Upper Bounds for Induced Operator Norms of Nonlinear Systems

Vahid Zahedzadeh, Horacio J. Marquez, and Tongwen Chen

Abstract—In this paper, new methods are introduced to compute the \( L_1, L_2 \) and \( L_\infty \) \((\ell_1, \ell_2 \) and \( \ell_\infty \)) induced operator norms of continuous (discrete) nonlinear systems. These methods are based on the so-called \( \zeta_A \) representation of nonlinear systems, which is introduced earlier by the authors. The methods are applicable if a certain condition is satisfied by the parameter of the \( \zeta_A \) representation. Two examples are provided to show the applicability of the methods. In addition, another method is suggested to derive an upper bound on the \( L_\infty \) \((\ell_\infty \)) norm of the system output. This method does not suffer from the limitations of the previous ones.

Index Terms—Nonlinear systems, induced operator norms, continuous-time systems, discrete-time systems.

I. INTRODUCTION

The problem of computing the \( L_p \) operator norm of a nonlinear system has been a challenging issue in the systems literature. The importance of this problem originates in the fact that the influence of various inputs on the various signals inside the system can be quantified by such a measure. One of the applications of this measure is in control systems, where the attenuation of disturbance signals is required.

For linear systems, computing the \( L_p \) norms has a well established solution. See references [1] and [2]. In [3], the \( L_\infty \)-gain of nonlinear systems is characterized by means of the value function of an associated variational problem. The \( L_p \) gain, also referred to as \( H_\infty \) gain of a nonlinear system, can be approximated using storage functions and the theory of dissipative systems. This approach is, however, conservative and finding storage functions is often very difficult or impossible. See also [5] for a numerical approximation of the \( H_\infty \) norm. Moreover, in [7], a computational method is suggested to compute the \( L_2 \) induced norm for single-input linear systems with saturation.

In this paper, we propose a method to compute an upper bound on the \( L_1, L_2, L_\infty, (\ell_1, \ell_2 \) and \( \ell_\infty \) \) norms of a class of continuous (discrete) nonlinear systems. Our method can be optimized based on some selected parameters. For continuous (discrete) nonlinear systems, which do not satisfy the condition required by the suggested approach, a method is also provided for computing the upper bounds on the \( L_\infty \) \((\ell_\infty \)) norm of the output with respect to the \( L_\infty \) \((\ell_\infty \)) norm of the input and \( \infty \)-norm of the initial condition.

The paper is organized as follows: In Section II we introduce the notation and present some preliminaries results. In Section III, the method is explained as well as two examples.

II. NOTATION AND PRELIMINARIES

A. Notation

Let \( R \) and \( C \) denote the fields of real and complex numbers, respectively. \( R^n \) denotes the space of \( n \times 1 \) real vectors. The Euclidean norm in \( R^n \) is denoted by \( \| \cdot \| \). \( I_{n \times n} \) denotes the \( n \times n \) identity matrix. Let \( B^p(c, r) \) denote the open ball with center \( c \) and radius \( r \) with norm \( p \), i.e. \( B^p(c, r) := \{ x \mid \| x - c \|_p < r \} \), \( L^p \) denotes Lebesgue \( p \)-space of \( r \)-vector valued functions on \([0, \infty)\), with norm defined as \( \| f \|_p := \left( \int_0^\infty \| f(t) \|_p^p \, dt \right)^{1/p} \) for \( 1 \leq p < \infty \) and \( \| f \|_\infty := \esssup_{t \in \mathbb{R}} \| f(t) \|_q \). In the aforementioned norm definitions, \( p \) and \( q \) are called the temporal and spatial norms, respectively. It is important to note that the definition of \( L_p \) is independent of the spatial norm. Usually \( r \) is a finite integer; we drop \( r \) and write \( L_p \) instead of \( L^p_r \). Similarly, let \( \ell_p \) denote the vector space of discrete-time signals with norm \( \| \cdot \|_p \). Let \( X_p \) denote either \( L_p \) or \( \ell_p \) and \( X \) denote \( X_p \) for any \( 0 \leq p < \infty \). To distinguish among various norm notation, we indicate the space as a subscript for the norm, such as \( \| \cdot \|_{R^n} \) or \( \| \cdot \|_{X_p} \). Whenever the space is not mentioned, norms with \( t \) argument denote Euclidean norm at \( t \) and without \( t \) denote the \( X_p \) norm where \( X_p \) is as a general space or can clearly be understood from the text. Let \( T_r \) denote the Truncation operator: for \( f(t), 0 \leq t < \infty \), \( T_r f(t) = f(t) \) on \([0, \tau] \), and zero otherwise. We also denote the truncation of \( f(t) \) by \( f_T(t) := T_r f(t) \). For an operator \( \lambda : X_p \rightarrow X_p \), let \( \gamma_p(\lambda) \) stand for the induced norm (gain) of the operator defined as

\[
\gamma_p(\lambda) := \sup_{0 \neq x \in X_p} \frac{\| \lambda(x) \|_{X_p}}{\| x \|_{X_p}} \tag{1}
\]

B. \( \zeta_A \) and \( \zeta_{AB} \) Representations

Our proposed method to compute the aforementioned norms is based on \( \zeta_A \) and \( \zeta_{AB} \) representations of nonlinear systems.
systems, which have recently been introduced in [9]. In this section, we briefly explain the $\zeta_A$ and $\zeta_{AB}$ representations. See [9] for further details.

C. Continuous-time systems

Assume that the nonlinear system of interest, $N$, is

$$N : \dot{x}(t) = f(t, x(t))$$

(2)

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz [6]. We also assume that the initial condition of the system is finite. Let $A \in \mathbb{R}^{n \times n}$. Define

$$\Phi(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$$

$$\Phi(t, x) := f(t, x) - Ax$$

(3)

$$\Gamma : \ell_p \to \ell_p, \quad \Gamma(z(t)) := \int_0^t e^{A(t-\tau)}z(\tau)\,d\tau,$$

(4a)

and

$$\Omega : \ell_p \to \ell_p, \quad \Omega(x(t)) := e^{At}x(0).$$

(4b)

The nonlinear system is equivalent to the structure represented in Fig. 1. This representation of the nonlinear system is called the $\zeta_A$ representation with ordered operator set $[\Phi, \Gamma, \Omega]$ [9]. For forced nonlinear systems, suppose that the system of interest is

$$N : \dot{x}(t) = f(t, x(t), u(t))$$

(5)

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Define

$$\Phi(x, u, t) := f(t, x, u) - Ax - Bu$$

(6)

Let

$$\Theta : \ell_p \to \ell_p, \quad \Theta(u(t)) := \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau,$$

(7)

and $\Gamma$ and $\Omega$ be defined in the same formulas as in (4). The nonlinear system is equivalent to the structure represented in Fig. 2. This representation of the nonlinear system is called the $\zeta_{AB}$ representation with ordered operator set $[\Phi, \Gamma, \Omega]$ [9].

It is important to note that $\begin{bmatrix} A & I \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}$ are state-space realizations for $\Gamma$ and $\Theta$, respectively. Since $A$ and $B$ are chosen arbitrary, $\zeta_A$ and $\zeta_{AB}$ representations are not unique. A useful choice for the $\zeta_{AB}$ representation is $B = 0$, which implies $\theta = 0$ and simplifies the $\zeta_{AB}$ structure as the structure shown in Fig. 3. For forced systems, this representation is also called $\zeta_A$ representation.

D. Discrete-time systems

Similar to the continuous-time case, $\zeta_A$ and $\zeta_{AB}$ representations are defined for discrete-time systems. An unforced nonlinear system with the following state equation

$$N : x(t + 1) = f(t, x(t))$$

(8)

is equivalent to the structure shown on Fig. 1, which is called $\zeta_A$ representation with ordered operator set $[\Phi, \Gamma, \Omega]$ [9]. Here, the operators are defined as

$$\Phi(t, x) : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$$

$$\Phi(t, x) := f(t, x) - Ax$$

(9)

$$\Gamma : \ell_p \to \ell_p, \quad \Gamma(z(t)) := \sum_{l=0}^{t} A^{t-l-1}z(l),$$

(10a)

and

$$\Omega : \ell_p \to \ell_p, \quad \Omega(x(t)) := A^tx(0)$$

(10b)
For forced nonlinear systems with the following state equation

\[ N: x(t + 1) = f(t, x(t), u(t)) \]  

(11)

the equivalent structures are shown in Figs. 2 and 3 where \( \Theta \) is defined as

\[ \Theta: \ell_p \to \ell_p, \quad \Theta(u(t)) := \sum_{t=0}^{l} A^{l-t-1} Bu(t) \]  

(12)

III. UPPER BOUNDS FOR THE INDUCED OPERATOR NORMS

In this section, we obtain a computable upper bound for induced operator norms. We will use the \( \zeta_A \) representation for forced system; namely the structure shown in Fig. 3. In this structure, it is trivial to show that

\[ \|x\|_{x_p} \leq \|u\|_{x_p} + \|d\|_{x_p} \leq \gamma_p(\Gamma)\gamma_p(\Phi) \left( \left\| \begin{array}{c} x \\ u \end{array} \right\|_{x_p} + \|d\|_{x_p} \right) \leq \gamma_p(\Gamma)\gamma_p(\Phi) \left( \left\| \begin{array}{c} x \\ u \end{array} \right\|_{x_p} \right) \]

(13)

\[ \gamma_p(\Omega)\|x_0\|_p \]

The computation of \( \gamma_p(\Gamma), \gamma_p(\Omega) \) and \( \gamma_p(\Phi) \) was extensively discussed in Reference [9].

It is easy to show that for any \( x, u \in X_p^r \):

\[ \left\| \begin{array}{c} x \\ u \end{array} \right\|_{x_p} \leq \|x\|_{x_p} + \|u\|_{x_p} \]  

(14)

Moreover, if \( x, u \in X_2 \):

\[ \left\| \begin{array}{c} x \\ u \end{array} \right\|_{x_2}^2 = \|x\|_{x_2}^2 + \|u\|_{x_2}^2 \]  

(15)

Equation (14) is true for all Banach spaces. However, (15) is true when the temporal norm is \( X_2 \) with Euclidean 2-norm chosen as the corresponding spatial norm.

**Theorem 3.1:** Let \( [\Phi, \Gamma, \Omega] \) be a \( \zeta_A \) representation for a forced system, \( N \). If

\[ \gamma_p(\Gamma)\gamma_p(\Phi) < 1 \]  

(16)

then

\[ \gamma_p(N) \leq \frac{\gamma_p(\Gamma)\gamma_p(\Phi)}{1 - \gamma_p(\Gamma)\gamma_p(\Phi)} \]  

(17)

**Proof:** Substituting (14) in (13), we have

\[ \|x\| \leq \gamma_p(\Gamma)\gamma_p(\Phi) \|x\| + \|u\| + \gamma_p(\Omega)\|x_0\| \]  

(18)

\[ (1 - \gamma_p(\Gamma)\gamma_p(\Phi))\|x\| \leq \gamma_p(\Gamma)\gamma_p(\Phi)\|u\| + \gamma_p(\Omega)\|x_0\| \]

(19)

Since \( \gamma_p(\Gamma)\gamma_p(\Phi) < 1 \),

\[ \|x\| \leq \frac{\gamma_p(\Gamma)\gamma_p(\Phi)}{1 - \gamma_p(\Gamma)\gamma_p(\Phi)}\|u\| + \frac{\gamma_p(\Omega)}{1 - \gamma_p(\Gamma)\gamma_p(\Phi)}\|x_0\| \]  

(20)

which implies (17).

Inequality (17) can be used as an upper bound for \( X_p \) induced norm. It is important to note that since the \( \zeta_A \) representation is not unique, the solution of the following minimization problem is the lowest upper bounds that can be obtained by our method:

\[ \gamma_p(N) \leq \min_A \frac{\gamma_p(\Gamma)\gamma_p(\Phi)}{1 - \gamma_p(\Gamma)\gamma_p(\Phi)} \]

(21)

where \( \Gamma(s) = \left[ \begin{array}{c} A \\ 0 \end{array} \right] \) and \( \Phi(x) = f(x) - Ax \).

The method provided by Theorem 3.1 is general in the sense of the induced norm, \( \gamma_p \). An interesting case occurs when the temporal norm is \( X_2 \) with Euclidean 2-norm chosen as the corresponding spatial norm. The reason is that a quite mature theory, called \( H_\infty \) optimization, has been developed for linear systems in this case. Suppose \( \Gamma \) is a continuous (discrete) linear time-invariant stable operator with impulse response \( g(t) : R^+ \rightarrow R^{n \times n} \) \( (g(t) : Z^+ \rightarrow R^{n \times n}) \). Let \( G(s) \) denotes the Laplace transform of \( g(t) \). We have

\[ \gamma_2(\Gamma) := \|G(s)\|_{H_\infty} \]

(22)

In this case, the following theorem provides lower upper bounds for the induced norm, \( \gamma_2 \).

**Theorem 3.2:** Let \( [\Phi, \Gamma, \Omega] \) be a \( \zeta_A \) representation for a forced system, \( N \). If \( \gamma_2(\Gamma)\gamma_2(\Phi) < 1 \) then

\[ \gamma_2(N) \leq \frac{\gamma_2(\Gamma)\gamma_2(\Phi)}{\sqrt{1 - \gamma_2(\Gamma)^2\gamma_2(\Phi)^2}} \]

(23)

**Proof:** In this proof, vector norms are Euclidean 2-norm for constant vectors and \( X_2 \)-norm for time-varying ones. It is trivial to show that for all \( a, b \geq 0 \)

\[ a^2 + b^2 \leq (a + b)^2 \]

(24)

Inequality (13) implies that

\[ (\|x\| - \gamma_2(\Omega)\|x_0\|)^2 \leq \left( \gamma_2(\Gamma)\gamma_2(\Phi) \left\| \begin{array}{c} x \\ u \end{array} \right\| \right)^2 \]

(25a)

Using (15),

\[ \|x\|^2 - 2\gamma_2(\Omega)\|x_0\||\|x\| + \gamma_2(\Omega)^2\|x_0\|^2 \leq \gamma_2(\Gamma)^2\gamma_2(\Phi)^2 (\|x\|^2 + \|u\|^2) \]

(25b)

For simplicity, let \( \alpha := \gamma_2(\Gamma)\gamma_2(\Phi) \)

\[ \|x\|^2 - 2\gamma_2(\Omega)\|x_0\||\|x\| + \gamma_2(\Omega)^2\|x_0\|^2 \leq \alpha^2 \frac{\|x\|^2}{1 - \alpha^2} \]

(25c)

\[ \left( \|x\| - \frac{\gamma_2(\Omega)\|x_0\|}{1 - \alpha^2} \right)^2 \leq \frac{\alpha^2\gamma_2(\Omega)^2}{1 - \alpha^2} \|x_0\|^2 + \frac{\alpha^2}{1 - \alpha^2}\|u\|^2 \]  

(25d)
Using (24),
\[
\|x\| - \gamma_2(\Omega)\|x_0\| \leq \frac{\alpha\gamma_2(\Omega)}{1 - \alpha^2}\|x_0\| + \frac{\alpha}{\sqrt{1 - \alpha^2}}\|u\| \tag{25e}
\]
Consequently
\[
\|x\| \leq \frac{\gamma_2(\Gamma)\gamma_2(\Phi)}{\sqrt{1 - \gamma_2(\Gamma)^2\gamma_2(\Phi)^2}}\|u\| + \frac{\gamma_2(\Omega)}{1 - \gamma_2(\Gamma)\gamma_2(\Phi)^2}\|x_0\| \tag{25f}
\]
which implies (23).

Similarly, the solution of the following minimization problem is the lowest upper bounds that can be obtained by our method:
\[
\gamma_2(N) \leq \min_A \frac{\gamma_2(\Gamma)\gamma_2(\Phi)}{\sqrt{1 - \gamma_2(\Gamma)^2\gamma_2(\Phi)^2}} \tag{26}
\]
where \(\Gamma(s) = [\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\) and \(\Phi(x) = f(x) - Ax\). Equivalently,
\[
\gamma_2(N) \leq \min_A \frac{1}{\sqrt{\| (sI - A)^{-1}\|_{\infty}^{-2}\gamma_2^{-2}(f(x) - Ax) - 1}} \tag{27}
\]

**Example 3.1:** (RLC circuit with non-ideal inductor)
The network of Fig. 4 represents a RLC circuit with a non-ideal inductor. The inductor has nonzero resistance and saturation characteristic as shown in Fig. 5(a), where \(\lambda\) is the flux linkage. The relationship of the magnetic flux linkage to terminal voltage of an inductor is given by Faraday’s law; namely \(v_L(t) = d\lambda(t)/dt\). The state equation for this network may be written as
\[
v_L = \dot{\lambda} = \frac{d\lambda}{dt}_L \frac{di_L}{dt} \tag{28a}
\]
\[
\frac{d\lambda}{dt}_L = \left( \frac{d\lambda}{dt}_L \right)^{-1} (v_C - R_2i_L) \tag{28b}
\]
where \(\left( \frac{d\lambda}{dt}_L \right)^{-1}\) is depicted in Fig. 5(b) versus \(i_L\).
\[
C\frac{dv_C}{dt} = i - \frac{v_C}{R_1} - i_L \tag{28c}
\]
Defining \(x_1 := i_L, x_2 := v_C\) and \(u := i\),
\[
\begin{cases}
\dot{x}_1 = (x_2 - R_2x_1) \left( \frac{d\lambda}{dt}_L \right)^{-1} \\
\dot{x}_2 = \frac{u}{R} - \frac{x_2}{L} - \frac{i_L}{C}
\end{cases} \tag{28d}
\]
Let \(R_1 = \frac{1}{2}, R_2 = 1\) and \(C = 2\). Assuming \(A = \begin{bmatrix} -1 & 0.5 \\ -0.5 & -1 \end{bmatrix}\), we have \(\Phi(x_1, x_2, u) = [\Phi_1(x_1, x_2, u) \Phi_2(x_1, x_2, u)]^T\) where \(\Phi_1(x_1, x_2, u) := x_1 - 0.5x_2 + (x_2 - x_1) \left( \frac{d\lambda}{dt}_L \right)^{-1}\) and \(\Phi_2(x_1, x_2, u) := -\frac{u}{C}\).

We use the computational methods that has been introduced in [9]. Since there are three independent variables in \(\gamma_\rho(\Phi)\), i.e., \(x_1, x_2\) and \(u\), we plot \(\|\Phi(x, u)\|\) versus \(\|\Phi\|\) instead of plotting versus \(x_1, x_2\) and \(u\). As shown in Fig. 6, \(\gamma_1(\Phi) \approx 0.50, \gamma_2(\Phi) \approx 0.50\) and \(\gamma_\infty(\Phi) \approx 0.50\). Computation also shows that \(\gamma_1(\Gamma) \approx 1.237, \gamma_2(\Gamma) \approx 1.00\) and \(\gamma_\infty(\Gamma) \approx 1.237\). Theorems 3.1 and 3.2 imply that \(\gamma_1(N) \leq 1.62, \gamma_2(N) \leq 0.577\) and \(\gamma_\infty(N) \leq 1.62\), respectively.

There is no doubt that the condition \(\gamma_\rho(\Gamma)\gamma_\rho(\Phi) < 1\) in Theorems 3.1 and 3.2 is restrictive. The following theorem might be used to overcome this shortcoming.

**Theorem 3.3:** Let \([\Phi, \Theta, \Gamma, \Omega]\) be a \(\zeta_{AB}\) representation for a nonlinear system. Let \(\eta > 0\) and \(M_p > \gamma_\infty(\Omega) + \eta\gamma_\infty(\theta)\) and
\[
\dot{D} := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m} \middle| \gamma_\Gamma^D(\Phi) < \frac{M_p - \gamma_\infty(\Omega) - \eta\gamma_\infty(\theta)}{(M_p + \eta)\gamma_\infty(\Gamma)} \right\} \tag{29}
\]
Let \(D := B^\infty(o, r_D)\) be an open ball inside \(\dot{D}\). Let \(D_x\) and \(D_u\) be the images of \(D\) under \(I_{n \times n} 0_{n \times m}\) and \(0_{n \times n} I_{m \times m}\), respectively. Consequently, \(D_x\) and \(D_u\) are also open balls in \(\mathbb{R}^n\) and \(\mathbb{R}^m\) respectively.

Let \(r_x\) and \(r_u\) denote respectively their radius, i.e., \(D_x = B^\infty(0, r_x)\) and \(D_u = B^\infty(0, r_u)\). Choose \(\epsilon\) and \(\delta\) such that \(0 < \epsilon < r_x\) and
\[
0 < \delta \leq \frac{1 - \gamma_\Gamma^D(\Phi)\gamma_\infty(\Gamma)}{\gamma_\infty(\Omega) + \eta(\gamma_\infty(\Theta) + \gamma_\Gamma^D(\Phi)\gamma_\infty(\Gamma))}\epsilon
\]
If \( \|u\|_{\infty} < \min (\eta, \delta, r_u) \) and \( \|x_0\|_{\infty} \leq \delta \), then
\[
\|x\|_{\infty} < \epsilon \quad (30)
\]

Proof: The proof for discrete-time systems is very similar and is omitted. In this proof, vector norms are Euclidean \( \infty \)-norm for constant vectors and \( \mathcal{X}_{\infty} \)-norm for time-varying ones.

It is trivial that \( M_p - \gamma_{\infty}(\Omega) - \eta \gamma_{\infty}(\theta) < M_p + \eta \); therefore \( \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma) < 1 \). We use contradiction to prove the theorem. Since we have assumed that systems of interest are locally Lipschitz, system trajectories are continuous. Consequently, if \( x \) were to leave the ball with radius \( \epsilon \), it should cross the boundary of the ball. Suppose that \( x \) crosses the boundary at \( t = \tau \). As a result, \( \|T_{\tau}x\| = \|x\|_{\tau} = \epsilon \). Since \( \epsilon < r_x \) and \( \|u\| < \min (\eta, \delta, r_u) \) guarantees that \( u \in D_u \), we have \( [x_{\tau}, u_{\tau}] \in D \) and consequently
\[
\|x_{\tau}\| \leq \|d_{\tau}\| + \|w_{\tau}\|
\leq \|d_{\tau}\| + \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma)(\|d_{\tau\tau}\| + \|x_{\tau}\|)
\leq \gamma_{\infty}(\Theta) \|u_{\tau}\| + \gamma_{\infty}(\Omega) \|x_0\|
+ \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma) \|x_{\tau}\| + \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma) \|u_{\tau}\|
\leq \gamma_{\infty}(\Omega) \|x_0\| + [\gamma_{\infty}(\Theta) + \gamma_{\infty}(\Phi) \gamma_{\infty}(\Gamma)] \|u_{\tau}\|
\]

Then
\[
\epsilon = \frac{\|x_{\tau}\|}{\gamma_{\infty}(\Omega) + \eta \left[ \gamma_{\infty}(\Theta) + \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma) \right]}
\leq \frac{\|x_0\|}{1 - \gamma_{\infty}^{D}(\Phi) \gamma_{\infty}(\Gamma)}
\]

(32)

Which is a contradiction. Therefore, \( x(t) < \epsilon \); \( \forall t \geq 0 \), i.e. \( \|x\| < r_x \).

The following example was studied in [9].

**Example 3.2:** Consider an example of continuous-stirred tank reactor (CSTR) system shown in Fig. 7, where an irreversible, first-order reaction takes place. CSTR is used to convert reactants to products. The reactant is fed constantly into a vessel where a chemical reaction takes place and yields the desired product. The heat generated by the chemical reaction is removed by the coolant medium that is circulated through a jacket. The following mathematical model is taken from [8].

\[
\begin{aligned}
\dot{x}_1 &= -\dot{x}_1 + D_a (1 - \dot{x}_1) e^{\frac{\dot{x}_2}{\dot{x}_1}} \\
\dot{x}_2 &= -\dot{x}_2 + B_h D_a (1 - \dot{x}_1) e^{\frac{\dot{x}_2}{\dot{x}_1}} + \beta_h (\dot{u} - \dot{x}_2)
\end{aligned}
\]

(33)

where \( \dot{x}_1, \dot{x}_2, \) and \( \dot{u}_1 \) are the dimensionless reagent conversion, the temperature (output), and the coolant temperature (input), respectively. The numerical values for the coefficients are \( D_a = 0.072 \), \( \varphi = 20 \), \( B_h = 8 \), and \( \beta_h = 0.3 \).

Three operating points are considered in [4]. One of them is an unstable point, \( \dot{u}_{10} = 0, \dot{x}_{10} = 0.4472 \), and \( \dot{x}_{20} = 2.7517 \). Let us transfer the origin of the state plane into...
this unstable point, which is investigated here. Therefore, we define \( x_1 := \hat{x}_1 - \hat{x}_{\text{eq}} \) and \( x_2 = \hat{x}_2 - \hat{x}_{\text{eq}} \). We study the closed-loop system which is depicted in Fig. 8 where \( K = 100 \) is a proportional controller and \( u \) is an exogenous input which can be interpreted as sensor noise or disturbance.

The state equation for the closed-loop system is

\[
\begin{align*}
\dot{x}_1 &= -x_1 - 0.4472 + 0.072(0.5528 - x_1)e^{20x_2 + 55.034} \\
\dot{x}_2 &= -31.3x_2 - 3.5772 + 0.576(0.5528 - x_1)e^{20x_2 + 55.034} + 30u
\end{align*}
\]

Let \( A = \begin{bmatrix} -1.81 & 0.357 \\ -6.474 & -28.143 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 30 \end{bmatrix} \) then

\[
\Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix}
\]

where

\[
\Phi_1(x) = -0.81x_1 - 0.357x_2 - 0.4472 + 0.072(0.5528 - x_1)e^{20x_2 + 55.034}
\]

and

\[
\Phi_2(x) = -3.157x_2 + 6.474x_1 - 3.5772 + 0.576(0.5528 - x_1)e^{20x_2 + 55.034}
\]

Computation with the given methods in [9] provides \( \gamma_{\infty}(\Gamma) < 0.5354, \gamma_{\infty}(\Theta) = 0.5423, \gamma_{\infty}(\Omega) < 1.221, \) and \( \gamma_{\infty}(\Omega) = 1 \). Let \( \eta = 0.1 \) and \( M_P = 3 > \gamma_{\infty}(\Omega) + \eta\gamma_{\infty}(\Theta) \). Since \( \Phi \) is independent from \( u \), \( \mathcal{D} \subset \mathbb{R}^2 \). For this example, since \( \mathcal{D} \) is simply connected, \( \mathcal{D} = \mathcal{D} \).

The surface of \( \|\Phi(x)\|_\infty \) as well as the boundary of \( \mathcal{D} \) is depicted in Fig. 9. The various subsets of \( \mathbb{R}^2 \) are depicted in Fig. 10. Since \( \mathcal{D} \) is independent of \( u \), \( r_u = \infty \). As shown in Fig. 10, \( r_x = 0.1519 \). Let \( \epsilon = 0.151 \) and \( \delta = 0.0402 \leq \frac{1 - \gamma_{\infty}(\Phi)\gamma_{\infty}(\Gamma)}{\gamma_{\infty}(\Omega) + \eta(\gamma_{\infty}(\Theta) + \gamma_{\infty}(\Omega))} \epsilon \).

According to Theorem 3.3, for any input \( u \) which satisfies \( \|u\|_{\mathcal{L}_\infty} < \min(\eta \delta, r_u) = 0.004 \) and any initial satisfying \( \|x_0\|_\infty < \delta = 0.0402 \), \( x \) is bounded as \( \|x\|_{\mathcal{L}_\infty} < \epsilon = 0.1519 \).

IV. CONCLUSION

In this paper, the computation of the \( \mathcal{L}_1, \mathcal{L}_2 \) and \( \mathcal{L}_\infty \) (\( \ell_1, \ell_2 \) and \( \ell_\infty \)) induced operator norms of continuous (discrete) nonlinear systems is studied. Based on the \( \zeta \) representation of nonlinear systems, methods are suggested to compute the aforementioned norms. To show the applicability of the methods, two examples are provided. The main limitation of the suggested methods is inequality (16) that restricts the usage of the method for a class of the nonlinear systems and the freedom on choosing the parameter \( \alpha \). In the \( \mathcal{L}_\infty \) case, to lessen the restrictions encountered in the computation of the \( \mathcal{L}_\infty \) norm, a method is provided to derive an upper bound on the \( \mathcal{L}_\infty (\ell_\infty) \) norm of the system output. This method does not suffer from the limitations of the pervious ones.

REFERENCES


