Abstract—Iterative joint detection of random CDMA is considered. It is shown that optimal power levels and optimal code choices can be found that achieve the capacity of this channel. Even few power levels can significantly improve the achievable capacity. Furthermore, it is shown that very low rate codes are optimal, but that practical codes, again, can achieve a large portion of the channel capacity.

I. INTRODUCTION

Iterative joint decoding of code-division multiple access (CDMA) systems using forward error control coding is based on the success of turbo coding [3], and has proven to be a very powerful decoding method which has the potential of realizing a significant portion of the multiple access channel capacity [7], [1], [6], [2], [12], [9]. Iterative decoding breaks the complex task of a multiuser decoder into two operations, viz. (i) a posteriori probability (APP) estimation of the coded symbols, or an approximation thereof, and (ii), parallel soft decoding of single-user forward error control (FEC) codes. Iterative joint decoding operates on all of the users jointly and concurrently.

The convergence behavior of an iterative decoder can conveniently be studied using a variance or signal-to-noise ratio transfer analysis. Recently, Boutros and Caire [4] have presented a single-parameter dynamical system interpretation of the decoder. Caire et. al. [5] have extended this analysis to different power levels and presented a linear programming solution to optimizing the power levels of different user groups.

Shi and Schlegel [10] have presented an analysis of the impact of error control coding on the performance of such iterative systems, in particular on the supportable system load. They found that weak error control codes, such as simple parity check codes or low-memory convolutional codes can support substantially higher system loads than stronger codes such as Shannon-limit achieving turbo codes for equal power levels among the users.

In this paper we view the iterative decoder as a non-linear layering device and consider the achievable capacities on each of these layers. We study the role of the error control codes via variance transfer methods (equivalent to density evolution), showing that the the combined transfer characteristics of using a number of different power levels approaches the transfer characteristics of “weak” FEC codes in an equal power setting, regardless of the codes used. We then calculate closed-form equations for the optimal power and system load distributions for strong FEC codes, to be defined below. We furthermore demonstrate that different power levels have the effect of migrating a parallel iterative decoder into an effective successive interference cancellation decoder, and that the optimized power levels coincide with the power levels dictated by the corner points of the capacity polytope which allows successive cancellation decoding. We show that, contrary to conventional successive cancellation decoding, iterative joint detection does not require Shannon-limit achieving FEC codes and functions well even with weak FEC codes. Finally, we study the equal-rate, equal-power multiple access channel, which represents the most difficult constellation for the joint decoder, by studying the sublayer mutual information values, and calculate achievable spectral efficiencies, which fall significantly short of those achievable with optimal power or rate distributions.

II. SYSTEM MODEL

The basic system model of asynchronous CDMA contains $K$ transmitters generate i.i.d. binary information symbols $u_k \in \{0, 1\}$, $k = 1, \ldots, K$ which are encoded by $K$ parallel error control encoders each with code rate $R$. Random interleavers separate the error control encoders from the spreading operation. The particular codes used are part of the topic of this paper and may be weak error control codes such as simple parity check codes, or very sophisticated turbo [3], [8], or low-density parity-check codes.

The signal from $k$th spreader is

$$x_k(t) = \sum_{l=0}^{L-1} \sqrt{P_k} d_{k,l} a_{k,l}(t - lT - \tau_k)$$

where $L$ is the number code symbols per user per frame, $d_{k,l}$ is the $l$-th symbol in the data stream of user $k$, and $\{\tau_k < T\}$, $k = 1, \ldots, K$, are the time delays of the different users, $P_k$ is the power of user $k$, and $a_{k,l}(t)$, supported on the interval $[0, T]$, is the $k$-th energy-normalized spreading

Christian Schlegel
Department of Electrical Engineering
University of Alberta
Edmonton, AB T6G 2V4, CANADA
Email: schlegel@ece.ualberta.ca

Zhenning Shi
Department of Electrical Engineering
University of Utah
Salt Lake City, UT 84060, USA
Email: zshi@ece.utah.edu
sequence waveform, given by

\[
a_{k,l}(t) = \sum_{n=0}^{N-1} a_{k,l,n} g(t - n T_c) \tag{2}
\]

where \(N\) is the spreading gain, \(T_c\) is the chip interval, \(a_{k,l,n} \in \{-1/\sqrt{N}, 1/\sqrt{N}\}\) is the \(n\)th spreading chip for user \(k\) at symbol time \(l\), and \(g(t)\) is the normalized chip waveform. In this paper we consider an AWGN channel multiple access channel and the received signal is

\[
y(t) = \sum_{k=1}^{K} x_k(t) + n(t) \tag{3}
\]

where \(n(t)\) is zero mean white Gaussian noise with double-sided noise power spectral density \(\sigma_d^2 = N_0/2\).

Assuming that timing and phase references have been established, the received signal sampled by chip matched filters can be written in the convenient discrete chip-based matrix model

\[
y = Ad + n \tag{4}
\]

where \(n\) is an \((L + 1)N\) vector of sampled white noise with variance \(\sigma_d^2\), \(A\) is an \((L + 1)N \times LK\) matrix whose \(j\)-th column is \(a_{k,l} = [0_{N+\tau_j/T_c}, a_{k,l,0}, \ldots, a_{k,l,N-1}, 0_{(L-l)N-\tau_j/T_c}]^T\), where \(l = \lfloor j/K \rfloor\) is the symbol time index, \(0_i\) is a length-\(i\) all-zero column vector, and \(d = [d_{1,0}, \ldots, d_{K,0}, d_{1,1}, \ldots, d_{K,1,L-1}]^T\) is the \(KL\)-sized vector of encoded (BFSK) symbols.

III. ITERATIVE (TURBO) JOINT DECODING

Figure 1 shows the block diagram of an iterative joint receiver. The CDMA interference resolution operation is used to generate soft outputs of the encoded symbols \(d_{k,l}\) given a received frame \(y\), and separate these into \(K\) parallel signal streams suitable for the \(K\) outer parallel FEC decoders.

As explained by Boutros and Caire [4], each user \(k\) sees a received signal, after cancellation, given by

\[
y_k = \sum_{i=1}^{L} \sqrt{P_{k}} d_{k,i} a_{k,i}^T + \sum_{i=1}^{L} \sum_{l=0}^{K} \sqrt{P_{l}} (d_{k,l} - \tilde{d}_{k,l}^i) a_{k,l} + n
\]

where \(\tilde{d}_{k,l}^i\) is a soft estimate of the coded symbol of user \(k\) at time \(l\), which was generated during a previous iteration, and is now used to partially cancel the interference from users \(k' \neq k\), which, after filtering by the user-specific filter \(w_{k,l}\), becomes

\[
z_{k,l} = w_{k,l}^T y_k = \sqrt{P_{k}} d_{k,l} + n_{k,l}^i \tag{6}
\]

Under some general conditions, \(n_{k,l}^i\) is well approximated by — and approaches in the limit — an independent Gaussian random variable [4].

The filter \(w_{k,l}\) can be a matched filter [1], or a more sophisticated filter, such as the conditional and unconditional MMSE filters [12], [4]. No matter which method is used, the decisive observation is that the variance of \(n_{k,l}^i\), denoted by \(\sigma_{eff}^2\) (for effective channel noise), is independent of \(l\) and \(k\) in the large system case where random spreading sequences are used. This enables us to describe the behavior of such systems by a single-parameter dynamical system, [1], [9], [4].
For simple cancellation with matched filtering,

\[ \sigma_{\text{eff}}^2 = \frac{\sigma^2}{P} + K - 1 + \alpha \sigma_d^2 \quad \text{as } K, N \to \infty \]

\[ \sigma_{\text{eff}}^2 \approx \frac{\sigma^2}{P} + \alpha \sigma_d^2 \] (7)

where \( \sigma_d^2 = E[(d - \bar{d})^2] \) is the variance of the residual interference, and \( \alpha = K/N \) is the constant system load as \( K, N \to \infty \).

We show that simple interference cancellation is quite sufficient for large systems, but more sophisticated approaches have been proposed, such as the conditional MMSE filter [12], [4], which can increase the load of an equal power system by the factor \( 1 + 1/\alpha \) [10]. The dynamical system shown Figure 1, i.e.,

\[ \sigma_{\text{eff}}^2(i) = f_{\text{IC}}(\sigma_d(i - 1)); \quad \sigma_d(i) = f_{\text{APP}}(\sigma_{\text{eff}}(i - 1)) \] (8)

Figure 2 shows the fundamental behavior of such iterative joint detection receivers. The left-hand plot shows the variance transfer function, documenting the exchange of cancellation variance \( \sigma_{\text{eff}}^2 \) and single-user APP output variance \( \sigma_d^2 \). The trajectory is a simulated iteration for a system with 45 users. Decoding starts at the right top portion of the diagram, and alternately switches between the transfer curves of the code and the canceller, until it reaches the noise fixpoint close to the lower left corner. These dynamical systems have either one or two stable fixpoints. The first one, the interference limitation, depends on the system load and the FEC codes used. If no interference fixpoint exists, the iterations proceed through a narrow “bottleneck” region and converge to the noise fixpoint. During this process, the BER drops almost instantaneously (with SNR) as illustrated on the right-hand side BER simulations. At low values of the BER, the error curves follow the single user performance curve, which acts as lower asymptote. This is the noise limitation on the left-hand side.

Also shown in the figure is the variance transfer curve of a serially concatenated turbo code of rate \( R = 1/3 \) using as inner code a recursive convolutional code of rate \( R_i = 1/2 \) with generator \( g = [1 + D + D^2 + D^3] \), and as outer code a rate \( R_o = 2/3 \) recursive convolutional code. It VT curve represent a general observation, namely that the stronger code has a much more “step-function” like variance transfer function, which causes it to be limited by the interference limitation at load values significantly below those of the weaker code [9].

IV. UNEQUAL POWER DISTRIBUTIONS AND OPTIMAL POWER PROFILES

Unequal powers of the different users were considered by Shi and Schlegel [9] who showed how to combine the transfer curves of user groups using different codes as well as different power levels. Caire et. al. studied optimal power levels by numerically optimizing the dynamical system equation of the joint iterative detector via linear programming methods for simple cancellation, MMSE cancellation, and using an APP decoder front-end.

As can be seen from the graph of the variance of a strong FEC code (see Figure 2), its VTC curve approaches a step-function. We can now lower-bound the variance transfer curve...
of such a code quite tightly by the staircase function
\[
\sigma_d^2 = f(\sigma_{\text{eff}}^2) = \begin{cases} 
1 & \text{if } \sigma_{\text{eff}}^2 \geq \tau P_j, \\
0 & \text{if } \sigma_{\text{eff}}^2 < \tau P_j,
\end{cases} 
\]
where \(1/\tau\) is the SNR threshold of the code, and \(P_j\) is the power of the \(j\)-th user group, and we call the hypothetical code whose variance transfer obeys (9) a “strong code”. Also we separate the \(K\) users into \(J\) groups of powers \(P_1, \ldots, P_J\), ordered as \(P_1 \leq P_2 \leq \cdots \leq P_J\), even though this is irrelevant in practice due to the parallel decoder architecture.

Now, let \(\alpha = \sum_{j=1}^J K_j/N = \sum\alpha_j\) be the system load, where \(\alpha_j\) is the partial load of the \(j\)-th group. Assuming then that \(K_j \gg 1\), and beginning with \(j = 1\) we observe that
\[
\tau P_1 \geq \alpha_1 P_1 + \sigma^2, 
\]
and recursively, we find
\[
P_j \geq \frac{\sum_{j=1}^{j-1} \alpha_j P_j + \sigma^2}{\tau - \alpha_j}; \quad 1 \leq j \leq J. 
\]
The minimum powers are achieved in (11) when the inequalities are met with equality, and the power levels are given by
\[
P_j = \frac{\sigma^2}{\tau - \alpha_j} \prod_{j=1}^{j-1} \frac{1}{\tau - \alpha_j}; \quad 1 \leq j \leq J. 
\]
Equation (11) optimizes the power levels such that maximal system load can be achieved. In order to maximize the spectral efficiency we minimize the average power per user
\[
\bar{P} = \min_{\{\alpha_j\}} \frac{\sum_{j=1}^J P_j \alpha_j}{\sum_{j=1}^J \alpha_j} 
\]
The solution to (13) is given by the following

**Lemma 1:** The average power in (13) is minimized by the uniform load distribution \(\alpha_j = \frac{\gamma}{J}; \forall j\). Proof: See [11].

Under these optimizing conditions, the group powers (12) are given by
\[
P_j = \frac{\sigma^2}{\tau - \alpha_j} \left(\frac{\tau}{\tau - \alpha_j}\right)^{j-1} \quad 1 \leq j \leq J, 
\]
i.e., by an exponential power distribution, and the optimal average power in (13) is given by
\[
\bar{P}_{\text{opt}} = \frac{\sigma^2}{\alpha} \left[\left(\frac{\tau}{\tau - \alpha/J}\right)^{J} - 1\right]. 
\]
In (15), the optimal power is a function of \(\sigma^2\), \(\alpha\) and \(\tau\). The corresponding signal-to-noise-ratio is then
\[
\left[ \frac{E_b}{N_0} \right]_{\text{opt}} = \frac{\bar{P}_{\text{opt}}}{2R\sigma^2} = \frac{1}{2R\alpha} \left[\left(\frac{\tau}{\tau - \alpha/J}\right)^{J} - 1\right], 
\]
where \(R\) is the common code rate of the FEC codes.

The attainable spectral efficiency is improved as the number of power groups is increased. This is shown by the following lemma

**Lemma 2:** The sequence \(\left(\frac{\tau}{\tau - \alpha/J}\right)^{J}\) is monotonically decreasing as \(J\) increases in the range \([J^*, \infty)\), where \(J^*\) denotes the smallest integer such that \(\frac{\tau}{\tau - \alpha/J} < 1\). Proof: See [11].

In the limit as \(J \to \infty\) the required \(E_b/N_0\) has to obey
\[
\left[ \frac{E_b}{N_0} \right]_{\lim} = \frac{e^{\frac{\gamma}{2}} - 1}{2R\alpha}. 
\]
The optimal SNR is approximately an exponential function of the total system load \(\alpha\). The minimum SNR for CDMA can be obtained by letting \(\alpha \to 0\) in (17), and equals \(\frac{\gamma}{2R\alpha}\), which is the threshold of the FEC code in single-user channels.

For \(R \to 0\) we obtain the following

**Theorem 1:** Shannon-type strong codes which obey (9) can achieve the AWGN Shannon bound as \(R \to 0\), i.e.,
\[
\left[ \frac{E_b}{N_0} \right]_{\lim} = \frac{2^{2C} - 1}{2C}; \quad C = R\alpha 
\]


**V. CONCLUSIONS**

Optimal selection of power levels and codes were found for iterative joint detection of random CDMA. It was shown that low-rate “strong codes” can achieve the capacity of the multiple access channel in the limit with many optimized power levels. Even few power levels and non-ideal FEC codes can achieve a significant portion of that capacity in more practicable circumstances. Additional figures and examples are presented at the conference and in references [11].

**REFERENCES**


