# Gaussian and Jointly Gaussian Probability Densities

**Gaussian Probability Density:** The regular Gaussian density function is given by

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

and, for an arbitrary mean $\mu_x = E(X)$, and variance $\sigma_x^2 = E(X^2) - (E(x))^2$

$$\boxed{\mathcal{N}(\mu_x, \sigma_x^2) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}}$$

as illustrated in the lecture notes.

**Gaussian Error Integral:** Of importance is the Gaussian error integral, which does not have a closed form solution, and is defined as

$$\boxed{Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} d\alpha}$$

that is, it is the integral of the tail of the unit variance, zero mean Gaussian PDF.

**Jointly Gaussian Random Variables**

The one-dimensional Gaussian PDF can be generalized to two dimensions, with two random variables, $X$, and $Y$. In this case the joint PDF is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left( -\frac{(x-\mu_x)^2}{2\sigma_x^2(1-\rho^2)} + \rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y(1-\rho^2)} - \frac{(y-\mu_y)^2}{2\sigma_y^2(1-\rho^2)} \right),$$

where

$$\begin{aligned} \mu_x &= E(X), \mu_y = E(Y) \\ \sigma_x^2 &= E((X-\mu_x)^2), \sigma_y^2 = E((Y-\mu_y)^2) \end{aligned}$$

are the mean and variances of the two random variables, and

$$\rho = \frac{E((X-\mu_x)(Y-\mu_y))}{\sigma_x\sigma_y}$$

is the normalized cross-covariance coefficient.
If $\rho = 0$, the jointly Gaussian PDF breaks into the usual product of two independent Gaussian PDFs.

**Conditional Gaussian Random Variables**

This equation can be written in the form:

$$
\begin{aligned}
f_{XY}(x,y) &= \frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)\frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho^2)}}\exp\left(-\frac{[y-\left(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)]^2}{2\sigma_y^2(1-\rho^2)}\right) \\
&= f_X(x)f_{Y|X}(y|x),
\end{aligned}
$$

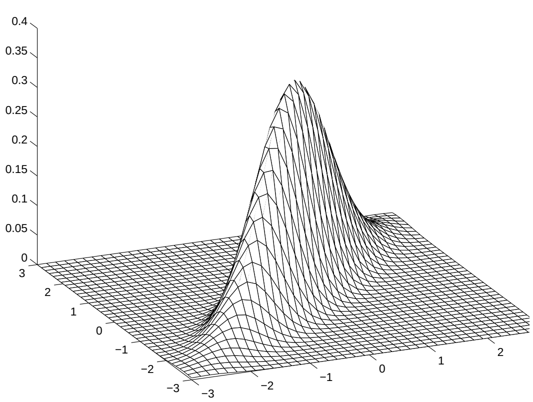
from which we find the conditional probability density function

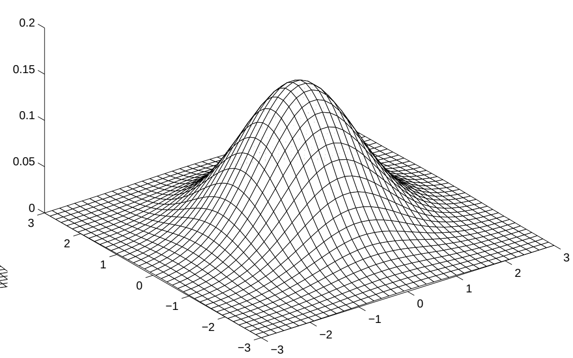
$$
\boxed{f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho^2)}}\exp\left(-\frac{[y-\left(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)]^2}{2\sigma_y^2(1-\rho^2)}\right)}
$$

with conditional mean and variance

$$
\begin{aligned}
\mu_{y|x} &= \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x) \\
\sigma_{y|x}^2 &= \sigma_y^2(1-\rho^2)
\end{aligned}
$$

The figures below show two examples of joint Gaussian distributions: The first joint distribution has a correlation factor of $\rho = 0.9$, and the second has $\rho = 0$, that is, the two RVs are independent.




**Multiple Joint Gaussian Random Variables**

The equation for two jointly Gaussian random variables is often written in matrix algebraic form:

$$
\begin{aligned}
f_{XY}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\left[\begin{array}{cc} x-\mu_x, & y-\mu_y \end{array}\right]\left[\begin{array}{cc} \sigma_x^2 & \sigma_x\sigma_y \\ \sigma_y\sigma_x & \sigma_y^2 \end{array}\right]^{-1}\left[\begin{array}{c} x-\mu_x \\ y-\mu_y \end{array}\right]\right) \\
&= \frac{1}{2\pi\det(\boldsymbol{C})^{1/2}}\exp\left(-\frac{1}{2}\left[\begin{array}{cc} x-\mu_x, & y-\mu_y \end{array}\right]\boldsymbol{C}^{-1}\left[\begin{array}{c} x-\mu_x \\ y-\mu_y \end{array}\right]\right),
\end{aligned}
$$

where

$$\boldsymbol{C} = \left[ \begin{array}{cc} E((X-\mu_x)^2) & E((X-\mu_x)(Y-\mu_y)) \\ E((X-\mu_x)(Y-\mu_y)) & E((Y-\mu_y)^2) \end{array} \right].$$

Clearly, this above equation can be generalized to $n$ random variables, $\boldsymbol{X} = [X_1, \cdots, X_n]^T$, whose jointly Gaussian PDF is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi \det(\boldsymbol{C}))^{n/2}} \exp\left(-\frac{1}{2}\left[\boldsymbol{x}-\boldsymbol{\mu}_x\right]^T \boldsymbol{C}^{-1}\left[\boldsymbol{x}-\boldsymbol{\mu}_x\right]\right),$$

and

$$\boldsymbol{C} = \left[ \begin{array}{ccc} E((X_1-\mu_1)^2) & \cdots & E((X_1-\mu_1)(X-\mu_n)) \\ \vdots & & \vdots \\ E((X_n-\mu_n)(X_1-\mu_1)) & \cdots & E((X_n-\mu_n)^2) \end{array} \right].$$

**Linear Transformations of Gaussian RVs**

If we transform a vector of Gaussian random variables

$$\boldsymbol{y} = \boldsymbol{T}\boldsymbol{x},$$

where $\boldsymbol{T}$ is an $n \times n$, nonsingular transformation matrix, such as a rotation. Then the PDF of the random vector $\boldsymbol{Y}$ is easily calculated from the statistic of $\boldsymbol{X}$. Note, that the new random variables are still Gaussian. In fact

$$\begin{aligned} \boldsymbol{\mu}_y &= \boldsymbol{T}\boldsymbol{\mu}_x \\ \boldsymbol{C}_y &= E(\boldsymbol{X}^T\boldsymbol{X}) = \boldsymbol{T}^T\boldsymbol{C}_x\boldsymbol{T} \end{aligned}$$

**Example:** Consider the rotation of the two dimensional zero-mean PDF

$$f_{X_1X_2}(x_1, y_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}\left(x_1^2 - 2\rho x_1 x_2 + x_2^2\right)\right),$$

whose correlation matrix is given by

$$\boldsymbol{C}_x = \sigma^2 \left[ \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right].$$

Let the transformation matrix

$$\boldsymbol{T} = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right] \Longrightarrow \boldsymbol{C}_y = \boldsymbol{T}\boldsymbol{C}_x\boldsymbol{T}^T = \left[ \begin{array}{cc} \sigma^2(1+\rho) & 0 \\ 0 & \sigma^2(1-\rho) \end{array} \right].$$

From this we can quickly find the joint PDF as

$$f_{X_1X_2}(x_1, y_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(-\frac{y_1^2}{2\sigma^2(1+\rho)} - \frac{y_2^2}{2\sigma^2(1-\rho)}\right),$$

that is, $Y_1$ and $Y_2$ are *independently* Gaussian distributed with

$$\begin{aligned} \mu_{y_1} &= \mu_{y_2} = 0 \\ \sigma_{y_1}^2 &= \sigma^2(1+\rho), \sigma_{y_2}^2 = \sigma^2(1-\rho) \end{aligned}$$

This corresponds to placing the axes through the "ridge" of the correlated Gaussian PDF on the previous page.