a given \( \beta \).

- The constants \( a \) and \( c \) should be selected as arbitrary small positive numbers, while \( b \) must approximately be equal to \( \omega_1 \) (the cutoff frequency). Different weightings could also be used for the different sensor channels. In this case \( A_w = \text{diag}(-c_1, \cdots, -c_p) \), \( C_w = \text{diag}(b_1 - a_1c_1, \cdots, b_p - a_p c_p) \) and \( D_w = \text{diag}(a_1, \cdots, a_p) \).

- A parameterization of all possible observers can be obtained using the approach in [24], Chapter 14. The difference between this parameterization and the one obtained in section 3 is that the matrices \( F \) and \( L \) must satisfy two Riccati equations in addition to the stability condition. Also, \( \hat{Q}(s) \) must satisfy \( \| \hat{Q} \|_\infty < \gamma^* \), where \( \gamma^* \) is the minimum possible solution for the problem in Definition 1.
In case of multiple frequency bands, a bank of observers can be used where each one estimates the faults vector in a specific range. These multiple estimates can then be used to restore the original fault vector. For example, in the case of simultaneous low frequency and narrow band faults and/or disturbances, the structure in Figure 6 can be used to solve the problem. This structure is similar to the one proposed in [7] for fault isolation. The difference in our case is that each observer uses all the system outputs, while filters are added to extract the frequency information.

Figure 6: Residual Set over a Frequency Range.

4.2 The high frequency range case

Similar to the low frequency range, a SISO weighting \( \hat{w}_h(s) = \frac{s+(a\times b)}{b} \) could be selected to emphasize the high frequency range \([w_h, \infty)\) with “\(b\)” selected as \(w_h\) and “\(a\)” as an arbitrary small number for \(|\hat{w}_h(j\omega)|\) as \(\omega \to 0\). Since this weighting is not proper, a modified weighting can be used, [24], as \(\hat{w}_{hmod}(s) = \frac{s+(a\times b)}{c + b} \) with an arbitrary small \(\epsilon > 0\).

With the help of \(\hat{w}_{hmod}(s)\), a suitable weighting \(W\) that emphasizes the high frequency range for the observer problem in Figure 5 can be designed. The augmented \(\bar{G}\) is also given from (23) (same as the low frequency case), but with \(A_w, C_w\) and \(D_w\) given as \(\text{diag}_{p}(-\frac{1}{\epsilon})\), \(\text{diag}_{p}(\frac{a\times b}{\epsilon} - \frac{k}{\epsilon})\) and \(\text{diag}_{p}(-\frac{1}{\epsilon})\) respectively.

It is straightforward that this augmented plant \(\bar{G}\) satisfies all of the regularity assumptions iff \(A\) has no eigenvalues on the imaginary axis and therefore, similar to the low frequency range, a regular \(H_{\infty}\) problem related to the high frequency range can be defined. Also, a theorem for high frequency optimal residual generator can be defined in a similar way to Theorem 3 (details are omitted due to similarity).
5 Experimental results

In this section we consider an illustrative example using the QUANSEF rotary inverted pendulum (ROTPEN) shown schematically in Figure 7, [13]. The angle that the perfectly rigid link of length $l_1$ and inertia $J_1$ makes with the $x$-axis of an inertial frame is denoted $\theta_1$ (rad). Also, the angle of the pendulum (of length $l_2$ and mass $m_2$) from the $z$-axis of the inertial frame is denoted $\theta_2$ (rad). The ROTPEN has a state space model of the form $\dot{x} = f(x) + g(x)u$ where $x = [\theta_1 \theta_2 \dot{\theta}_1 \dot{\theta}_2]^T$

![Figure 7: The Rotary Inverted Pendulum (ROTPEN).](image)

is the state vector and $u$ is the scalar servomotor voltage input (Volt). The output is assumed to be $\theta_2$ (the pendulum angle), i.e $y = x_2$. The system parameters are [13]: $l_1 = 0.215$ m, $l_2 = 0.335$ m, $m_2 = 0.1246$ Kg and $J_1 = 0.0064$ Kg.m$^2$.

In the experiments, the results in sections 3 and 4 are assessed on the linearized model around the inverted position. A small operating range for the pendulum is guaranteed by using a linear state feedback controller and a feedforward gain as follows:

$$u = -F \cdot x + p \cdot r$$  \hspace{1cm} (24)

where “$r$” is the reference motor angle to be tracked, which is assumed as 30 degrees. First, the linear model is obtained as (see Appendix C):

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (25)

The control parameters in (24) are computed by placing the poles of $(A - BF)$ at $\{-18 + 18i, -18 - 18i, -1.5 + 1.5i, -1.5 - 1.5i\}$ (see Appendix C). In the experiments, different fault scenarios are assumed to affect the pendulum sensor. The linear design technique is used to diagnose these faults, and their effect on the tracking performance is also considered.
5.1 Case Study 1

An important fault that affects the ROTPEN in real-time is a sensor fault introduced by the pendulum encoder. The encoder returns the pendulum angle relative to the initial condition, assuming this initial condition to be \( \theta_2 = 0 \). This constitutes a source of bias when the pendulum initial condition is unknown or is deviated from the inverted position. In this case study, we use the design procedure in section 3 to accurately estimate and tolerate these faults. The faults are assumed to affect the pendulum measurement, and therefore the reduced LTI model is used for the design (see Appendix C for all models and parameters of this case study). Two observers are compared in this case study:

1) A high gain Luenberger observer of the form (3)-(4). The static gain \( L_1 \) is computed by placing the poles of \((A - L_1C)\) at \{-70, -20 + 5i, -20 - 5i\}.

2) A dynamic observer of the form (9)-(13). The dynamic gain \( K_1 \) is computed using the design procedure in section 3 for the special case \( \omega_o = 0 \). To ensure fast convergence of the dynamic observer, and avoid delay in the detection time, the matrices \( L \) and \( F \) are computed with the eigen values \([-2 - 50 - 70]\) and \([-10 - 150 - 200]\) respectively.

Figure 8 compares the two observers with respect to one sensor bias that results from the error in the initial condition of the pendulum. As can be seen in the figure, the dynamical observer outperforms the static observer in the bias estimation. This shows an important application of the dynamic observer in the problem of estimating the initial condition of the pendulum. In Figure 9, a sensor bias of 10 degrees is added to the pendulum measurement at the time 10 seconds. The two observers are used in observer-based control. The effect that the correct fault estimation has on the tracking performance is clear in the case of the dynamic observer. This is also illustrated in Figure 10 for a different fault scenario (a time-varying bias of 5 degrees that starts at 10 seconds, then increases to 10 degrees at 40 seconds, and decreases back to 0 degrees at 70 seconds).

Figure 8: Case 1 - Sensor Bias Estimation.

5.2 Case Study 2

We consider the case of low frequency sensor faults (in the range \([0, 1 \text{ rad/sec}]\)). Using the design in section 4 (and with \( a = 0.1 \), \( b = 1 \) and \( c = 0.001 \)), the optimal observer gain \( K_2 \) is obtained by
solving the $H_\infty$ problem in Definition 1 using the command \textit{hinfsyn} in MATLAB, with minimum $\gamma$ as 10 and with $\beta = 1$. Using this observer for fault diagnosis, a correct estimation of a low frequency sensor fault is shown in Figure 11. Figure 12 compares the response of the dynamic observer-based controller with the Luenberger-based controller in this case. The improvement in the tracking performance is clear in this figure.

6 Conclusion

In this paper, we considered the use of a dynamic observer structure for the sensor faults diagnosis problem. This structure offers extra degrees of freedom over the classical Luenberger structure and we showed how this freedom can be used for the sensor faults and state estimations problems. For the narrow frequency band case, the problem was shown to be equivalent to an output zeroing problem for which a dynamic gain is necessary. The use of appropriate weightings to transform this problem into a standard $H_\infty$ optimal control problem was also demonstrated. The introduced strategies were applied through experiments to the QUANSE\textsuperscript{TM} rotary inverted pendulum. Future work includes studying the robustness properties of the introduced fault diagnosis strategies, and extending this framework to more general system faults.
Figure 11: Case 2 - Low Frequency Estimation.

Figure 12: Case 2 - Observer-based Response.

A Algorithm for $\hat{Q}(s)$ computation

If $j\omega_o$ is not an eigenvalue of $A$, then (from Lemma 2) the matrices $\hat{T}_2(j\omega_o)$ and $\hat{T}_3(j\omega_o)$ are invertible, and the matrix equation $\hat{T}_{ef}(j\omega_o) = 0$ can be solved for $Q(j\omega_o)$ as follows:

$$\hat{Q}(j\omega_o) = -\hat{T}_2^{-1}(j\omega_o) \hat{T}_1(j\omega_o) \hat{T}_3^{-1}(j\omega_o) = \hat{Q}_{re} + j\hat{Q}_{im}$$

where $\hat{Q}_{re}$ and $\hat{Q}_{im}$ are $n \times p$ matrices that represent the real and imaginary parts respectively.

Let $\hat{Q}(s) = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$, where $A_q \in \mathbb{R}^{\ell \times \ell}$, $B_q \in \mathbb{R}^{\ell \times p}$, $C_q \in \mathbb{R}^{n \times \ell}$ and $D_q \in \mathbb{R}^{n \times p}$ and where $\ell$ is the order of $\hat{Q}(s)$. Then the problem of computing $\hat{Q}(s) \in RH_\infty$ reduces to solving:

$$C_q (j\omega_o I_\ell - A_q)^{-1} B_q + D_q = \hat{Q}_{re} + j\hat{Q}_{im}$$

(26)
for a stable \( A_q \) with a suitable order \( \ell \), and for \( B_q, C_q \) and \( D_q \). By choosing \( \ell = n \), \( C_q = I_n \) and \( A_q = -I_\ell \), the problem in (26) reduces to:

\[
\frac{1}{1 + \omega_o^2} B_q^{(ij)} + D_q^{(ij)} = \hat{Q}_{te}^{(ij)}; \ i = 1, \ldots, n; \ j = 1, \ldots, p
\]

(27)

\[
-\omega_o \frac{1}{1 + \omega_o^2} B_q^{(ij)} = \hat{Q}_{te}^{(ij)}; \ i = 1, \ldots, n; \ j = 1, \ldots, p
\]

(28)

where \( B_q^{(ij)} \) and \( D_q^{(ij)} \) are the elements in row \( i \) and column \( j \) of the \( n \times p \) matrices \( B_q \) and \( D_q \) respectively. Equations (27) and (28) can be solved simultaneously for all the elements of \( B_q \) and \( D_q \). This completes the solution for \( \hat{Q}(s) \in RH_\infty \).

**B Proof of Lemma 3**

Using the definitions in Lemma 3, the transfer matrices \( \hat{R}_1(s) \) and \( \hat{R}_2(s) \) can be computed as the Linear Fractional Transformations (LFT) of \( G \) and \( K \) as follows (where \( k \) is the controller order):

\[
\hat{R}_1(s) = \begin{bmatrix}
A - D_L C & -C_L \\
B_L C & A_L \\
I_n & 0nk \\
D_L & B_L
\end{bmatrix}
\]

(29)

\[
\hat{R}_2(s) = \begin{bmatrix}
A - D_L C & -C_L \\
B_L C & A_L \\
I_n & 0nk \\
\beta D_L C & \beta C_L \\
\beta D_L & 0np
\end{bmatrix}
\]

(30)

We will define \( \hat{A} \) as the common state matrix \( \begin{bmatrix} A - D_L C & -C_L \end{bmatrix} \), and we will hence refer to the state transition matrix of \( \hat{R}_1(s) \) and \( \hat{R}_2(s) \) as: \( \hat{H}(s) \equiv (sI_{n+k} - \hat{A})^{-1} = \begin{bmatrix} \hat{H}_{11}(s) & \hat{H}_{12}(s) \\
\hat{H}_{21}(s) & \hat{H}_{22}(s) \end{bmatrix} \).

From (29)-(30) and the definition of \( \hat{H}(s) \), we have:

\[
\hat{R}_1(s) = -\hat{H}_{11}(s) D_L + \hat{H}_{12}(s) B_L
\]

(31)

\[
\hat{R}_2(s) = \begin{bmatrix}-\hat{H}_{11}(s) D_L + \hat{H}_{12}(s) B_L \end{bmatrix}
\]

(32)

where \( \hat{N}(s) = \begin{bmatrix} A - D_L C & -C_L \\
D_L C & C_L \\
D_L & D_L \end{bmatrix} \).

(Sufficiency) For the “two input/one output” setup \( R_2 \), let \( \exists \beta > 0 \) and a stabilizing controller \( K \) such that \( \| \hat{R}_2(s) \|_\infty < \gamma \). But \( \| \hat{R}_2(s) \|_\infty = \max \left( \beta \| \hat{N}(s) \|_\infty, -\hat{H}_{11}(s) D_L + \hat{H}_{12}(s) B_L \|_\infty \right) \).

Therefore, \( \| \hat{R}_1(s) \|_\infty < \gamma \); from (31).

(Necessity) Let \( \exists \) a controller \( K \) such that \( \| \hat{R}_1(s) \|_\infty < \gamma \).

It follows that \( \| -\hat{H}_{11}(s) D_L + \hat{H}_{12}(s) B_L \|_\infty = \sigma < \gamma \). Therefore, \( \| \hat{R}_2(s) \|_\infty = \max \left( \beta \| \hat{N}(s) \|_\infty, \sigma \right) \).

But since \( K \) is a stabilizing controller, then \( \| \hat{N}(s) \|_\infty = \rho \) (where \( \rho \) is a finite number).

Therefore, \( 0 < \beta < \frac{\gamma}{\rho} \Rightarrow \| \hat{R}_2(s) \|_\infty < \gamma \). \( \triangle \)
C  Models and parameters of the ROTPEN experiments

*Linear model parameters:*

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -25.14 & -17.22 & 0.2210 \\
0 & 68.13 & 16.57 & -0.599 \\
\end{bmatrix}; \quad B = \begin{bmatrix}
0 \\
0 \\
26.3370 \\
-25.3596 \\
\end{bmatrix}
\]

*Control parameters:*

\[
F = \begin{bmatrix}
-2.5207 & -35.9896 & -2.4878 & -3.4189 \\
\end{bmatrix}; \quad p = -2.5207
\]

*Reduced-order model for observer design (\( \ddot{x} = [\theta_2 \dot{\theta}_1 \dot{\theta}_2]^T \)):*

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 1 \\
-25.14 & -17.22 & 0.2210 \\
68.13 & 16.57 & -0.599 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
26.3370 \\
-25.3596 \\
\end{bmatrix} u \\
\ddot{y} = \begin{bmatrix}
1 & 0 & 0 \\
\end{bmatrix} \dddot{x}
\]

*Luenberger observer:*

\[
L_1 = \begin{bmatrix}
92.2 & 95.7 & 1643.9 \\
\end{bmatrix}^T
\]

*Linear dynamic observer for sensor bias: (\( K_1 \))*

\[
A_{L1} = \begin{bmatrix}
93 & 0 & 0 & 0 & 0 \\
11850 & -150 & 0 & 0 & 0 \\
20836 & 0 & -200 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}, \quad B_{L1} = \begin{bmatrix}
-103 \\
-11850 \\
-20836 \\
0 \\
0 \\
\end{bmatrix}, \\
C_{L1} = \begin{bmatrix}
-197 & 0 & 1 & 0 & 0 \\
-10280 & 133 & 0 & 0 & 0 \\
-22713 & 17 & 199 & 0 & 0 \\
\end{bmatrix}, \quad D_{L1} = \begin{bmatrix}
207 \\
10255 \\
22781 \\
\end{bmatrix}
\]

*Linear dynamic observer for low frequencies: (\( K_2 \), obtained for \( a = 0.1, \ b = 1, \ c = 0.001, \beta = 1, \gamma = 10 \))

\[
A_{L2} = \begin{bmatrix}
10.6178 & 10.6118 & 0.0049 & -0.0349 \\
-34.9216 & -63.2524 & -2.9453 & -3.3428 \\
34.1085 & 60.292 & -17.5710 & -0.3241 \\
-240.9994 & -177.2030 & 16.0249 & -0.9341 \\
\end{bmatrix}, \quad B_{L2} = \begin{bmatrix}
-47.5 \\
-155.9 \\
-152.6 \\
1077.7 \\
\end{bmatrix}, \\
C_{L2} = \begin{bmatrix}
0.0126 & 6.3495 & 0.6567 & 0.9843 \\
0.0013 & 0.6567 & 0.0803 & 0.1090 \\
0.0020 & 0.9843 & 0.1090 & 0.1661 \\
\end{bmatrix}, \quad D_{L2} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

References


