Shannon's Channel Capacity

Shannon derived the following capacity formula (1948) for an additive white Gaussian noise channel (AWGN):

$$C = W \log_2 (1 + S/N) \quad \text{[bits/second]}$$

- W is the bandwidth of the channel in Hz
- S is the signal power in watts
- N is the total noise power of the channel watts

Channel Coding Theorem (CCT):
The theorem has two parts.

1. Its **direct part** says that for rate $R < C$ there exists a coding system with arbitrarily low block and bit error rates as we let the codelength $N \to \infty$.
2. The **converse part** states that for $R \geq C$ the bit and block error rates are strictly bounded away from zero for any coding system

The CCT therefore establishes rigid limits on the maximal supportable transmission rate of an AWGN channel in terms of power and bandwidth.

Bandwidth Efficiency characterizes how efficiently a system uses its allotted bandwidth and is defined as

$$\eta = \frac{\text{Transmission Rate}}{\text{Channel Bandwidth } W} \quad [\text{bits}/s/Hz].$$

From it we calculate the *Shannon limit* as

$$\eta_{\max} = \log_2 \left(1 + \frac{S}{N}\right) \quad [\text{bits}/s/Hz]. \tag{1}$$

Note: In order to calculate η, we must suitably define the channel bandwidth W. One commonly used definition is the 99% bandwidth definition, *i.e.,* W is defined such that 99% of the transmitted signal power falls within the band of width W.

Average Signal Power S can be expressed as

$$S = \frac{kE_b}{T} = RE_b,$$

- E_b is the energy per bit
- k is the number of bits transmitted per symbol
- T is the duration of a symbol
- $R = k/T$ is the transmission rate of the system in bits/s.
- S/N is called the signal-to-noise ratio
- $N = N_0W$ is the total noise power
- N_0 is the one-sided noise power spectral density

we obtain the Shannon limit in terms of the bit energy and noise power spectral density, given by

$$\boxed{\eta_{\max} = \log_2\left(1 + \frac{RE_b}{N_0W}\right).}$$

This can be resolved to obtain the minimum bit energy required for reliable transmission, called the **Shannon bound**:

$$\frac{E_b}{N_0} \geq \frac{2^{\eta_{\max}} - 1}{\eta_{\max}},$$

Fundamental limit: For infinite amounts of bandwidth, *i.e.*, $\eta_{\max} \to 0$, we obtain

$$\boxed{\frac{E_b}{N_0} \geq \lim_{\eta_{\max}\to 0} \frac{2^{\eta_{\max}} - 1}{\eta_{\max}} = \ln(2) = -1.59\text{dB}}$$

This is the absolute minimum signal energy to noise power spectral density ratio required to reliably transmit one bit of information.

Normalized Capacity

The dependence on the arbitrary definition of the bandwidth W is not satisfactory. We prefer to normalize our formulas per signal dimension. It is given by [5]. This is useful when the question of waveforms and pulse shaping is not a central issue, since it allows one to eliminate these considerations by treating signal dimensions [2].

$$C_d = \frac{1}{2}\log_2\left(1 + 2\frac{R_d E_b}{N_0}\right) \quad \text{[bits/dimension]}$$

$$C_c = \log_2\left(1 + \frac{R E_b}{N_0}\right) \quad \text{[bits/complex dimension]}$$

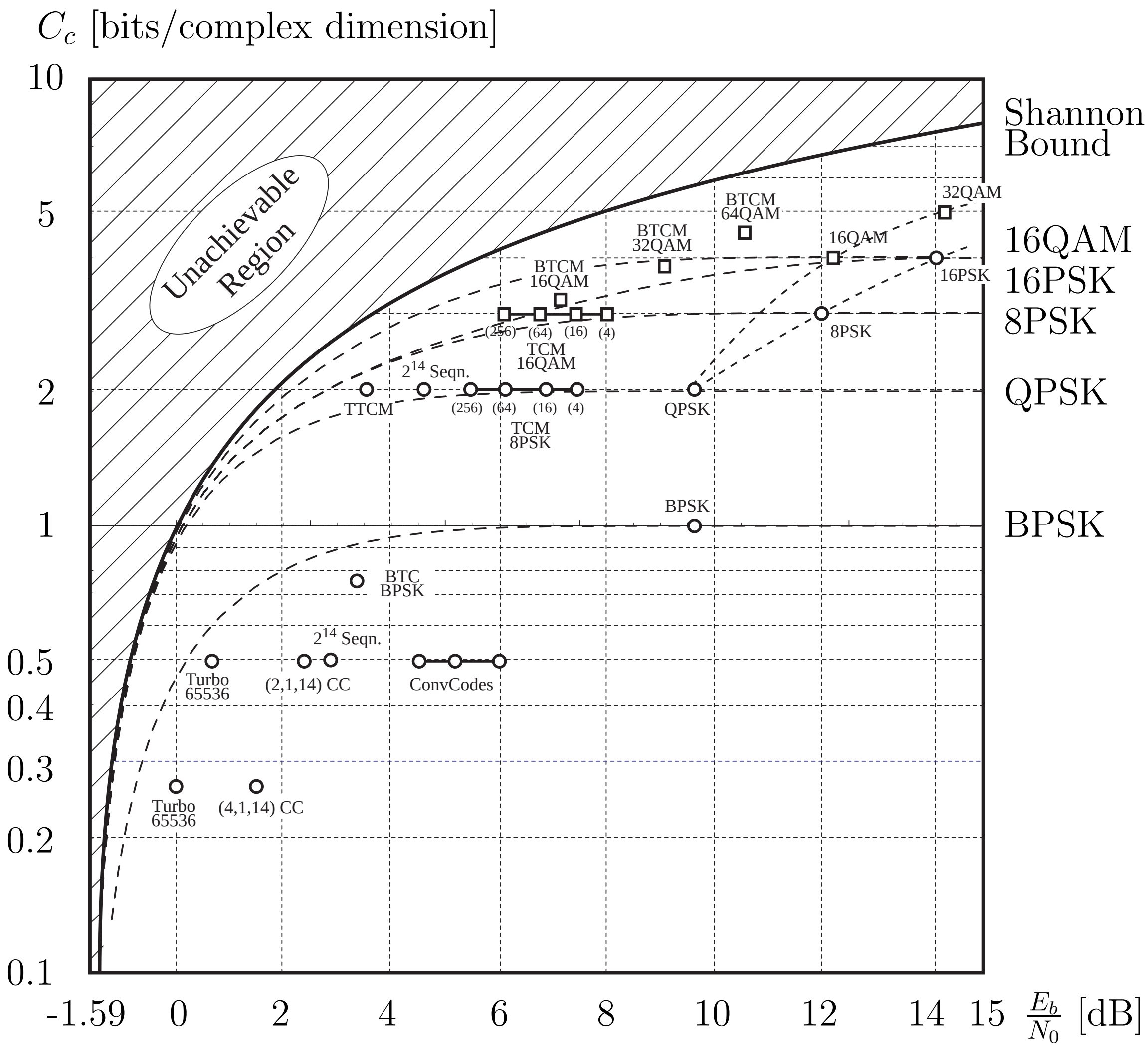
Applying similar manipulations as above, we obtain the Shannon bound normalized per dimension as

$$\frac{E_b}{N_0} \geq \frac{2^{2C_d} - 1}{2C_d}; \qquad \frac{E_b}{N_0} \geq \frac{2^{C_c} - 1}{C_c}.$$

System Performance Measure In order to compare different communications systems we need a parameter expressing the performance level. It is the information bit error probability P_b and typically falls into the range $10^{-3} \geq P_b \geq 10^{-6}$.

Examples:

Spectral Efficiencies versus power efficiencies of various coded and uncoded digital transmission systems, plotted against the theoretical limits imposed by the discrete constellations.



Discrete Capacities

In the case of discrete constellations, Shannon's formula needs to be altered. We begin with its basic formulation:

$$\begin{aligned} C &= \max_{\boldsymbol{q}} \left[H(Y) - H(Y|X) \right] \\ &= \max_{\boldsymbol{q}} \left[-\int_y \sum_{a_k} q(a_k) p(y|a_k) \log \left(q(a_k') \sum_{a_k'} p(y|a_k') \right) \right. \\ &\qquad \left. - \int_y \sum_{a_k} q(a_k) p(y|a_k) \log \left(p(y|a_k) \right) \right] \\ &= \max_{\boldsymbol{q}} \left[\sum_{a_k} q(a_k) \int_y \log \left(\frac{p(y|a_k)}{\sum_{a_k'} q(a_k') p(y|a_k')} \right) \right] \end{aligned}$$

where $\{a_k\}$ are the K discrete signal points, $q(a_k)$ is the probability with which a_k is selected, and

$$p(y|a_k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-a_k)^2}{2\sigma^2} \right)$$

in the one-dimensional case, and

$$p(y|a_k) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(y-a_k)^2}{2\sigma^2} \right)$$

in the complex case.

Symmetrical Capacity: When $q(a_k) = 1/K$, then

$$C = \log(K) - \frac{1}{K} \sum_{a_k} \int_n \log \left[\sum_{a_k'} \exp\left(-\frac{(n - a_k' + a_k)^2 - n^2}{2\sigma^2} \right) \right]$$

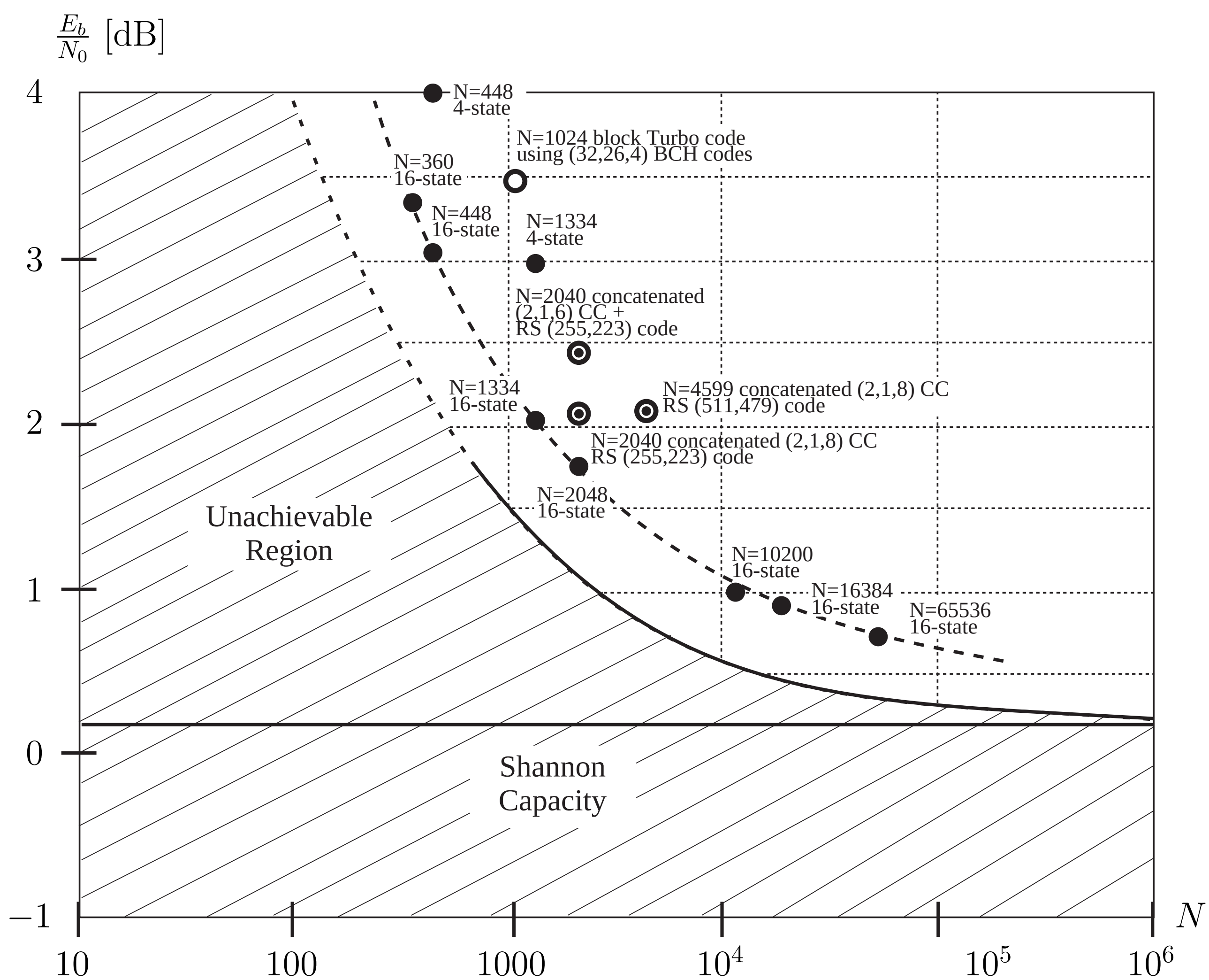
This equation has to be evaluated numerically.

Code Efficiency

Shannon et. al. [4] proved the following *lower bound* on the codeword error probability P_B:

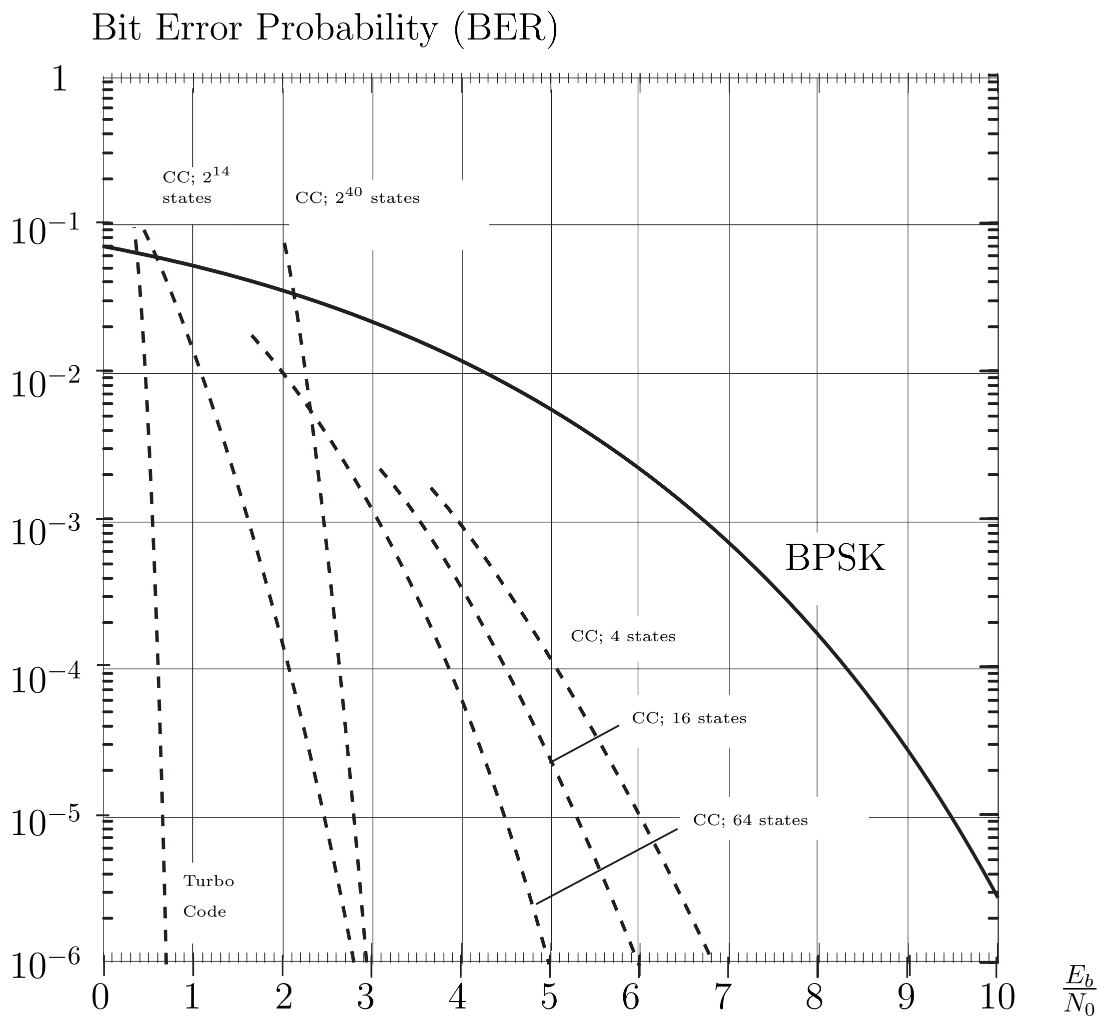
$$P_B > 2^{-N(E_{\rm sp}(R)+o(N))}; \quad E_{\rm sp}(R) = \max_{\boldsymbol{q}} \max_{\rho>1} \left(E_0(\boldsymbol{q},\rho) - \rho R\right).$$

The bound is plotted for rate $R = 1/2$ for BPSK modulation [3], together with selected Turbo coding schemes and classic concatenated methods:



Code Performance

The performance of various coding schemes is shown below (see [2]). Note the steep behavior of the turbo codes known as the **Turbo cliff**.



References

[1] R.G. Gallager, *Information Theory and Reliable Communication,* John Wiley & Sons, Inc., New-York, 1968.

[2] C. Schlegel, *Trellis Coding*, IEEE Press, Piscataway, NJ, 1997.

[3] C. Schlegel and L.C. Perez, "On error bounds and turbo codes,", *IEEE Communications Letters*, Vol. 3, No. 7, July 1999.

[4] C.E. Shannon, R.G. Gallager, and E.R. Berlekamp, "Lower bounds to error probabilities for coding on discrete memoryless channels," *Inform. Contr.,* vol. 10, pt. I, pp. 65–103, 1967, Also, *Inform. Contr.,* vol. 10, pt. II, pp. 522-552, 1967.

[5] J.M. Wozencraft and I.M. Jacobs, *Principles of Communication Engineering*, John Wiley & Sons, Inc., New York, 1965, reprinted by Waveland Press, 1993.