LMI-based sensor fault diagnosis for nonlinear Lipschitz systems

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Abstract

The problem of sensor fault diagnosis in the class of nonlinear Lipschitz systems is considered. A dynamic observer structure is used with the objective to make the residual converge to the faults vector achieving detection and estimation at the same time. It is shown that, unlike the classical constant gain structure, this objective is achievable by minimizing the faults effect on the estimation error of the dynamic observer. The use of appropriate weightings to solve the design problem in a standard convex optimization framework is also demonstrated. An LMI design procedure solvable using commercially available software is presented.

Keywords: Dynamic observers; Lipschitz systems; Faults; Sensors; LMI

1. Introduction

The fault diagnosis problem is gaining increasing consideration worldwide in both theory and application. This is due to the growing demand for higher reliability in control systems, and hence the importance of having a monitoring system to detect the existing faults and specify their locations and significance in the control loop. The observer-based approach is one of the most popular techniques used for fault diagnosis. Many standard observer-based techniques exist in the literature providing different solutions to both the theoretical and practical aspects of the problem for the linear time-invariant (LTI) case (see Frank, 1990; Willsky, 1976 for good surveys). The basic idea behind this approach is to estimate the outputs of the system from the measurements by using either static gain observers in a deterministic framework (Zhong, Ding, Lam, & Wang, 2003) or Kalman filters in a stochastic framework (Chen, Mingori, & Speyer, 2003). The output estimation error is then used as the residual. In contrast to the LTI case, however, the nonlinear problem lacks a universal approach and is currently an active area of research (see Adjallah, Maquin, & Ragot, 1994; Garcia & Frank, 1997; Hammouri, Kinnaret, & Elyaagoubi, 1999; Kabore & Wang, 2001; Vemuri, 2001; Wang, Huang, & Daley, 1997; Yu & Shields, 1996 for some important nonlinear results). The main obstacle in the solution of the observer-based nonlinear fault detection problem is the lack of a universal approach for nonlinear observer synthesis. In this paper, we focus on the class of Lipschitz systems of the form

\[ \dot{x}(t) = Ax(t) + f(t), \]

\[ y(t) = Cx(t) + f(t), \quad A \in \mathbb{R}^{n \times n}, \quad C \in \mathbb{R}^{p \times n}, \]

where \((A, C)\) is detectable, \(f(t)\) represent sensor faults, and \(\Phi(x, u, t)\) satisfies

\[ \|\Phi(x_1, u, t) - \Phi(x_2, u, t)\| \leq \|x_1 - x_2\| \]

\(\forall u \in \mathbb{R}^m\) and \(t \in \mathbb{R}\) and \(\forall x_1\) and \(x_2 \in D\), where \(D\) is a closed and bounded region containing the origin (see Pertew, Marquez, & Zhao, 2006 for more details about Lipschitz systems, their importance, and previous works on the Lipschitz observer design problem). In this paper, we make use of the dynamic observer structure introduced in Pertew et al. (2006) for the sensor fault estimation problem where the objective of estimating the fault magnitude is considered in addition to detection.
and isolation. We show that, using the extra degrees of freedom in the new structure, this objective is achievable by minimizing the fault effects in a narrow frequency band on the observer’s estimation error. We also show that the classical constant gain structure cannot solve the problem in this case. The sensor fault diagnosis problem is then modeled as a convex optimization problem and an LMI design procedure, solvable using commercially available software packages, is presented. The use of weightings to consider different frequency patterns is also illustrated.

2. Preliminaries and notation

The classical observer structure for Lipschitz systems of form (1)–(3), with no faults (see Pertew et al., 2006), falls in the class of Luenberger-like observers, namely:

$$\dot{x} = A\dot{x} + \Gamma(u, t) + \Phi(\dot{x}, u, t) + L(y - \hat{y}), \quad L \in \mathbb{R}^{p \times p}, \quad (4)$$

$$\dot{y} = C\dot{x}. \quad (5)$$

In this paper, following Pertew et al. (2006), a different approach is adopted, by making use of dynamical observers of the form

$$\dot{\hat{x}} = A\hat{x} + \Gamma(u, t) + \Phi(\hat{x}, u, t) + \eta, \quad \eta = C_L\hat{\zeta} + D_L(y - \hat{y}), \quad A_L \in \mathbb{R}^{k \times k}, \quad B_L \in \mathbb{R}^{k \times p}, \quad C_L \in \mathbb{R}^{n \times k}, \quad D_L \in \mathbb{R}^{n \times p}, \quad (6)$$

$$\hat{\zeta} = A_L\hat{\zeta} + B_L(y - \hat{y}), \quad (7)$$

$$\eta = C_L\hat{\zeta} + D_L(y - \hat{y}), \quad (8)$$

$$\hat{y} = C\hat{x}. \quad (9)$$

We will write $K = \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}$ to represent the dynamic observer gain in (7)–(8). It was shown in Pertew et al. (2006) that $K$, sufficient to achieve observer convergence, can be represented by a set of controllers. In this paper, we make use of this freedom in the sensor fault diagnosis problem using the residual as

$$r(t) = y(t) - \hat{y}(t). \quad (10)$$

To this end, the following definitions and notation will be used in the paper:

**Definition 1 (Fault detection).** The residual in Eq. (10) achieves fault detection (strong fault detection) if the following condition is satisfied:

$$r(t) = 0; \quad \forall t \iff f(t) = 0; \quad \forall t.$$  

**Definition 2 (Fault isolation).** The residual in (10) achieves fault isolation if

$$(r_i(t) = 0; \quad \forall t \iff f_i(t) = 0; \quad \forall t) \quad \text{for} \quad i = 1, \ldots, p.$$  

**Definition 3 (Fault identification).** Fault identification is satisfied by (10) if

$$(r_i(t) = f_i(t); \quad \forall t) \quad \text{for} \quad i = 1, \ldots, p.$$  

(The previous definitions are borrowed from Chen & Patton, 1999. Also note that in these definitions the residuals transient period is not considered.)

**Definition 4 ($\mathcal{L}_2$ space).** Space $\mathcal{L}_2$ consists of all $\mathcal{L}$-based measurable functions $u : \mathbb{R}^r \to \mathbb{R}^q$, with finite $\|u\|_{\mathcal{L}_2}$, where $\|u\|_{\mathcal{L}_2} = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2 \, dt}$.  

For a system $H : \mathcal{L}_2 \to \mathcal{L}_2$, we will represent by $\gamma(H)$ the $\mathcal{L}_2$ gain of $H$ defined by $\gamma(H) = \sup_{u \in \mathcal{L}_2} \|Hu\|_{\mathcal{L}_2}/\|u\|_{\mathcal{L}_2}$. It is well known that, for an LTI system $H : \mathcal{L}_2 \to \mathcal{L}_2$ (with a transfer matrix $H(s)$), $\gamma(H) \equiv \|H(s)\|_{\mathcal{L}_2} = \sup_{r \in \mathbb{R}} e^{\sigma_{\max}(\hat{H}(jo))}$.

The matrices $I_n, 0_n$, and $0_n m$ will represent the identity matrix of order $n$, the zero square matrix of order $n$, and the zero $n \times m$ matrix, respectively. $\text{Diag}(a, b, \ldots, a_p)$ represents the diagonal square matrix of order $p$ with $[a \ a \ \cdots \ a]_x$ as its diagonal vector, while $\text{diag}(a_1, a_2, \ldots, a_p)$ represents the one with $[a_1 \ a_2 \ \cdots \ a_p]$ as its diagonal vector. $T_{yu}$ represents the transfer matrix from input $u$ to output $y$. $RH_\infty$ denotes the space of all proper real rational stable transfer matrices. The partitioned matrix $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (when used in $y = Hu$) represents

$$(\hat{\zeta} = A\hat{\zeta} + Bu, \quad y = C\hat{\zeta} + Du), \quad \text{and} \quad \hat{H}(s) = C(sI - A)^{-1}B + D.$$  

We will make use of (11) (Zhou & Doyle, 1998), if $s$ is not an eigenvalue of $A$:

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{rank}(\hat{H}(s)), \quad n \text{ being the dimension of } A. \quad (11)$$

The setup in Fig. 1 will also be used throughout the paper along with

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}. \quad (12)$$

We will also make use of the following result (Gahinet & Apkarian, 1994):

**Theorem 1.** Assume stabilizability and detectability of $(A, B_2, C_2)$ and that $D_{22} = 0$, and let $N_{12}$ and $N_{21}$ denote orthonormal bases of the null spaces of $(B_2^T, D_{12}^T)$ and $(C_2, D_{22})$. There exists a controller $K$ such that $\|T_{\infty} \|_\infty < \gamma$ if there exist symmetric $R, S \in \mathbb{R}^{n \times n}$ satisfying the following

$$\begin{bmatrix} \tau & \nu \\ \varphi & \varphi \end{bmatrix}$$

Fig. 1. Standard setup.
system of LMIs:

\[
\begin{bmatrix}
I_n & 0 \\
0 & f
\end{bmatrix}^T \begin{bmatrix}
AR + RA^T RC^T & B_f \\
B_f^T & D_{11}^T - \gamma I
\end{bmatrix} \begin{bmatrix}
I_n & 0 \\
0 & I
\end{bmatrix} < 0,
\]

(13)

\[
\begin{bmatrix}
I_n & 0 \\
0 & f
\end{bmatrix}^T \begin{bmatrix}
A^T S + SA SB - B_f C^T & 0 \\
0 & D_{11}^T - \gamma I
\end{bmatrix} \begin{bmatrix}
I_n & 0 \\
0 & I
\end{bmatrix} < 0,
\]

(14)

\[
\begin{bmatrix}
R & I \\
I & S
\end{bmatrix} \geq 0.
\]

(15)

2.1. Observer-based sensor fault detection

In this section, general sensor faults are assumed. The objective is to develop conditions that guarantee fault detection according to Definition 1. To this end, it can be seen that the residual dynamics are given by

\[
\dot{e}(t) = Ae(t) - \eta(t) + \tilde{\phi}(t),
\]

(16)

\[
r(t) = Ce(t) + f(t),
\]

(17)

where \( e = x - \hat{x} \) is the estimation error, \( \tilde{\phi} = \Phi(x, u, t) - \Phi(x, \hat{x}, \hat{u}, \hat{t}) \). Defining \( \xi = [\eta, \zeta] = [\tilde{\phi}, v, \varphi = r] \), the residual can be represented by

\[
\xi = [A] z + [I_n, 0_{np}] \gamma - I_n \xi,
\]

(18)

\[
[\xi, \varphi] = [I_n, C] z + [0_n, 0_{np}, 0_n, 0_{pm}, I_o, 0_{pm}] \gamma
\]

(19)

which can also be represented by the standard form in Fig. 1 where \( G \) has the representation in (12) with the matrices in (18)–(19) and where \( K \) is the dynamic observer gain. The following theorem is an extension of the observer asymptotic convergence result obtained in Pertew et al. (2006) to the fault detection problem:

**Theorem 2.** Given system (1)–(2), the residual in (6)–(10) achieves fault detection \( \forall \Phi \) satisfying (3) if \( K \) is chosen s.t. \( \sup_{t \in [0, T]} \sigma_{max}(\dot{\tilde{T}}_{\xi1}(j\omega)) < 1/2 \).

**Proof.** The proof is similar to Theorem 4 in Pertew et al. (2006) and is hence omitted. However, it is important to note that the observer’s error can be represented by Fig. 2, where \( \Lambda \) is the static nonlinear operator mapping \( e \) to \( \tilde{\phi} \) and satisfying \( \gamma(\Delta) \leq \mu \) due to (3). The loop is locally asymptotically stable when \( f = 0 \) (see Pertew et al., 2006, Corollary 1), completing the proof. \( \square \)

3. The Lipschitz sensor fault identification problem

We here present our main results by developing conditions to solve the fault identification problem (according to Definition 3) with sensor faults in different frequency ranges, presenting a numerical LMI design procedure for its solution.

3.1. The narrow frequency band case

Since the residual “r” is given from Eq. (17), it is then clear that the observer estimation error “e” constitutes a part of the residual response, and that by minimizing “e” the residual converges to f which guarantees fault identification in this case. However, it was seen in Section 2.1 that the estimation error “e” can be represented by the feedback interconnection in Fig. 2 where \( f \) is the sensor fault vector that affects the system. Hence, minimizing “e” is equivalent to minimizing the effect of “f” on the feedback interconnection of Fig. 2. We consider this minimization problem in \( L^2 \) sense by assuming the sensor fault to have finite energy, and applying the small gain theorem to Fig. 2. This assumption is with no loss of generality, since it is guaranteed over any finite time operation of the observer. In this section, we consider the solution of this minimization problem when \( f \) is in a narrow frequency band around a nominal frequency \( \omega_0 \). From the special cases of interest is the case of sensor bias and the case of faults of known harmonics that frequently appear in power systems. We first show that the problem is not tractable for the classical structure in (4)–(5), and then present a solution using the dynamic observer. Towards that goal, we will first assume that the Fourier transform of the sensor fault \( F(j\omega) \) has a frequency pattern restricted to the narrow band \( \omega_0 \pm \Delta \omega \) as described by

\[
|F(j\omega)| \leq \begin{cases}
A, & |\omega - \omega_0| < \Delta \omega, \\
\delta, & \text{otherwise},
\end{cases}
\]

(20)

where \( \delta \) is a small neglected number for the frequency magnitudes outside the region of interest, and where \( A \) is a positive upper bound on these magnitudes inside the considered domain. We will then define an observer gain \( K \) as optimal if \( ||e||_{L^2} \) can be made arbitrarily small for all possible sensor faults satisfying (20). But by applying the small gain theorem to Fig. 2 when fault detection is satisfied (i.e., when \( K \) satisfies \( \|\tilde{T}_{ef}\|_{\infty} \leq \mu < 1/\pi \)) we have: \( ||e||_{L^2} \leq (1/(1 - \mu \pi)) ||e||_{L^2} \). And since (as \( \Delta \omega \to 0 \)), \( \tilde{T}_{ef}(j\omega) \to \tilde{T}_{ef}(j\omega_0) \) then we have \( ||e||_{L^2} \leq \|e||_{L^2} \leq \sigma_{max}(\tilde{T}_{ef}(j\omega_0))||f||_{L^2} \tilde{T}_{ef}(j\omega_0) \), and, therefore, it is easy to see that an optimal gain \( K \) is one that satisfies \( \tilde{T}_{ef}(j\omega_0) = 0 \). By assuming that the fault detection objective is satisfied (as stated in Theorem 2), it follows that fault identification according to Definition 3 is satisfied if the following two conditions are satisfied: (i) \( \|\tilde{T}_{ef}\|_{\infty} < 1/\pi \), (ii) \( \tilde{T}_{ef}(j\omega_0) = 0 \), where the first one is a sufficient condition in order to achieve fault detection.
Based on the previous discussion, we define an optimal residual generator as:

**Definition 5 (Optimal residual generator for narrow frequency band).** An observer of the form (6)–(10) is said to be an optimal residual generator for the sensor fault identification problem (with faults in a narrow frequency band around \( \omega_0 \)) if the observer gain \( K \) satisfies \( \| \hat{T}_{ef} \|_\infty < 1/\alpha \) and \( \hat{T}_{ef}(j\omega_0) = 0 \), for the standard setup in Fig. 1 where the plant \( G \) has the state space representation in (12) with the matrices defined in (18)–(19).

Now we present the main result of this section in the form of a theorem showing that the classical observer structure can never be an optimal residual generator. This shows the importance of having a dynamic observer generator:

**Theorem 3.** An observer of form (4)–(5) can never be an optimal residual generator according to Definition 5.

**Proof.** First, using (18)–(19) and the dynamic observer gain \( \hat{T}_{ef} \) it can be shown that \( \hat{T}_{ef} \) is given from

\[
\hat{T}_{ef} = \hat{T}_{ef}(s) = \begin{bmatrix} A - D_L C - C_L & -D_L \\ B_L & A_L \\ I_n & 0_{nk} & 0_{np} \end{bmatrix}
\]

(21)

The proof follows by noting that (when the observer gain \( K \) is replaced by the static gain \( L \)) the transfer matrix from \( f \) to \( \hat{y}_f \) in (21) is given by \( \hat{T}_{ef} = \begin{bmatrix} A - LC & -L \\ I_n & 0_{np} \end{bmatrix}. \) And since the gain \( L \) is chosen to stabilize \( A - LC \), then \( \langle \forall \omega_0 \rangle j \omega_0 \) is not an eigenvalue of \( A - LC \). Therefore, by using (11), we have

\[
\text{rank}(\hat{T}_{ef}(j\omega_0)) = \text{rank} \begin{bmatrix} A - LC - j\omega_0 I_n & -L \\ I_n & 0_{np} \end{bmatrix} - n.
\]

But

\[
\text{rank} \begin{bmatrix} A - LC - j\omega_0 I_n & -L \\ I_n & 0_{np} \end{bmatrix} = \text{rank} \begin{bmatrix} L & 0_n \\ 0_n & L \end{bmatrix} = n + \text{rank}(L).
\]

Therefore, \( \text{rank}(\hat{T}_{ef}(j\omega_0)) \neq 0 \) unless \( L = 0 \). This implies that no gain \( L \) can satisfy \( \hat{T}_{ef}(j\omega_0) = 0 \), and therefore the static observer structure can never be an optimal residual generator according to Definition 5. 

In the following section (Section 3.2), we provide a numerical approach based on LMIs by modeling the problem as a convex optimization problem using the dynamic observer structure in (6)–(10).

### 3.2. An LMI design procedure

We now show that the second objective, i.e., \( \hat{T}_{ef}(j\omega_0) = 0 \), can also be modeled as a weighted \( H_\infty \) problem solvable using the dynamic observer formulation. To this end, we first note that for an observer gain \( K \) that satisfies the fault detection condition (as stated in Theorem 2), the following two statements are equivalent: (i) \( \hat{T}_{ef}(j\omega_0) = 0 \) (ii) \( W(s)\hat{T}_{ef}(s) \in RH_\infty \), where \( W(s) = \text{diag}_{p_1}(1/\alpha) \) if \( \omega_0 = 0 \) and \( W(s) = \text{diag}_{p_2}(1/(s^2 + \omega_0^2)) \) if \( \omega_0 \neq 0 \). The equivalence of these two statements can be seen by first noting that the condition in Theorem 2 implies that \( \| \hat{T}_{ef} \|_\infty < 1/\alpha \) and hence that \( \hat{T}_{ef} \in RH_\infty \). It then follows that \( \hat{T}_{ef}(s) \in RH_\infty \) since \( \hat{T}_{ef} \) and \( \hat{T}_{ef} \) both have the same state transition matrix. Finally, once \( \hat{T}_{ef}(j\omega_0) = 0 \) corresponds to \( j \omega_0 \) being a zero of \( \hat{T}_{ef}(s) \) (which is equivalent to canceling the poles of \( W(s) \) on the imaginary axis), it is then easy to see that \( \hat{T}_{ef}(j\omega_0) = 0 \) is equivalent to having \( W(s)\hat{T}_{ef}(s) \in RH_\infty \).

According to the previous discussion, it follows that the objective \( \hat{T}_{ef}(j\omega_0) = 0 \) can be restated as follows: \( \exists \omega_0 > \alpha \) such that \( \| \hat{T}_{ef}(s) \|_\infty < 1/\alpha \), where the scalar \( \alpha \) is used for compatibility with the first objective (i.e., \( \| \hat{T}_{ef} \|_\infty < 1/\alpha \)). It then follows that the two objectives can be combined in the unified framework in Fig. 3, where the plant \( G \) has the state space representation in (12) with the matrices defined in (18)–(19).

It can be seen that the augmented plant \( \hat{G} \) in Fig. 3 is given by

\[
\hat{G} = \begin{bmatrix} A & 0_n & 0_n \\ B_1 & A & 0_n \\ C_1 & D_{11} & D_{12} \end{bmatrix}
\]

(22)

where

\[
\begin{align*}
\ell &= p, A_\theta = 0_p, B_\theta = I_p, C_\theta = I_p & \text{if } \omega_0 = 0, \\
\ell &= 2p, A_\theta = \text{diag}_p \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, & C_\theta = \text{diag}_p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } \omega_0 \neq 0.
\end{align*}
\]

Based on the previous results, the following theorem gives necessary and sufficient conditions for solving the problem in Definition 5:

**Theorem 4.** Given system (1)–(2), there exists an optimal residual according to Definition 5, \( \Phi \) satisfying (3) with a Lipschitz constant \( \alpha \), if and only if \( \exists \omega_0 > 0 \) and a gain \( K \) satisfying \( \| \hat{T}_{ef} \|_\infty < 1/\alpha \) for the setup in Fig. 3.
Proof. A direct result of Definition 5 and the previous discussion. □

However, standard $H_{\infty}$ tools cannot be directly applied for the $H_{\infty}$ problem defined in Theorem 4. For instance, the Riccati approach in Zhou and Doyle (1998) cannot be implemented since the augmented plant $\tilde{G}$ in (22) does not satisfy the needed regularity assumptions. Also, the LMIs in Eqs. (13)–(15) are not feasible due to the poles that $\tilde{G}$ has on the imaginary axis, making the use of the LMI approach in Gahinet and Apkarian (1994) impossible. However, by replacing the weightings $W(s)$ by the modified weightings $\tilde{W}(s)$ where $\tilde{W}(s) = \text{diag}_p(1/(s + \lambda))$ if $\omega_0 = 0$ and $\tilde{W}(s) = \text{diag}_p(1/(s^2 + 2\lambda\omega_0s + \omega_0^2))$ if $\omega_0 \neq 0$, with $\lambda \in \mathbb{R}^+$, the augmented plant $\tilde{G}$ in Fig. 3 is still given by Eq. (22), but with $A_\theta$ as

$$A_\theta = \begin{bmatrix} \text{diag}_p(-\lambda) & 0 \\ 0 & -\omega_0^2 - 2\lambda \omega_0 \end{bmatrix} \begin{bmatrix} \omega_0 = 0, \\ \omega_0 \neq 0 \end{bmatrix} \quad (24)$$

which has no poles on the imaginary axis. Using this modified plant and the result in Theorem 1, we propose the following convex optimization problem to solve the problem introduced in Theorem 4:

$$\min_{\lambda, R, S} \lambda$$

subject to “the 3 LMIs in (13)–(15) with $\gamma = \frac{1}{\lambda}$

with matrices in (13)–(15) replaced by the corresponding ones in (22)–(24).

The set of all admissible observer gains $K$ for a given $\lambda$ can then be parameterized using $R, S$ by using the result in Gahinet and Apkarian (1994). It can also be seen that these LMIs are feasible for all $\lambda > 0$, and that minimizing $\lambda$ in this case is equivalent to minimizing $\sigma_{\text{max}}(\tilde{T}_{\text{ef}}(j\omega_0))$. This guarantees that the proposed optimization problem converges to the existing solution as $\lambda \to 0$. It also guarantees that standard software packages can be used to solve this optimization problem.

Comment: According to the previous definition, an optimal residual generator guarantees sensor fault estimation and at the same time state estimation. An advantage of having state estimation in the presence of sensor faults is the possibility to use the observer in fault tolerant output feedback control (i.e., if a reconfiguration control action is involved). Also, from the special cases of interest is the case of sensor bias, where the previous approach can be used to get an exact estimation of all sensor biases at the same time. An important advantage over the adaptive approaches used to diagnose sensor bias in nonlinear systems, such as Venuri (2001) and Wang et al. (1997), is the ability to diagnose piecewise constant bias with the same observer. Moreover, the proposed approach is not limited to sensor biases and can be used to diagnose faults of any harmonics.

3.3. The low- and high frequency ranges

We now consider sensor faults of low frequencies determined by a cutoff frequency $\omega_c$. The SISO weighting $\hat{w}_h(\omega) = (a + b)/\omega$ (Zhou & Doyle, 1998) emphasizes this range with “$b$” selected as $\omega_c$ and “$a$” as an arbitrary small number for the magnitude of $\hat{w}_h(\omega)$ as $\omega \to \infty$. With a diagonal transfer matrix $\hat{W}(s)$ that consists of these SISO weightings (and similar to Section 3.2), the detection and identification objectives can be combined in the unified framework represented by the weighted setup of Fig. 3. In this case, the augmented plant $\tilde{G}$ is given by

$$\tilde{G} = \begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hat{C}_1 \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} = \begin{bmatrix} A_\theta & 0_m & 0_m \\ 0_m & B_\theta & 0_m \\ 0_m & 0_m & C_\theta \end{bmatrix} \begin{bmatrix} 0_m & 0_m & 0_m \\ I_m & 0_m & -I_m \\ 0_m & 0_m & 0_m \end{bmatrix} \quad (25)$$

where $A_\theta = 0_p, B_\theta = I_p, C_\theta = \text{diag}_p(b), \text{ and } D_\theta = \text{diag}_p(a)$. However, this form violates the assumptions of Theorem 1 (note that $(\hat{A}, \hat{B}_2)$ is not stabilizable). Therefore, we introduce the modified weighting $\hat{w}_h_{\text{mod}}(\omega) = (a + b)/\omega (s + \lambda)$, with arbitrary small positive “$\lambda$”. With this modification, the augmented plant $\tilde{G}$ is the same as (25) except for $A_\theta$ which is now given by the stable matrix $\text{diag}_p(-\lambda)$ and $C_\theta$ given by $\text{diag}_p(b - a\lambda)$. Similar to the narrow frequency band case, the assumptions of Theorem 1 are now satisfied and the LMI approach in Gahinet and Apkarian (1994) can be used to solve the $H_{\infty}$ problem. To this end, we define the $H_{\infty}$ problem associated with the low-frequency range as follows:

**Definition 6 (Low-frequency $H_{\infty}$).** Given $\lambda > 0, \varepsilon > 0$, find $\mathcal{F}$, the set of admissible controllers $K$ satisfying $\|\hat{T}_{\text{ef}}\|_{\infty} < \gamma$ for the setup in Fig. 3 where $\tilde{G}$ has the state space representation (25) with $A_\theta = \text{diag}_p(-\lambda), B_\theta = I_p, C_\theta = \text{diag}_p(b - a\lambda)$, and $D_\theta = \text{diag}_p(a)$.

Based on the above, we now present the main result of this section in the form of the following definition for an optimal residual generator in $\mathcal{F}_2$ sense:

**Definition 7 (Optimal residual for low frequencies).** An observer of the form (6)–(10) is an optimal residual generator for the sensor fault identification problem (low-frequency faults below the cutoff frequency $\omega_c$) if the dynamic gain $K$ in $\mathcal{F}_2^*$ (the set of controllers solving the $H_{\infty}$ problem in Definition 6 with $\gamma = 1/\lambda$ and with the minimum possible $\lambda$).

Similar to the low-frequency range, a proper weighting $\hat{w}_h_{\text{mod}}(\omega) = (s + (a \times b))/\omega (s + b)$ (Zhou & Doyle, 1998) with an arbitrary small $\lambda > 0$ could be selected to emphasize the high-frequency range $[w_h, \infty)$ with “$b$” selected as $\omega_h$ and “$a$” as an arbitrary small number for $\hat{w}_h(\omega)$ as $\omega \to 0$. With the help of $\hat{w}_h_{\text{mod}}(\omega)$, a suitable weighting $W$ that emphasizes the high-frequency range can be designed. The augmented $G$ is also given from (25) (same as the low-frequency case), but with $A_\theta, B_\theta, C_\theta$, and $D_\theta$ given as $\text{diag}_p(-b/\lambda), I_p, \text{diag}_p(a \times b)/\lambda - b/\lambda$, and $\text{diag}_p(1/\lambda)$, respectively. It is straightforward that $\tilde{G}$ satisfies all of the assumptions of Theorem 1 and, therefore, similar to the low-frequency range, an $H_{\infty}$ problem related to the high-frequency range [33x239]
range can be defined. An optimal residual generator can be defined in a similar way to Definition 7 (details are omitted due to similarity).

4. Conclusion

A new LMI observer design for Lipschitz nonlinear systems is proposed and is applied in the sensor fault diagnosis problem. This design offers extra degrees of freedom over the classical static gain structure and we show how this freedom can be used to solve both the sensor fault and state estimation problems. For the narrow frequency band case, the problem is shown to be equivalent to an output zeroing problem for which a dynamic gain is necessary. The use of appropriate weightings for different frequency patterns is demonstrated. A systematic design procedure that can be carried out using standard software products is presented. Experimental results illustrating the application of the proposed techniques in robotic systems will be published in a separate article due to space reasons.

References


