Multi-loop control synthesis for unstable systems and its application: An approach based on $\mu$ interaction measure

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SUMMARY

This paper presents a new practical framework for multi-loop controller design in which controllers are designed independently, i.e. a controller in one loop is designed without exploiting information of other controllers. The method is based on the (block) diagonal approximation of a system that is different from its (block) diagonal elements. The focus of this work is on unstable systems and the approximated systems are obtained by minimizing an upper bound of a scaled $\mathcal{L}_\infty$ norm for the error systems. This extends the applicability of conventional $\mu$-interaction measure to a more general scenario. The proposed approach is applied to a numerical example and to a simulated industrial boiler system. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the chemical process industry and large-scale systems (power systems, vehicle platooning), it is common to avoid centralized architectures in favor of simple decentralized or block decentralized controllers, see, e.g. [1–4] and references therein. Although, it is not always possible to respect this desideratum, the reasons for this preference include: ease of understanding by control engineers, less modeling effort, tuning of fewer parameters than the multi-variable controllers and loop failure

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tolerance of the resulting control system, to mention just a few. The design technique of the
demonstrated control system usually involves two steps [5]: (1) selection of suitable pairings
and (2) design of single input single output (SISO) or block decentralized controllers. So far in
process industries, the relative gain array (RGA) [6] and Rijnsdrop’s interaction measure [7] are
found to be the tools for removing impracticable pairings in step 1. For the case of control design
in step 2, all the past research efforts can be grouped into the following three categories [5, 8–14]:

- Simultaneous design: In this approach, the controller is assumed to have a fixed structure
  (such as state space form, PID, PI, etc.) with unknown parameters to be designed. By using
different optimization techniques, parameters are obtained by minimizing the aposite norm
of the closed-loop system. A shortcoming of this approach is its relative complexity and, in
some cases, the non-convexity of the resulting optimization problem [11, 13, 14].

- Sequential loop closing: In this case, each element of the controller (or block) is designed
  sequentially. At first, the controller in the fast loop (inner loop) is designed and the loop is
closed. Then, the controller corresponding to the next loop is designed based on this closed-
loop system information. Owing to its simplicity, this method is now being widely used in
industry. However, when the lower-level loops fail, the failure tolerance of the remaining
loops cannot be guaranteed [5, 8–10, 12–14].

- Independent design: In this method, the control design is based on the (block) diagonal
elements of the system. The controller in each loop is designed by stipulating the form of
each CL transfer function, resulting in an IMC-PID type controller. If the interaction is less
than a certain bound, this method can maintain stability of the overall closed-loop system.
Since information about the controllers in other loops is not used, the design is conservative
but the nominal stability of the remaining loops is guaranteed if any loop fails [5, 12–15].

In this paper, the work is focussed on control by independent designs. Though simpler than other
design techniques, one practically important question may always attract the attention of many
engineers: ‘How much is the performance deterioration of the overall closed-loop system caused
by ignoring the off-diagonal system blocks, and what should be the upper bound on the interactions
such that the overall closed-loop stability can be maintained?’ To answer these questions, a number
of interaction measures are available in the literature, which points out under what conditions the
stability of (block) diagonal parts guarantees stability of the overall closed-loop system. They also
predict closed-loop stability and quantify the performance loss due to the use of different structures
in controllers [1].

Among the different interaction measures available so far, namely, Rijnsdrop’s interaction
measure [7], RGA [6], μ-interaction measure (μ-IM) [1, 15] and direct Nyquist array [16], the μ-IM
is noteworthy. This is because, it offers a dynamic measure of interactions, and is also applicable
to high-order systems. Its equal applicability to block pairings and other elegant properties have
attracted the attention of many researchers in the field of decentralized control [5, 12, 14, 15, 17–19].

Based on the μ-IM, the method proposed by Grosdidier and Morari [15] utilize an independent
design approach. Sufficient conditions for the synthesis are provided under which the controllers
can maintain nominal stability of the overall closed-loop system. This approach, however, suffers
from the shortcoming that it requires that the system and its diagonal version to possess the same
number of RHP poles. Since this criterion is not general, and in most of the cases cannot be
satisfied, it restricts the approach to only open-loop stable systems.

In spite of these restrictions, Skogestad and Morari [5] generalized the results of Grosdidier
and Morari [15] by providing sufficient conditions to guarantee robust performance of the overall
system. Pairing rules for unstable plants, based on $\mu$-IM are then introduced in [20] and their relationship with RGA and NI are explicited. In [12], phase stability conditions are presented, which claimed to remove some conservativeness associated with the $\mu$-IM (since it constrains only the magnitude of each SISO loop). An independent robust decentralized control design approach for unstable as well as non-square systems was carried out in [17, 18], respectively. Several examples in chemical industry where this method can be of practical use are highlighted, e.g. in exothermic reactions there is a need of control strategy for retaining operation at an unstable steady-state. Many other ideas also came into picture and subsequently improved, some were really interesting and innovative, but finding a (block) diagonal-approximated system that possesses the same number of unstable poles as the system itself still remains an open question. In [14, 19], a step toward solving this approximation problem for unstable systems was carried out. It presents a numerical approach, where the decentralized controller is designed based on the (block) diagonal approximation of an unstable plant. The method is interesting because it extended the validity of $\mu$-IM to unstable systems and the outcome of numerical example given in the paper is also good. However, the algorithm bears some complexity and includes approximations, iterations in the frequency-wise approximation step as well as in the parametric identification step, which have no guarantee of convergence.

In this paper, the authors utilize Smith–McMillan decompositions, properties of norms, congruence transformations and reciprocal variant of the projection lemma to provide an easily understandable and programmable approach of obtaining (block) diagonal-approximated systems. By using a constant scaling matrix ($D_r$), the design algorithm is converted into an optimization problem, which can be directly solved by the available numerical software. There is no trial and error involved and, in some cases, the optimization problem involves linear matrix inequalities (LMIs) and only one semidefinite constraint. This quasi-convex optimization can be readily solved using another LMI parser (YALMIP) [21], which is a parser, namely, the interface between different solvers (including LMILab) and Matrix Laboratory (MATLAB). An upper bound on the closed-loop performance due to the decentralized architecture is derived and special attention is paid on the effect of non-minimum phase transmission zeros. It is well known that in multi-loop control systems, because of the inherent interactions between different loops, closing the loop around one subsystem moves the transmission zeros of other subsystems across the imaginary axis [6, 22]. This is possible even if the subsystems are minimum phase and it reflects performance limitation due to the use of decentralized controllers [22]. They are responsible for sensitivity peak as well as bandwidth limitation of the resulting closed-loop system. To overcome this problem, some conditions are developed, so that these zero crossings can be prevented.

The proposed approach is applied to a numerical example and to a simulated industrial utility boiler. The utility part of this plant consists of boilers, electricity-generating systems and headers at four different pressures (6.306, 4.24, 1.068 and 0.372 MPa). The existing decentralized PI controllers of the plant work well under normal load conditions; however, there are problems of poor performance under load fluctuations, and when a lower-level loop fails the failure tolerance of remaining loops cannot be guaranteed. To this end, a nonlinear model of the utility boilers–header system is first explained, where the pressure equation and the drum water level are based on data fitting and physical laws, respectively. Throughout the modeling and control design phase, a nonlinear simulation package of Syncrude utility system called SYNSIM is used. The proposed design strategy is simulated in SYNSIM under the aforementioned load variations and loop failures.

The remainder of this paper is organized as follows. Section 2 deals with some background results on the independent control design approach. In Section 3, an algorithm for obtaining a (block)
diagonal-approximated system that has the same number of unstable poles as the original system is first developed. This is followed by an independent design method for each controller based on skewed-$\mu$ condition that satisfies some constraints because of process interactions. An upper bound on the closed-loop performance is also obtained and a sufficient condition is derived under which the zero crossings can be prevented. Section 4 provides a numerical example and the proposed design strategy is applied to a steam-generating unit. Finally, Section 5 concludes the paper.

2. BACKGROUND

In the following, matrices are denoted by boldface upper case, vectors by boldface lower case and all other variables are in italics. The notation ‘$\star$’ represents terms that are induced by symmetry.

Consider an open-loop system $G(s)$, which is partitioned as $G(s) = (I+E(s))\tilde{G}(s)$ (Figure 1). Here, $\tilde{G}(s)$ consists of the (block) diagonal elements of $G(s)$, such that the number of RHP poles of $G(s)$ and $\tilde{G}(s)$ is the same [15]. The term $E(s) = (G(s) - \tilde{G}(s))\tilde{G}^{-1}(s)$ represents the relative error and let the decentralized controller $K(s)$ be designed such that the transfer matrix $\tilde{H}(s)$ is stable, where $\tilde{H}(s) = GK(s)(I+\tilde{G}K(s))^{-1}$.

The condition under which $K(s)$ also stabilizes the original unstable system $G(s)$ is given by the following lemma:

**Lemma 2.1 (Grosdidier and Morari [15])**
Assume that $\tilde{H}(s)$ is stable. With this assumption, the closed-loop system $H(s) = GK(s)(I+GK(s))^{-1}$ is stable iff the following condition holds:

$$N(0,\text{det}(I+E\tilde{H}(s))) = 0$$

Here, $N(\alpha,F(s))$ denotes the net number of clockwise encirclements of the point $(\alpha,0)$ by the image of Nyquist $D$ contour under $F(s)$.

Based on this lemma, the following measure of interaction results, which is also called the $\mu$-IM condition.

**Theorem 2.1 (Grosdidier and Morari [15])**
Under the assumption of Lemma 2.1, $H(s)$ is stable if,

$$|\tilde{h}_I(j\omega)| < \mu^{-1}(E(j\omega)) \quad \forall \omega$$

(1)

---

Figure 1. A general closed-loop system.
where \( \mu(.) \) is the structured singular value with respect to (w.r.t.) the structure of \( \tilde{H}(s) \) [23], and \( \tilde{h}_1(s) \) is the closed-loop transfer matrix of the \( i \)th loop.

Although the result is influential, the requirement that \( G(s) \) and \( \tilde{G}(s) \) have the same number of RHP poles limits its validity to only open-loop stable systems. The initiative to solve this problem was taken by Kariwala et al. [13, 14, 19], in which the \( \mu \)-IM condition was represented in terms of control sensitivity function. With slight modifications, the following result was obtained.

**Proposition 2.1 (Kariwala [13], Kariwala et al. [14])**

Partition \( G(s) \) as \( G(s) = \tilde{G}(s) + G_1(s) \), such that the number of RHP poles of \( \tilde{G}(s) \) and \( G(s) \) are the same. Then, the decentralized controller \( K(s) \) that stabilizes \( \tilde{G}(s) \) can also stabilize \( G(s) \) if

\[
\tilde{s}(K\tilde{S}(j\omega)) < \mu^{-1}(G_1(j\omega)) \quad \forall \omega \in \mathbb{R}
\]

(2)

where \( \tilde{S}(s) = (I + \tilde{G}K(s))^{-1} \) is the sensitivity function and \( G_1(s) = G(s) - \tilde{G}(s) \) represents the interactions.

Since \( G_1(s) \) is independent of \( K(s) \), the design proceeds to find \( \tilde{G}(s) \) that has the same number of unstable poles as \( G(s) \), such that \( \mu(G_1(s)) \) is minimized. Next, the decentralized controller is designed by using the relation in (2).

3. A SOLUTION TO (BLOCK) DIAGONAL APPROXIMATION AND CONTROLLER DESIGN

The \( \mathcal{H}_\infty \) norm of a stable transfer matrix \( G(s) \) is defined by \( \| G(s) \|_\infty = \sup_{\omega \in \mathbb{R}} \tilde{s}[G(j\omega)] \) and \( \mathcal{L}_\infty \) norm is similar to the \( \mathcal{H}_\infty \) norm, but for an unstable system \( G(s) \) and with no poles on the imaginary axis. In this section, an algorithm is developed, which finds (block) diagonal approximation \( \tilde{G}(s) \) for a given unstable system \( G(s) \) by minimizing the following scaled \( \mathcal{L}_\infty \) distance between the system and its approximation:

\[
\min_{\tilde{G}(j\omega)} \tilde{s}[D_r(G(j\omega) - \tilde{G}(j\omega))D_r^{-1}] \quad \forall \omega \in \mathbb{R}
\]

(3)

where the number of unstable poles in \( \tilde{G}(s) \) and \( G(s) \) is the same, and \( D_r \) is a constant scaling matrix. Clearly, as \( G(s) \) contains both stable and unstable poles, achieving an optimal solution to this problem is a very difficult task. However, a suboptimal solution can be obtained by proceeding in the following way:

1. Separate the stable and anti-stable part of \( G(s) \) by

\[
G(s) = G_1(s) + G_2(s) = L^{-1}(s)[G^{sm}]_rR^{-1}(s) + L^{-1}(s)[G^{sm}]_uR^{-1}(s)
\]

where \( G^{sm}(s) \) is the Smith–McMillan form [23] of \( G(s) \) and \( L(s), R(s) \) are unimodular matrices. \( [G^{sm}]_r \) and \( [G^{sm}]_u \) contain stable and unstable poles of \( G(s) \), respectively. Note that Smith–McMillan decomposition is just one way to separate the stable and anti-stable part of \( G(s) \). One can use other methods, for example, \texttt{sdecomp} function in MATLAB mui-analysis and synthesis toolbox decomposes a system matrix as the sum of stable and unstable systems. It utilizes Schur’s decomposition and orthogonal–triangular decomposition for this operation. Another useful function is \texttt{stabsep} in control system toolbox.
2. Now, \( G_1^T(-s), G_1(s) \in \mathcal{RH}_\infty \) (both have poles on the left half of the \( s \)-plane, i.e. stable), and \( \tilde{G}(s) \) can be parameterized as \( \tilde{G}(s) = G_1(s) + \tilde{G}_2(s) \), which gives

\[
\| D_r [G(s) - \tilde{G}(s)] D_r^{-1} \|_{\mathcal{L}_\infty} \leq \| D_r [G_1(s) - \tilde{G}_1(s)] D_r^{-1} \|_{\mathcal{L}_\infty} + \| (D_r^{-1})^T [G_2^T(-s) - \tilde{G}_2^T(-s)] D_r^T \|_{\mathcal{L}_\infty}
\]

(4)
since \( \| D_r [G_2(s) - \tilde{G}_2(s)] D_r^{-1} \|_{\mathcal{L}_\infty} = \| (D_r^{-1})^T [G_2^T(-s) - \tilde{G}_2^T(-s)] D_r^T \|_{\mathcal{L}_\infty} \) and \( \tilde{G}_2(s) \) is the \( \mathcal{L}_\infty \) optimal approximation of \( G_2(s) \) with \( n_p \) (say) unstable poles.

3. Solve the following optimization problem for decision variables \( \gamma \), \( X_P \), \( K_d \), \( M \), \( H \) and \( Q \):

\[
\begin{align*}
\min \ & \gamma \\
\text{s.t.} \ & X_P > 0 \\
& \begin{bmatrix}
-Q & A_{cl}^T + M^T & 0 & C_{cl,h}^T & \sqrt{2}X_P & M^T \\
* & -I & B_{cl} & 0 & 0 & 0 \\
* & * & -\gamma H & D_{cl,h}^T & 0 & 0 \\
* & * & * & -\gamma H & 0 & 0 \\
* & * & * & * & -I & 0 \\
* & * & * & * & * & -I
\end{bmatrix} < 0 \\
& \begin{bmatrix}
-\gamma H & D_{cl,h}^T \\
* & -\gamma H
\end{bmatrix} < 0, \begin{bmatrix}
Q & X_P + M^T \\
* & I
\end{bmatrix} \geq 0
\end{align*}
\]

(5) (6) (7)

where

\[
A_{cl} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} = A_0 + \tilde{B}K_d \tilde{C}
\]

\[
B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = B_0 + \tilde{B}K_d \tilde{D}_{21}
\]

\[
C_{cl} = \begin{bmatrix} C & 0 \end{bmatrix} + \begin{bmatrix} 0 & -I \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} = C_0 + \tilde{D}_{12}K_d \tilde{C}
\]

\[
D_{cl} = D + \begin{bmatrix} 0 & -I \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = D + \tilde{D}_{12}K_d \tilde{D}_{21}
\]

\[
C_{cl,h}^T = C_{cl,h}^T, \quad D_{cl,h}^T = D_{cl,h}^T, \quad H = D_r^T D_r
\]

Here, \( (A, B, C, D) \) is a minimal state space realization of \( G_1(s) \) and \( (A_d, B_d, C_d, D_d) \) is the state space realization of \( \tilde{G}_1(s) \). This optimization problem, which minimizes the scaled \( \mathcal{H}_\infty \) norm of the error system, should also be solved to obtain structured \( \tilde{G}_2^T(-s) \) from \( G_2^T(-s) \) (except that \( H \) is now \( H = D_r^T D_{inv} \), where \( D_{inv} = (D_r^{-1})^T \)).

\textit{Derivation of the optimization in (5), (6) and (7): Please see appendix.}
A locally optimal solution is obtained by the following steps:

1. Solve a parametric optimization problem to find $G_{bd}(j\omega)$ st $\sum_{i=1}^{n_p} G_{bd}(j\omega)^{-1}(\omega_i)$, where $n_p$ is the number of unstable poles.

2. Solve a nonlinear optimization problem for $\min_{G_{bd}(j\omega)} \bar{\sigma}(D(\omega)(G(j\omega) - \tilde{G}(j\omega))D^{-1}(\omega))$ in a set of pre-selected frequencies.

3. Solve a parametric optimization problem to find $G_{bd}(s)$ that has at least $n_p$ unstable poles by minimizing the worst case error between $G_{bd}(s)$ and $G_{bd}(j\omega)$.

4. If $G_{bd}(s)$ possesses more than $n_p$ unstable poles, then reduce the order of $G_{bd}(s)$ to $n_p$ through optimal Hankel norm approximation to achieve $\tilde{G}(s)$.

This method is meritorious; however, the frequency-wise optimization in the first step and the parametric identification in the second step involve approximations and iterations, which have no guarantee of convergence. Moreover, in the first step, the proposed method is only selecting large number of frequencies around the peak of $\bar{\sigma}(G(j\omega))$.

### 3.1. Controller design

The algorithm of finding a (block) decentralized controller $K(s)$ to satisfy the $\mu$-IM condition [1, 15], $\bar{\sigma}(\tilde{H}(j\omega)) < \mu^{-1}(E(j\omega))$, $\forall \omega$, can be reduced to solving a skewed-$\mu$ problem. Assume $\tilde{H}(s)$ is stable, and that $G(s)$ and $\tilde{G}(s)$ have the same number of unstable poles. Then the closed-loop
system $H(s)$ is stable (all loops are closed) if

$$\tilde{\sigma} \left( \frac{1}{c_H} \tilde{H}(j\omega) \right) \leq 1 \quad \forall \omega \in \mathbb{R}$$

(8)

where at each frequency $c_H$ solves

$$\mu_{\Lambda} \begin{bmatrix} 0 & (G(j\omega) - \tilde{G}(j\omega))\tilde{G}^{-1}(j\omega) \\ c_H \Gamma & 0 \end{bmatrix} = 1$$

(9)

Here, $\mu$ is computed w.r.t. the structure $\Lambda = \text{diag}(\tilde{H}(j\omega), \tilde{H}(j\omega))$. This condition can be easily derived by using the $\mu$-IM condition and properties of singular values in [5, 23] (see Theorem 1 of [5]). The advantage is that it is easy to program. One can calculate $c_H$ from (9) and design the controller for each loop independently using (8). The only shortcoming is that $c_H$ gives equal partiality to all loops; however, if some roll offs in $\tilde{h}_f(s)$ is not required then they can be overcome by introducing a weighting matrix $W(s)$ whose structure coincides with the structure of $\tilde{H}(s)$.

**Remark 3.3**

It is interesting to note that condition (9) is equivalent to

$$\det \left( I - \begin{bmatrix} 0 & (G(j\omega) - \tilde{G}(j\omega))\tilde{G}^{-1}(j\omega) \\ c_H \Gamma & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \right) = 0$$

which can be written as

$$\det[I - (G(j\omega) - \tilde{G}(j\omega))\tilde{G}^{-1}(j\omega) \times c_H \Lambda_1(j\omega)] = 0.$$  

Therefore, if the interactions are large at low frequencies, namely, $\mu_1[(G(j\omega) - \tilde{G}(j\omega))\tilde{G}^{-1}(j\omega)]$, then $c_H$ has to be small to satisfy this condition. This can lead to a poor performance, because $\tilde{H}(j\omega)$ in (8) has to be small to satisfy the nominal stability condition.

The following proposition reveals that the condition in (1) (which is equivalent to (2)) leads to minimization of an upper bound of the closed-loop performance.

**Proposition 3.1**

Under the assumption that $G(s)$ and $\tilde{G}(s)$ have the same number of RHP poles and (1) holds

$$\tilde{\sigma}(H(j\omega)) \leq \frac{\kappa(D(\omega))\tilde{\sigma}(G\tilde{G}^{-1}(j\omega))}{\tilde{\sigma}^{-1}(\tilde{H}(j\omega)) - \mu(E(j\omega))} \quad \forall \omega \in \mathbb{R}$$

(10)

where $\kappa(D)$ is the Euclidean condition number, $D(\omega)$ is the frequency-dependent scaling matrix and $E(j\omega) = (G(j\omega) - \tilde{G}(j\omega))\tilde{G}^{-1}(j\omega)$.

**Proof**

It is clear that

$$(I + G\hat{K}(s))K^{-1}(s)\tilde{G}^{-1}(s) = (I + \tilde{G}\hat{K}(s))K^{-1}(s)\tilde{G}^{-1}(s) + (G(s) - \tilde{G}(s))\tilde{G}^{-1}(s)$$

$$= \tilde{S}^{-1}(s)K^{-1}(s)\tilde{G}^{-1}(s) + E(s)$$

where $\tilde{S}(s) = (I + \tilde{G}\hat{K}(s))^{-1}$ is the sensitivity function of the approximated system $\tilde{G}(s)$. Pre- and post-multiplying by $D(\omega)$ and $D^{-1}(\omega)$, respectively, and using the properties of the singular
values [13, 14, 23]
\[ \sigma(D(\omega)S^{-1}K^{-1}G^{-1}(j\omega)D^{-1}(\omega)) \geq \sigma(D(\omega)H^{-1}(j\omega)D^{-1}(\omega)) - \sigma(D(\omega)E(j\omega)D^{-1}(\omega)) \]  
(11)
where \( S(s) = (I + GK(s))^{-1} \) is the sensitivity function of the overall closed-loop system and \( H^{-1}(j\omega) = S^{-1}K^{-1}G^{-1}(j\omega) \). Now,

\[ D(\omega)S^{-1}K^{-1}G^{-1}(j\omega)D^{-1}(\omega) = D(\omega)S^{-1}K^{-1}G^{-1}G^{-1}(j\omega)D^{-1}(\omega) = DH^{-1}(j\omega)(I + E(j\omega))D^{-1}(\omega) \]

where \( H(s) \) is the closed-loop transfer function and

\[ \sigma[D(\omega)H^{-1}(j\omega)(I + E(j\omega))D^{-1}(\omega)] \leq \sigma(D(\omega))\sigma[H^{-1}(j\omega)(I + E(j\omega))D^{-1}(\omega)] \]

\[ \leq \sigma(D(\omega))\sigma(H^{-1}(j\omega))\sigma(I + E(j\omega))\sigma(D^{-1}(\omega)) \]

\[ = \kappa(D(\omega))\sigma(I + E(j\omega))\sigma(H^{-1}(j\omega)) \]
(12)

Here, \( \kappa(D(\omega)) = \sigma(D(\omega))\sigma(D^{-1}(\omega)) \) is the Euclidean condition number. Since \( D(\omega)H^{-1}(j\omega)D^{-1}(\omega) = H^{-1}(j\omega) \), and suppose that \( D(\omega) \) is chosen to minimize \( \sigma(D(\omega)E(j\omega)D^{-1}(j\omega)) \), then from (11) and (12)

\[ \kappa(D(\omega))\sigma(I + E(j\omega))\sigma(H^{-1}(j\omega)) \geq \sigma(H^{-1}(j\omega)) - \mu(E) \]

which gives,

\[ \sigma(H(j\omega)) \leq \frac{\kappa(D(\omega))\sigma(I + E(j\omega))}{\sigma^{-1}(H(j\omega)) - \mu(E(j\omega))} \]

\[ \square \]

Hence, stabilization of the closed loop system satisfying (8) leads to minimization of an upper bound (loose) of the closed loop performance. However, the maximization of \( \sigma^{-1}(H(j\omega)) - \mu(E(j\omega)) \) provides an advantage of maximizing the robustness of the closed-loop system against unmodeled dynamics represented by an output multiplicative uncertainty. Similar condition in terms of control sensitivity function was also derived in [14]. The results in (8) and (9) can be augmented with the robust performance conditions in [5] to design a robust controller for unstable systems.

3.2. Performance limitations due to RHP zero crossings

In the following, a method of multi-loop controller design is presented, which prevents the movement of the transmission zeros of open-loop subsystems across the imaginary axis (when some other loops are closed). The main concern is to find an upper bound for the interactions, such that zero crossings can be prevented. For the sake of brevity, a stable open-loop square system is considered.

**Theorem 3.1**

Assume that \( G(s) \) is divided into two blocks and the subsystems \( G_{11}(s) \) and \( G_{22}(s) \) are minimum phase. When the first loop is closed with a (block) decentralized controller \( K_1(s) \), such that \( H_1(s) = G_{11}K_1(s)(I + G_{11}K_1(s))^{-1} \) is stable, then the transmission zeros of the remaining subsystem
This derivation utilizes \( G_{22}(s) \) will not cross the imaginary axis, if
\[
\hat{\sigma}(\hat{H}_1(j\omega)) < \mu^{-1} \left[ G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}(j\omega) \right] \quad \forall \omega \in \mathbb{R}
\]

**Proof**
Consider the following system
\[
y_1 = G_{11}(s)u_1 + G_{12}(s)u_2, \quad y_2 = G_{21}(s)u_1 + G_{22}(s)u_2
\]
When a negative feedback \( u_1 = -K_1(s)y_1 \) is applied around the subsystem \( G_{11}(s) \), then the remaining subsystem \( \hat{G}_{22}(s) \) is represented by \( \hat{G}_{22}(s) = G_{22}(s) - G_{21}(s)(I + G_{11}K_1(s))^{-1}K_1 \)
\( G_{12}(s) = G_{22}(s)[I - G_{22}^{-1}G_{21}(s)(I + G_{11}K_1(s))^{-1}K_1G_{12}(s)]. \) Now,
\[
\text{det}(\hat{G}_{22}(s)) = \text{det}(G_{22}(s)) \text{det}[I - G_{22}^{-1}G_{21}(s)K_1(s)(I + G_{11}K_1(s))^{-1}G_{12}(s)]
\]
\[
= \text{det}(G_{22}(s)) \text{det}[I - \hat{H}_1G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(s)]
\]
(13)

Therefore, if the overall system is stable, the zeros of the second subsystem will not cross the imaginary axis if and only if the nyquist plot of \( \text{det}[I - \hat{H}_1G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)], \forall \omega \in \mathbb{R} \) does not encircle the origin. Using the spectral radius stability condition [23], the zero crossing can be prevented if \( \rho(\hat{H}_1G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)) < 1, \forall \omega \in \mathbb{R} \). Since \( \hat{H}_1(j\omega) \) has a structure and
\[ \rho(\hat{H}_1G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)) \leq \mu(\hat{H}_1G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)), \forall \omega \in \mathbb{R}, \] the sufficient condition is given by
\[
\hat{\sigma}(\hat{H}_1(j\omega)) < \mu^{-1} \left[ G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}(j\omega) \right] \quad \forall \omega \in \mathbb{R}
\](14)
This derivation utilizes \( \mu(A)B \leq \mu(A)\hat{\sigma}(B) \) and \( \mu \) is computed w.r.t. the structure of \( \hat{H}_1(j\omega) \).

**Remark 3.4**
For a \( 2 \times 2 \) system with scalar loops, Theorem 3.1 boils down to \( \hat{\sigma}(\hat{H}_1(j\omega)) < \mu^{-2}(E(j\omega)), \forall \omega \in \mathbb{R} \).

In general, for controllers designed independently
\[
\hat{\sigma}(\hat{H}_1(j\omega)) < \min(\mu^{-1}(E(j\omega)), \mu^{-1}[G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)])
\]
(15)
and
\[
\hat{\sigma}(\hat{H}_2(j\omega)) < \mu^{-1}(E(j\omega)) \quad \forall \omega \in \mathbb{R}
\]
(16)
guarantee overall closed-loop stability and also prevent the movement of transmission zeros in \( G_{22}(s) \) across the imaginary axis, when the first loop is closed. This has an important effect on the closed-loop system performance. For scalar loops, (15) can be represented by
\[
\hat{\sigma}(\hat{H}_1(j\omega)) < \min(\mu^{-1}(E(j\omega)), \mu^{-2}(E(j\omega))), \forall \omega \in \mathbb{R}. \] If interactions are large, then designing controller based on (15) and (16) leads to low-frequency performance deterioration in the first channel. This is because \( \hat{\sigma}(\hat{H}_1(j\omega)) \) has to be reduced at low frequencies
\[
(\mu^{-1}[G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(j\omega)]) < \mu^{-1}(E(j\omega)) < 1.
\]
For systems with integral action in all channels, the upper bound of the interaction is given by
\[
\max(\mu(E(0), \mu[G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}(0)] < 1
\]
since $\tilde{H}_1(0) = \tilde{H}_2(0) = I$. This result also gives some idea of pairing, because the loops should at least be paired in such a way that $\mu(E(0)) < 1$ is satisfied for closed-loop stability. As this condition depends on the steady-state gain information, it can be easily verified by some experiments. Moreover, it is also a sufficient condition for decentralized integral controllability (closed-loop stability can be maintained when the loops are detuned arbitrarily) [25].

It should be noted that for unstable systems, the situation is complex. Here, it is not possible to separate $\det(G_{22}(s))$ from $\det[I - \tilde{H}_1 G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}(s)]$ leading to inapplicability of multi-input multi-output Nyquist stability criteria [23]. However, under the assumption that only $G_{22}(s)$ is stable and minimum phase, RHP zero crossing can be prevented if the Nyquist plot of $\det[I - \tilde{H}_1 G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}(s)]$

(a) does not pass through the origin and
(b) makes $P_{ol}$ anti-clockwise encirclements of the origin, where $P_{ol}$ is the number of open-loop unstable poles in $\tilde{H}_1 G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}(s)$.

Nevertheless, it is not easy to come up with a condition in terms of maximum singular value or structured singular value (similar to (15) and (16)).

4. SIMULATION RESULTS

Consider the following system [14]:

$$
G(s) = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0.5 & 0.5 \\
0 & 2 & 0 & 0 & 0.5 & 1 & 0.5 \\
0 & 0 & 3 & 0 & 0.5 & 0.5 & 1 \\
0 & 0 & 0 & -4 & 1 & 0.4 & 0.4 \\
1 & 0.1 & 0.1 & 1 & 0 & 0 & 0 \\
0.1 & 1 & 0.1 & 0.6 & 0 & 0 & 0 \\
0.1 & 0.1 & 1 & 0.6 & 0 & 0 & 0
\end{bmatrix}
$$

The system has unstable poles at 1, 2 and 3. Application of the (block) diagonal approximation algorithm gives

$$
\tilde{G}(s) = \begin{bmatrix}
\frac{0.7923s - 6.009}{s^2 - 7.179s - 37.82} & 0 & 0 \\
0 & \frac{0.6135s - 0.05103}{s^2 - 1.402s - 27.43} & 0 \\
0 & 0 & \frac{0.7911s + 0.9201}{s^2 - 0.8216s - 24.77}
\end{bmatrix}
$$
which has poles at 10.71, −3.53, 5.98, −4.58, 5.40 and −4.58, i.e. the same number of unstable poles as the original system \( G(s) \). The singular values of the error system \( G(j\omega) - G_{bd}(j\omega) \), where \( G_{bd}(j\omega) = \tilde{G}(j\omega) \), are shown in Figure 2. The optimum \( \gamma \) is 1.3.

Next, consider the controller design. Since \( E(s) = (G(s) - \tilde{G}(s))\tilde{G}^{-1}(s) \) is improper, the algorithm in (8) and (9) is slightly modified to

\[
\begin{bmatrix}
0 & (G(j\omega) - \tilde{G}(j\omega)) \\
\mu_c & I
\end{bmatrix} \leq 1,
\forall \omega \in \mathbb{R}
\]

where at each frequency \( c_H \) solves

\[
\hat{D} = \begin{bmatrix} 0 & 0 \\
\tilde{H}(j\omega), & \tilde{R}(j\omega) \end{bmatrix}.
\]

Application of this algorithm gives \( c_H = 0.8026 \) and the decentralized controller

\[
K(s) = \text{diag} \left( \begin{array}{ccc}
123.2s + 435 & 34.94s + 160.1 & 20.78s + 95.24 \\
s - 72.64 & s - 4.881 & s - 1.046
\end{array} \right)
\]

The stabilizing effect of this controller for simultaneous reference step inputs of amplitude 1 (at \( t = 1s \)) and unit step disturbances occurring at \( t = 3s \) are shown in Figure 3. In [14] \( \gamma \) was 0.45, however, the design procedure in the present paper is straightforward. When the unstable poles in the LMI optimization are kept fixed, the approximated system is given by

\[
\tilde{G}(s) = \text{diag} \left( \begin{array}{ccc}
1.067s + 0.9654 & 0.9054s + 2.366 & 0.9953s + 2.508 \\
s^2 + 2.531s - 3.51 & s^2 + 2.583s - 9.167 & s^2 + 1.583s - 13.75
\end{array} \right)
\]
with $\gamma = 0.9$. Application of the design algorithm to this system gives $c_H = 1.1233$ and the stabilizing controller

$$K(s) = \text{diag}\left(\frac{4.459s + 15.75}{s + 0.7735}, \frac{6.304s + 28.89}{s + 2.876}, \frac{8.283s + 37.96}{s + 2.34}\right)$$

### 4.1. Utility boilers

The boilers at Syncrude consist of three utility boilers (UB 201-UB 203), two CO-type boilers (C.O.1 and C.O.2 in addition to UE-1 system) and two once through steam generators (OTSG1 and OTSG2), as shown in Figure 4. Owing to the difference of time constants and usage of different fuels, utility boilers regulate the 900# header pressure, CO boilers maintain their own steam flow and OTSGs the steam temperature.

In the following, a nonlinear model of the utility boilers (UB 201-203) and the 6.306 MPa header is first explained. Inputs to the model are feedwater flow rate, firing rate and attemperator spray flow rate; outputs are: drum level, header pressure and steam temperature.

The differential equations for the drum pressure, steam flow rate and fuel flow rate are given by [26]

$$\dot{x}_1 = -0.0157x_1 + 1.866\left(\frac{10^{-4}x_2}{x_1 + x_4}\right) + 0.00157x_5 - 0.0000395u_1 - 3.545\left(\frac{10^{-4}x_2}{x_1 + x_4}\right)$$

$$\dot{x}_2 = x_3 + 0.009151u_1 + 0.02988x_5 + 0.2239u_3$$

$$\dot{x}_3 = -0.001864x_2 - 0.1533x_3 - 0.001987u_1 + 0.03634x_5 - 0.03288u_3$$
\[
\begin{align*}
\dot{x}_4 &= -0.08x_4 + 0.006u_3 \\
\dot{x}_5 &= x_6 + 0.1758u_2 \\
\dot{x}_6 &= -0.001833x_5 - 0.1731x_6 - 0.0177u_2,
\end{align*}
\]

\[y_{\text{drum}} = x_1 + x_4\]

Here, \(x_1\) is the drum pressure when the effect of spray flow \(u_3\) is negligible, \(x_2\) is the steam flow rate, \(x_4\) is representing the effect of spray flow on drum pressure, \(x_5\) is the fuel flow rate, \(u_1\) represents the feedwater flow rate and \(u_2\) is the firing rate. The intermediate variables \(x_3\) and \(x_6\) are effecting the dynamics of steam flow rate and the fuel flow rate, respectively. The modeling of the drum water level takes into account the following physical relations, which are developed in [27–30]:

\[
\begin{align*}
\dot{x}_7 &= \frac{u_1 - x_2}{V_T} - \frac{u_1 - x_2}{155.1411} \\
q_e &= \frac{1}{1 + K}\left[k_b\epsilon + ru_1\right] + \frac{K}{1 + K}\epsilon^2 \\
y_{\text{level}} &= \frac{1}{A_d}\left[v_w V_T x_7 + k_1 x_T + T_s q_e\right]
\end{align*}
\]
Here, \( x_7 \) is the fluid density, \( V_T \) is the total volume of the drum, the downcomers and the risers. The constant \( K \) can be considered as a measure of the change in mass of steam generated in the boiler per unit mass lost from the steam space, \( \epsilon_f \) is the energy flow rate, \( A_d \) is drum area at normal operating level, \( v_w \) is the specific volume of water, \( T_s \) is the increase in water volume per unit increase in evaporation rate, \( k_b = 1/h_{fg} \) and \( r = h_w - h_f/h_{fg} \), where \( h_{fg} \) is the latent heat of evaporation, \( h_f \) is the enthalpy of saturated water and \( h_w \) that of feedwater.

Along with some heuristic adjustments, the measured outputs: header pressure, drum level (deviation about mean) and steam temperature are given by [26]

\[
\gamma_{\text{header}} = \sqrt{x_1^2 + x_4^2 + 2x_1x_4 - \frac{x_2^2}{900}}
\]

\[
\Delta y_{\text{level}} = 0.01028x_7 + 0.0044963x_2 + 0.035154x_5 - 5.71107u_{w1} - 7.2741
\]

\[
\dot{x}_8 = x_9 - 0.002324u_1 + 0.5772x_5 + 2.194u_3
\]

\[
\dot{x}_9 = x_{10} + 0.002323u_1 - 0.08838x_5 - 1.859u_3
\]

\[
\dot{x}_{10} = x_{11} - 0.001799u_1 + 0.06898x_5 + 1.436u_3
\]

\[
\dot{x}_{11} = -2.35 \times 10^{-6}x_8 - 0.000531x_9 - 0.0346x_{10} - 0.8159x_{11} + 0.001391u_1 - 0.05352x_5 - 1.108u_3
\]

\[
y_{\text{steam}} = x_8
\]

where

\[
u_{w1} = 10^{-5}u_1
\]

Linearization of the overall model at normal operating load gives one pole at the origin (associated with water dynamics), one RHP zero at 0.0619 and decentralized controllers are designed to maintain the pressure of the 6306 kPa header, and to keep the water level and steam temperature at their set points. For this system, application of the block diagonal approximation algorithm yields \( \gamma = 0.28 \) and for implementation the controller is then discretized with a sampling period of 6 s, which gives the following form:

\[
K(z) = \text{diag}[K_{11}(z), K_{22}(z), K_{33}(z)]
\]  \hspace{1cm} (20)

where

\[
K_{11}(z) = \frac{60.64z^3 - 41.41z^2 - 91.75z + 72.96}{z^3 - 1.875z^2 + 0.8786z + 7.242 \times 10^{-18}}
\]

\[
K_{22}(z) = \frac{0.0003238z^3 - 0.0009514z^2 + 0.0009314 - 0.0003039}{z^3 - 2.97z^2 + 2.941z - 0.9708}
\]

\[
K_{33}(z) = \frac{-0.02555z^3 - 0.006817z^2 - 0.01441z + 2.1 \times 10^{-4}}{z^3 - 0.008143z^2 + 0.001135z + 1.598 \times 10^{-24}}
\]

With this controller, the condition in (2) can be satisfied at all frequencies (Figure 5) and RHP zero crossings can be prevented (conditions (15) and (16) are satisfied). When this controller is
implemented in an interconnected system consisting of boilers (UBs and CO-type), once through steam generators and headers (Figure 6), the responses can be smooth. However, when the same controller is implemented in the overall system (Figure 4), which takes care of interactions from
turbines, generators, tie-lines, etc., the responses of the system under perturbed conditions are shown in Figures 7–10. Figure 7 represents the measurements during a sudden load change of 100 kpph in the 6.306 MPa steam header and Figure 8 shows the corresponding inputs that are required to overcome the load variations.

Figure 7. Controlled variables during load change.

Figure 8. Inputs during load change.
The interesting feature of Figure 9 is that when the plant is controlled by a multi-variable $H_{\infty}$ controller and if the firing rate master controller fails, then overall system becomes unstable. However, the decentralized controller in (20) is capable of maintaining the stability. This is an
important property of control by independent designs. Figure 10 shows the system response for a step change in steam temperature from high to normal load condition.

5. CONCLUSIONS

This paper develops an algorithm to apply the concept of $\mu$-IM to unstable systems. It is shown that the (block) diagonal approximation can be obtained by solving a quasi-convex optimization problem using the available numerical softwares. In addition to this, the paper presents other results, which includes: (a) deriving an upper bound of the closed-loop performance due to decentralized architecture and (b) some sufficient conditions to prevent zero crossings across the imaginary axis.

The proposed approach is applied to a numerical example and to a simulated industrial boiler system. Future work will be concentrated on obtaining a low-order nonlinear model and controlling the firing rate of CO-boilers along with the UBs to reduce the fuel consumption during load variations (economy consideration).

APPENDIX A

Derivation of the optimization in (5), (6) and (7): It is straightforward to show that finding a structured $\tilde{G}_1(s)$ such that $\bar{\sigma}[D_r(G_1(j\omega) - \tilde{G}_1(j\omega))D_r^{-1}] < \gamma$ is equivalent to solving the following optimization problem (after applying some transformations on the standard $H_\infty$ result [23])

$$\min \gamma$$

s.t. $P > 0$

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} D_r^{-1} & C_{cl}^T D_r^T \\ * & -\gamma I & (D_r^{-1})^T D_r \\ * & * & -\gamma I \end{bmatrix} < 0$$

(A1)

Pre- and Post-multiplying (A1) by $\text{diag}(I, D_r^T, D_r^T)$ and $\text{diag}(I, D_r, D_r)$, respectively

$P > 0, \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & C_{cl}^T H \\ * & -\gamma H & D_{cl}^T H \\ * & * & -\gamma H \end{bmatrix} < 0$ \hspace{1cm} (A2)

where $H = D_r^T D_r$. It is clear that the optimization problem involves bilinear terms at each position. Pre- and post-multiplying (A2) by $\text{diag}(P^{-1}, I, I)$ and $\text{diag}(P^{-1}, I, I)$, respectively, the conditions are equivalent to

$Y > 0$ and

$$\begin{bmatrix} Y A_{cl}^T + A_{cl} Y & B_{cl} & Y C_{cl}^T H \\ * & -\gamma H & D_{cl}^T H \\ * & * & -\gamma H \end{bmatrix} < 0$$
where $Y = P^{-1}$. This can be expanded using Schur’s complement method [31] as

$$
\begin{bmatrix}
YA^T_{cl} + A_{cl}Y + \frac{YC^T_{cl}HC_{cl}Y}{\gamma} & B_{cl} + \frac{YC^T_{cl}HD_{cl}}{\gamma} \\
\star & -\gamma H + \frac{D^T_{cl}HD_{cl}}{\gamma}
\end{bmatrix} < 0
$$

and further into

$$
YA^T_{cl} + A_{cl}Y + \frac{YC^T_{cl}HC_{cl}Y}{\gamma} + \left(B_{cl} + \frac{YC^T_{cl}HD_{cl}}{\gamma}\right)\left(\gamma H - \frac{D^T_{cl}HD_{cl}}{\gamma}\right)^{-1}
\times \left(B^T_{cl} + \frac{D^T_{cl}HC_{cl}Y}{\gamma}\right) < 0
$$

(A3)

$$
\left(-\gamma H + \frac{D^T_{cl}HD_{cl}}{\gamma}\right) < 0
$$

(A4)

Now, it is well known that according to the reciprocal projection lemma [32], the following statements are equivalent:

1. $\Psi + S + S^T < 0$.
2. For any given positive-definite matrix $X$, the LMI problem

$$
\begin{bmatrix}
\Psi + X - (W + W^T) & S^T + W^T \\
\star & -X
\end{bmatrix} < 0
$$

is feasible w.r.t. $W$, where $W$ is a decision variable.

Therefore, applying this lemma to the inequality in (A3)

$$
\begin{bmatrix}
\mathcal{F}_{11} + X - (W + W^T) & YA^T_{cl} + \mathcal{F}_{12} + W^T \\
\star & -X
\end{bmatrix} < 0
$$

(A5)

where

$$
\mathcal{F}_{11} = \frac{YC^T_{cl}HD_{cl}}{\gamma}\left(\gamma H - \frac{D^T_{cl}HD_{cl}}{\gamma}\right)^{-1}D^T_{cl}HC_{cl}Y + \frac{YC^T_{cl}HC_{cl}Y}{\gamma} + B_{cl}\left(\gamma H - \frac{D^T_{cl}HD_{cl}}{\gamma}\right)^{-1}B^T_{cl}
$$

$$
\mathcal{F}_{12} = \frac{YC^T_{cl}HD_{cl}}{\gamma}\left(\gamma H - \frac{D^T_{cl}HD_{cl}}{\gamma}\right)^{-1}B^T_{cl}
$$
Since, $X$ can be any given positive-definite matrix and $B_{cl}(\gamma H - (D_{cl}^T H D_{cl})/\gamma))^{-1}B_{cl}^T$ is symmetric, in most of the cases, the parameters $(B_d, D_d, \gamma, H)$ and $X$ can be designed such that

\[
X + B_{cl} \left( \frac{\gamma H - D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} B_{cl}^T = I \tag{A6}
\]

This selection is done to decouple the design variables from the positive-definite matrix $Y$. If, in some cases, this is not satisfied, then similar to [33] a large decision variable $\lambda$ can always be selected such that positive-definiteness of

\[
X = \lambda I - B_{cl} \left( \frac{\gamma H - D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} B_{cl}^T \tag{A7}
\]

is guaranteed.

Case I: With the selection in (A6), (A5) is equivalent to

\[
\begin{bmatrix}
I + \mathcal{D}_{11} - (W + W^T) & YA_{cl}^T + \mathcal{D}_{12} + W^T \\
* & -I + B_{cl} \left( \frac{\gamma H - D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} B_{cl}^T
\end{bmatrix} < 0
\]

where

\[
\mathcal{D}_{11} = \frac{YC_{cl}^T H D_{cl} \gamma}{\gamma} \left( \frac{\gamma H - D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} \frac{D_{cl}^T H C_{cl} Y}{\gamma} + \frac{Y C_{cl}^T H C_{cl} Y}{\gamma}
\]

and

\[
\mathcal{D}_{12} = \frac{YC_{cl}^T H D_{cl} Y}{\gamma} \left( \frac{\gamma H - D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} B_{cl}^T
\]

Pre- and post-multiplying by $\text{diag}(Y^{-1}, I)$ and $\text{diag}(Y^{-1}, I)$, respectively and expanding the above inequality

\[
\begin{bmatrix}
Y^{-1} Y^{-1} + \frac{C_{cl}^T H C_{cl} \gamma}{\gamma} - Y^{-1} M - M^T Y^{-1} & A_{cl}^T + M^T \\
* & -I
\end{bmatrix} - \begin{bmatrix}
C_{cl}^T H C_{cl} \gamma \gamma \\
B_{cl}
\end{bmatrix}
\times \left( -\frac{\gamma H + D_{cl}^T H D_{cl} \gamma}{\gamma} \right)^{-1} \begin{bmatrix}
D_{cl}^T H C_{cl} \gamma \\
B_{cl}
\end{bmatrix} < 0
\]
where \( M = WY^{-1} \). Using Schur’s complement

\[
\begin{bmatrix}
    Y^{-1}(Y^{-1} - M)M^TY^{-1} + \frac{C_{cl}^THC_{cl}}{\gamma} & A_{cl}^T + M^T & -\frac{C_{cl}^THD_{cl}}{\gamma} \\
    * & -I & B_{cl} \\
    * & * & -\gamma H + \frac{D_{cl}^THD_{cl}}{\gamma}
\end{bmatrix} < 0
\]

This inequality is equivalent to

\[
\begin{bmatrix}
    2XPX_P + M^TM - Q & A_{cl}^T + M^T & 0 & C_{cl}^TH^T \\
    * & -I & B_{cl} & 0 \\
    * & * & -\gamma H & D_{cl}^TH \\
    * & * & * & -\gamma H
\end{bmatrix} < 0 \tag{A8}
\]

where \( X_P = Y^{-1} \) and \( Q = (X_P + M)^TX_P + M \). Finally, applying the Schur’s complement method to the inequality in (A8) and relaxing the equality constraint

\[
\begin{bmatrix}
    -Q & A_{cl}^T + M^T & 0 & C_{cl,\hat{h}}^T & \sqrt{2X_P} & M^T \\
    * & -I & B_{cl} & 0 & 0 & 0 \\
    * & * & -\gamma H & D_{cl,\hat{h}}^T & 0 & 0 \\
    * & * & * & -\gamma H & 0 & 0 \\
    * & * & * & * & -I & 0 \\
    * & * & * & * & * & -I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
    Q & X_P + M^T \\
    * & I
\end{bmatrix} \geq 0 \tag{A9}
\]

where \( Q = (X_P + M)^TX_P + M \) corresponds to the boundary of the convex set in (A9). Moreover, the inequality

\[
\left( -\gamma H + \frac{D_{cl}^THD_{cl}}{\gamma} \right) < 0
\]

is equivalent to

\[
\begin{bmatrix}
    -\gamma H & D_{cl}^TH \\
    * & -\gamma H
\end{bmatrix} < 0
\]
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