Solving Traveling Salesman Problems Using Ising Models with Simulated Bifurcation

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***Abstract*—Many combinatorial optimization problems can be solved by numerically simulating classical nonlinear Hamiltonian systems based on the Ising model. Solving the traveling salesman problem (TSP) using the Ising model requires a quadratically increasing number of spins with strict constraints. Unlike classical simulated annealing, simulated bifurcation (SB) can update the states of spins in parallel. This feature can potentially accelerate the convergence of Hamiltonian in the Ising model by taking advantage of modern multi-core processors. As an improved SB algorithm, the ballistic SB (bSB) algorithm is considered for solving the TSP in this paper. Two different mapping methods are proposed by converting the TSP to an Ising problem with or without external magnetic fields. bSB is then expanded by introducing a time-dependent factor or redundant spins depending on the mapping methods. Experiments on benchmark datasets show that the bSB-based Ising solvers offer superior performance in solution quality and convergence speed.**

***Index Terms*—Traveling salesman problem, simulated bifurcation, parallel processing, Ising model**

I. Introduction

Combinatorial optimization is ubiquitous in various social and industrial applications. Such a problem can be solved by using the Ising model with heuristic algorithms, e.g., simulated annealing (SA) [1]. The Ising model is mathematically constructed for a set of electron spins connected with each other by simulating the ferromagnetic interactions among them. The Hamiltonian (H) in an Ising model (or Ising problem) with the external magnetic field is given by [1]

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \tag{1}$$

where σ_i (or σ_j) $\in \{-1, +1\}$ denotes the state of the ith (or jth) spin; J_{ij} denotes the coupling coefficient between the ith and jth spins, and h_i denotes the external field of the ith spin.

Unlike SA, which cannot simultaneously update the states of neighbor spins, simulated bifurcation (SB) [2] can realize massive processing parallelism. Inspired by the quantum adiabatic optimization [3] using Kerr-nonlinear parametric oscillators, SB simulates the adiabatic evolutions of classical nonlinear Hamiltonian systems. To restrain the errors introduced due to the use of continuous variables (for the positions of oscillators) to represent the discrete states of spins, the ballistic SB (bSB) was developed by solving (2) and (3) to quickly find suboptimal solutions for an Ising problem defined by (1) but without external fields [4]:

$$\dot{x_i} = a_0 y_i, \tag{2}$$

$$\dot{y_i} = -\{a_0 - a(t)\}x_i + c_0 \sum_j J_{ij}x_j, \tag{3}$$

where x_i and y_i are the position and the momentum of the ith oscillator (seen as the ith spin); $\dot{x_i}$ and $\dot{y_i}$ denote time derivatives; $a(t)$ is a control parameter; a_0 and c_0 are positive constants. x_i is replaced with its sign if $|x_i| > 1$. At the end of search, the sign of x_i indicates the state of the ith spin.

The traveling salesman problem (TSP) is to find the shortest route to visit all cities once and then return to the origin city. An n-city TSP using n^2 spins in a lattice is formulated as [5]

$$\begin{aligned} H_{tsp} = \ & \tfrac{A}{4}\textstyle\sum_{k\neq l}\sum_i W_{kl}\sigma_{ik}\sigma_{(i+1)l} + \tfrac{A}{2}\sum_{k\neq l}\sum_i W_{kl}\sigma_{ik} \\ & + \tfrac{B}{4}\textstyle\sum_i\sum_k\sum_l \sigma_{ik}\sigma_{il} + \tfrac{(n-2)B}{2}\sum_i\sum_k \sigma_{ik} \\ & + \tfrac{C}{4}\textstyle\sum_i\sum_k\sum_j \sigma_{ik}\sigma_{jk} + \tfrac{(n-2)C}{2}\sum_i\sum_k \sigma_{ik}, \end{aligned} \tag{4}$$

where $\sigma_{ik} = +1$ (or -1) indicates whether the kth city is visited (or not) at the ith step; W_{kl} denotes the distance between the kth and lth cities. A, B and C are the hyperparameters that balance the significance between the objective function (by A) and constraints (by B and C).

The TSP is notoriously difficult to solve due to quadratically increasing spin counts and strong constraints placed on the spins by using the Ising model [5]. These constraints do not allow visiting multiple cities in a single step or visiting a city in multiple steps. Moreover, using the Ising model to update the states of all spins in parallel has not been reported to solve the TSP. In this paper, the parallelizable algorithm bSB is considered for the efficient solving of TSPs. The mapping of TSP into an Ising model with or without external fields is formulated. Experimental results of the solution quality are discussed by using the proposed bSB based methods and a recent SA method [6] to update the states of spins.

II. Solving TSP using the Ising Model with bSB

A. Reformulation of TSP to the Ising Model

1) With external fields: By considering the interaction between spins as a coupling coefficient and the bias applied to a spin as an external field, (4) can be reformulated as:

$$H_{tsp} = \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} J_{ikjl}\sigma_{ik}\sigma_{jl} + \sum_{i=1}^{n}\sum_{k=1}^{n} h_{ik}\sigma_{ik}, \tag{5}$$

where

$$J_{ikjl} = \begin{cases} \frac{A}{8}W_{kl} & j = i+1 \quad or \quad i = j+1 \quad or \\ & i = 1, j = n \quad or \quad i = n, j = 1 \\ \frac{B}{4} & i = j, k \neq l \\ \frac{C}{4} & k = l, i \neq j \\ \frac{B}{4} + \frac{C}{4} & k = l, i = j \end{cases}, \tag{6}$$

$$h_{ik} = \frac{A}{2}\sum_{l\neq k} W_{kl} + \frac{(n-2)B}{2} + \frac{(n-2)C}{2}. \tag{7}$$

2) Without external fields: By expanding n^2 spins to $(n+1)^2$ spins in a lattice, the n-city TSP can be formulated as (8) by considering h_{ik} as the coupling coefficient between σ_{ik} and $\sigma_{(n+1)(n+1)}$ $(i,k \in [1,n])$:

$$H_{tsp} = \sum_{i=1}^{n+1}\sum_{k=1}^{n+1}\sum_{j=1}^{n+1}\sum_{l=1}^{n+1} J'_{ikjl}\sigma_{ik}\sigma_{jl}, \tag{8}$$

where

$$J'_{ikjl} = \begin{cases} J_{ikjl} & i,k,j,l \in [1,n] \\ \frac{h_{ik}}{2} & i,k \in [1,n]\ and\ j=l=n+1 \\ \frac{h_{jl}}{2} & j,l \in [1,n]\ and\ i=k=n+1 \\ 0 & others \end{cases} \tag{9}$$

and $\sigma_{(n+1)(n+1)}$ is fixed to "1" and the states of other spins in the $(n+1)$th dimension are fixed to "0".

B. TSP Solving with bSB

1) With external fields: A positive dimensionless parameter $b(t)$ is introduced to guarantee the adiabatic evolution of the Hamiltonian system. Adapted form [2], the Hamiltonian for bSB is given by

$$\begin{aligned} H_{bSB_{tsp}} &= \sum_{i,k}\frac{a_0}{2}{y_{ik}}^2 + V(\boldsymbol{x},t) \\ &= \sum_{i,k}\frac{a_0}{2}{y_{ik}}^2 + \frac{a_0-a(t)}{2}\sum_{i,k}{x_{ik}}^2 + \\ &\quad c_0\sum_{i,k,j,l}J_{ikjl}x_{ik}x_{jl} + c_0 b(t)\sum_{i,k}h_{ik}x_{ik} \end{aligned} \tag{10}$$

where $V(\boldsymbol{x},t)$ is the potential energy related to $\boldsymbol{x}$.

The differential equations to be solved are given by the derivatives of x_{ik} and y_{ik} in (10) with respect to time as

$$\dot{x_{ik}} = a_0 y_{ik}, \tag{11}$$

$$\dot{y_{ik}} = -\{a_0 - a(t)\}x_{ik} - 2c_0\sum_{j=1}^{n}\sum_{l=1}^{n}J_{ikjl}x_{jl} - c_0 b(t)h_{ik},$$

where x_{ik} is replaced with its sign if $|x_{ik}| > 1$. (12)

When $a(t)$ gradually increases from zero to a sufficiently large value a_f at the end of search, the position of each oscillator (x_{ik}) is approximately given by $\sigma_{ik}\sqrt{(a_f - a_0)}$ (where $\sigma_{ik} = \pm 1$) [2]. Thus, $V(\boldsymbol{x},t)$ is derived (from [2]), given by

$$\begin{aligned} V(\boldsymbol{x},t) &= c_0(a_f - a_0)\sum_{i,k,j,l}J_{ikjl}\sigma_{ik}\sigma_{jl} \\ &\quad + c_0 b(t)\sqrt{(a_f - a_0)}\sum_{i,k}h_{ik}\sigma_{ik} - \frac{n^2(a_f-a_0)^2}{2} \end{aligned} \tag{13}$$

To ensure that the ground state can provide an approximate solution, the first and second terms in (13) must be proportional to the total energy in (5). Thus, at the end of search, $b(t)$ is manually set as approximately $\sqrt{a_f - a_0}$.

2) Without external fields: Extended from (2) and (3), the differential equations using bSB to solve the TSP are given by (11) and (14).

$$\dot{y_{ik}} = -\{a_0 - a(t)\}x_{ik} - 2c_0\sum_{j=1}^{n+1}\sum_{l=1}^{n+1}J_{ikjl}x_{jl}, \tag{14}$$

where x_{ik} is replaced by its sign if $|x_{ik}| > 1$.

Since the states of spins in the $(n+1)$th dimension are fixed, there is no need to update them. The update of each momentum y_{ik} is related to all $(n+1)^2$ positions. The positions in the $(n+1)$th dimension ($i = n+1$ or $k = n+1$) are initialized to"$+1$" for $x_{(n+1)(n+1)}$ and "-1" for the others.

TABLE I: Comparison of the TSP solving using SA- and the proposed bSB-based Ising models

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	bSB1	**bSB2**	**SA[6]**	**bSB1**	**bSB2**	**SA[6]**	**bSB1**	**bSB2**	**SA[6]**
$\boldsymbol{Ave}$	3786	4006	6451	8019	8474	12040	8859	9481	16435
$\boldsymbol{Max}$	4713	4292	8009	10638	10342	14669	10612	10043	18862
$\boldsymbol{Min}$	3323	3511	4945	6974	7331	8815	7808	8358	13000
$\boldsymbol{Std}$	405	212	696	698	453	1240	735	565	1156

III. Experimental Results of TSP solving

We set $a_0 = 1$. $a(t)$ is slowly increased from "0" to "2"; $b(t) = \frac{a(t)}{2}$. $\boldsymbol{x}$ and $\boldsymbol{y}$ are initialized to a zero matrix and a random matrix with entries within $[-0.1, +0.1]$, respectively. The semi-implicit Euler method uses a first-order integrator to solve the differential equations [4]. The time step is fixed to '1' to simplify the hardware implementation. The parameters A, B, and C in (4) are set to "1", "$\max\{W\}$" and "$\max\{W\}$", where $\max\{W\}$ obtains the maximum value in W. The solution quality is evaluated by the average (Ave), the maximum (Max), the minimum (Min), the standard deviation (Std) of the route distances from 100 trials.

Table I shows the quality of solving the TSP by using bSB with two different mapping methods (referred to as bSB1 with external fields and bSB2 without external fields) and SA. Three datasets are used from the TSPLIB benchmark [7]. Let the iteration be $50k$ in the improved SA [6] and $2k$ in the bSB1 and bSB2. In each iteration, all the spins are updated once. Compared with SA, bSB1 and bSB2 can obtain a better solution with at least an improvement by 29% in AE and 36% in Std with about 129× shorter runtime due to the parallel processing and faster convergence of Hamiltonian via Matlab. Compared with bSB2, although bSB1 is more likely to find a better solution (with a smaller Min), it is also prone to a worse solution (with a larger Max). Therefore, the solution quality using bSB2 is more stable (with a smaller Std).

IV. Conclusion

This paper discusses TSP solving using a bSB-based Ising model. The bSB can realize the parallel update of spins' states, which enables an efficient implementation using multi-core processors. The TSP is solved by a bSB-based Ising model using two different mapping methods. Using the proposed methods can improve the solution quality by at least 29% with 129× shorter runtime than using the SA method.

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