Dynamic Spectrum Leasing with Two Sellers

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Abstract—This paper studies dynamic spectrum leasing in a cognitive radio network. There are two spectrum sellers, who are two primary networks, each with an amount of licensed spectrum bandwidth. When a seller has some unused spectrum, it would like to lease the unused spectrum to secondary users. A coordinator helps to perform the spectrum leasing stage-by-stage. As the two sellers may have different leasing periods, there are three epochs, in which seller 1 has spectrum to lease in Epochs II and III, while seller 2 has spectrum to lease in Epochs I and II. Each seller needs to decide how much spectrum it should lease to secondary users in each stage of its leasing period, with a target at revenue maximization. It is shown that, when the two sellers both have spectrum to lease (i.e., in Epoch II), the spectrum leasing can be formulated as a non-cooperative game. Nash equilibria of the game are found in closed form. Solutions of the two sellers in their leasing periods are then derived.

Index Terms—Cognitive radio, dynamic pricing, Nash equilibrium.

I. INTRODUCTION

Cognitive radio has been considered as a promising solution to the spectrum shortage problem in the near future. In cognitive radio, spectrum access of unlicensed users (referred to as secondary users), which is required not to affect the communication of licensed users (referred to as primary users), is permitted [1]. To implement cognitive radio, two typical modes are overlay mode and underlay mode. In overlay mode, spectrum access of secondary users is permitted only when primary users are idle. Thus, secondary users are required to perform periodical spectrum sensing to detect possible transmissions of primary users. In underlay mode, secondary users can access the spectrum when primary users are transmitting. Secondary users should carefully manage their transmit power such that their generated interference to primary users is below a threshold limit.

It is difficult to guarantee quality-of-service (QoS) of secondary users in overlay or underlay mode. In overlay mode, secondary users have to wait until primary users do not have traffic to transmit. If primary users have high traffic load, secondary users would have little chance to transmit. In underlay mode, due to the transmit power constraint, secondary users may only achieve short-range low-rate communications.

To better serve secondary users, spectrum leasing has been introduced, in which if a primary user (also called spectrum seller) has some unused spectrum for a certain amount of time, it leases the unused spectrum to secondary users. During the leasing period, the secondary users can use the spectrum exclusively, which guarantees their communication QoS. Furthermore, the leasing revenue can motivate the spectrum seller to use its spectrum more efficiently so as to collect more unused spectrum for leasing. Optimal spectrum leasing that maximizes the spectrum seller’s revenue is an interesting topic, which is also the focus of this paper.

In the literature, spectrum leasing has been well investigated under the modes of monopoly spectrum leasing (in which there is one spectrum seller) and oligopoly spectrum leasing (in which multiple spectrum sellers exist). In monopoly spectrum leasing, e.g., the works in [2]–[4], the major target is to achieve the maximal revenue of the seller. In oligopoly spectrum leasing, e.g., the works in [5]–[12], the major target is to achieve an equilibrium in the competition among multiple spectrum sellers. In these works, spectrum leasing is performed only once, and the spectrum price is fixed for the whole spectrum leasing duration, referred to as static spectrum leasing. On the other hand, dynamic spectrum leasing, in which the spectrum price may change over time, is more appropriate for the cases that the secondary users may need spectrum at different time instants. There are limited research efforts in the literature on dynamic spectrum leasing, including the works in [13]–[15] that consider a single spectrum seller and the work in [16] that considers multiple spectrum sellers.

In this paper, we study dynamic spectrum leasing problem in a duopoly market with two sellers. As the two sellers may have different leasing periods, the system has three epochs, in which seller 1 has spectrum to lease in Epochs II and III, while seller 2 has spectrum to lease in Epochs I and II. The main contributions in this paper are summarized as follows.

1) We show that, the spectrum leasing problems of the sellers in

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Epoch I and Epoch III are convex optimization problems. For Epoch II, we formulate spectrum leasing of the two sellers as a non-cooperative game. We derive closed-form expressions for the Nash equilibria of the non-cooperative game. 2) The amount of spectrum that seller 1 would like to lease in Epoch III affects the non-cooperative game in Epoch II, and thus, affects the total revenues of the two sellers. By analyzing properties of seller 1’s revenue in Epoch II and Epoch III, we propose a method that finds the optimal amount of spectrum that seller 1 should lease to secondary users in Epoch III.

The rest of this paper is organized as follows. In Section II, related works are reviewed. In Section III, the system model is presented, and the spectrum leasing problems for the two sellers are formulated. In Section IV, Nash equilibria of the non-cooperative game in Epoch II are derived. Section V discusses how seller 1 should distribute its spectrum to be leased in Epoch II and Epoch III. Numerical results are given in Section VI, and finally the paper is concluded in Section VII.

II. RELATED WORKS

Monopoly spectrum leasing: In the work of [2], there are a spectrum provider, a broker, and a number of secondary users. By a Stackelberg game modeling, the broker optimally decides on the number of channels it should purchase from the spectrum provider as well as the price it should use to sell the purchased spectrum to secondary users. The work in [3] also considers a broker. It is assumed that for a given spectrum price, the amount of spectrum demand from secondary users is random. The work in [4] considers the impact of spectrum leasing on primary user performance (such as possible extra interference to the primary system). An optimal solution is given for the primary user, which strikes a balance between the earned revenue and the cost.

Oligopoly spectrum leasing: Two brokers are assumed in [5]. Each broker decides on the amount of spectrum that it should purchase from spectrum providers and on the spectrum price that it should announce to secondary users, with a target at profit maximization. The work in [6] also considers two brokers, by assuming that the leased spectrum may be shared by multiple secondary users simultaneously. Therefore, interference among secondary users needs to be taken into account. The works in [7]–[9] consider a duopoly market, in which the price competition of two spectrum sellers is investigated by using game theoretical approaches. The work in [10] discusses the case with multiple sellers. By using an evolutionary game model, a solution is given to secondary users for their spectrum selection and to sellers for price setting. The work in [11] considers multiple sellers as well as one broker, in which the impact of spectrum leasing on sellers’ performance (i.e., service quality degradation) is taken into account. The work in [12] considers heterogeneous secondary users, i.e., different secondary users may have different criteria on their spectrum leasing decisions.

Note that in all above works on monopoly and oligopoly spectrum leasing, the spectrum price is fixed for the whole spectrum leasing duration, and thus, the works all consider static spectrum leasing.

Dynamic pricing: In [13], dynamic pricing in monopoly spectrum leasing is performed over infinite time horizon. The spectrum price is set dynamically, with a target at long-term average revenue maximization. In [14], dynamic pricing in monopoly spectrum leasing is performed over a finite duration. The finite duration is divided into a number of stages, and the price in each stage is set up so as to maximize the overall revenue. In [15], dynamic spectrum leasing is investigated for a monopoly market. Among all channels, some are allocated as leased channels, and the others are called unleased channels. Secondary users can access the leased channels (with higher priority if primary users are also allowed to access), as well as the unleashed channels with a lower priority than that of primary users. The number of leased channels is adjusted following the arrival/departure events of the primary and secondary users. In [16], dynamic spectrum leasing is investigated, considering the competition among multiple spectrum sellers. A stochastic Cournot game model is used to derive the leasing strategies (i.e., the amount of spectrum to lease) of the spectrum sellers.

In this paper, we also consider dynamic spectrum leasing. The difference of our work from those in [13]–[15] lies in that the works in [13]–[15] consider a monopoly market while we consider a duopoly market. The difference of our work from that in [16] is as follows. In [16], each spectrum seller’s available spectrum for leasing is determined by the market (in other words, the spectrum seller sets the amount of spectrum that it could lease so as to maximize its profit). In our work, each spectrum seller’s available spectrum is determined by traffic load of its own users. After all its users’ traffic has been accommodated, the unused spectrum can be leased to secondary users.

III. SYSTEM MODEL AND PROBLEM FORMULATION

Consider two spectrum sellers (seller 1 and seller 2), one coordinator, and multiple secondary users. Similar to [15], the coordinator is a centralized unit (e.g., a base station controller) which is trusted by the spectrum sellers and secondary users. It is responsible to collect information (for example, amount of spectrum to lease) from and get back to spectrum sellers, post spectrum price to secondary users, lease spectrum to secondary users, manage secondary users’ access to the leased spectrum, etc. The two sellers are primary networks with a certain amount of licensed spectrum bandwidth. For each seller, when the data traffic from its own users is light, the seller may partition its spectrum bandwidth into two portions: primary portion and secondary portion. The primary portion will be assigned to the seller’s own users, and the secondary portion can be leased to secondary users. In specific, consider that seller 1 and seller 2 have bandwidth $Q_1$ and $Q_2$ in their secondary portion, respectively. For each seller, the bandwidth in its secondary portion can be leased to secondary users for a duration (called leasing period). Consider that the two sellers’ leasing periods are not identical, and overlap with each other. 2

2If the two leasing periods are identical, it is a special case of the problem considered in this paper.
Without loss of generality, we assume that the leasing period of seller 2 starts earlier than the leasing period of seller 1. We also assume that the leasing period of seller 2 ends earlier than that of seller 1. An illustration of the two leasing periods is given in Fig. 1. Here the union of the two leasing periods contains $N$ fixed-length stages. For presentation simplicity, the last stage of seller 1’s leasing period is called stage 1, while the first stage of seller 2’s leasing period is called stage $N$. Seller $i$ ($i = 1, 2$) would distribute its spectrum bandwidth $Q_i$ to be leased in the stages of its leasing period. In other words, it needs to decide on the amount of spectrum bandwidth to be leased in each stage in its leasing period, with a constraint that the total amount of leased spectrum bandwidth in the stages is bounded by $Q_i$. For seller $i$, denote the amount of spectrum bandwidth it would like to lease to secondary users in stage $n$ as $d_{n,i}$. At the beginning of stage $n$, seller $i$ should report to the coordinator the information of $d_{n,i}$.

At the beginning of stage $n$, after the coordinator gets the information of $d_{n,1}$ and $d_{n,2}$, it would set up a spectrum unit price (the price per unit bandwidth per stage) and lease the spectrum bandwidth $(d_{n,1} + d_{n,2})$ to secondary users. In other words, the coordinator should set up the unit price to attract $(d_{n,1} + d_{n,2})$ spectrum bandwidth demand from secondary users. Denote the price $p$ to attract $d$ spectrum bandwidth demand as $P(d)$, which is a function of $d$. Economics analysis [17], [18] has shown that price and demand typically follow a linear model, and thus, price $p$ and spectrum bandwidth demand $d$ satisfy the following feature:

$$
p = P(d) = C_0 - C_1 \cdot d
$$

in which $C_0$ and $C_1$ are coefficients. $P(d)$ is a decreasing function of $d$. In addition, $d \cdot P(d)$ should be an increasing function of $d$ (as the total revenue for more leased spectrum bandwidth should be higher), based on which we have

$$
C_0 > 2C_1 (Q_1 + Q_2).
$$

From Fig. 1, the union of the two sellers’ leasing periods can be divided into three epochs: In Epoch I, only seller 2 has spectrum to lease; in Epoch II, both sellers have spectrum to lease; and in Epoch III, only seller 1 has spectrum to lease. Denote the set of stages in Epoch I, II, and III as $N_1$, $N_{II}$, and $N_{III}$, respectively. Denote the set of stages in the leasing period of seller 1 and seller 2 as $N_1$ and $N_2$, respectively. Thus, we have $N_1 = N_{II} \cup N_{III}$ and $N_2 = N_1 \cup N_{III}$.

Seller $i$ ($i \in \{1, 2\}$) aims at maximizing its total revenue over all the stages by deciding on $d_{n,i}$, $n \in N_i$. Next, the spectrum leasing problem in each epoch is discussed.

A. Spectrum Leasing Problem in Epoch I

In Epoch I, only seller 2 has spectrum to lease, and it does not know when seller 1 will join the spectrum leasing market and how much spectrum bandwidth seller 1 will offer for spectrum leasing. In other words, in Epoch I, seller 2 does not know when Epoch II will start. So seller 2 assumes a monopoly market in Epoch I. At a stage in Epoch I, once an amount of spectrum is leased to secondary users, the spectrum will be used by secondary users until the last stage of seller 2’s leasing period.

For seller 2’s spectrum leasing, it has the following two constraints for $d_{n,2}$:

$$
0 \leq d_{n,2} \leq Q_2, \forall n \in N_2,
$$

$$
\sum_{n \in N_2} d_{n,2} \leq Q_2.
$$

Seller 2’s collected revenue at stage $n$ is $(C_0 - C_1 d_{n,2}) d_{n,2} (n - |N_{III}|)$, in which $\cdot | \cdot$ means $\sum$.

The rationale for this setting is as follows. It is possible that a secondary user may finish its transmission before the last stage of seller 2’s leasing period. The secondary user’s actual transmission duration depends on the user’s traffic load as well as its instantaneous channel quality during the transmission. However, when a secondary user decides to lease the spectrum, it is difficult to predict the instantaneous channel quality during future transmission over the spectrum, as the instantaneous channel quality may vary dynamically during future transmission. Thus, when a secondary user decides to lease the spectrum, it is unaware when its transmission will finish. Then, at the spectrum seller’s side, when some spectrum is leased to a secondary user, the seller assumes that the leased spectrum will be used by the secondary user until the end of the seller’s leasing period. If the secondary user finishes its transmission earlier, say at stage $k$, then the seller will take the leased spectrum back, update its spectrum stock, and re-run our proposed scheme at stage $k$.

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3Note that the method in this paper can be straightforwardly extended to deal with the case when the leasing period of seller 2 ends later than that of seller 1.

4When there are highly demanding secondary users, $C_0$ can be set higher and $C_1$ can be set lower. With less demanding secondary users, $C_0$ can be set lower and $C_1$ can be set higher, to attract more spectrum requests from secondary users.
cardinality of a set. To maximize its overall revenue, seller 2 should solve the following optimization problem:

\[
\begin{align*}
\text{Problem 1:} & \\
\max_{\{d_{n,2} | n \in N_2\}} & \sum_{n \in N_2} (C_0 - C_1 d_{n,2}) d_{n,2} \ (n - |N_{\text{III}}|) \\
\text{s.t.} & \sum_{n \in N_2} d_{n,2} \leq Q_2 \\
& d_{n,2} \geq 0, \forall n \in N_2.
\end{align*}
\]

Problem 1 is a convex optimization problem, because 1) its objective function is a concave function with respect to the vector of variables \(\{d_{n,2} | n \in N_2\}\), and 2) its feasible region is a convex set. Thus, the global optimal solution of Problem 1 can be achieved by existing numerical optimization methods.

### B. Spectrum leasing Problem in Epoch II

At Epoch II’s first stage (denoted as stage 1), seller 1 has available spectrum bandwidth \(Q_1\), while we denote the remaining spectrum bandwidth of seller 2 as \(Q_2\) (in other words, spectrum bandwidth with amount \((Q_2 - Q^II_2)\) has been leased out by seller 2 in Epoch I). At the beginning of stage 1, each seller does not know the presence of the other seller, and thus, assumes a monopoly spectrum leasing. So each seller reports to the coordinator the amount of spectrum bandwidth it would like to lease to secondary users in the stage. In specific, seller 1 has the following constraints for \(d_{n,1}\):

\[
0 \leq d_{n,1} \leq Q_1, \forall n \in N_1,
\]

\[
\sum_{n \in N_1} d_{n,1} \leq Q_1.
\]

So seller 1 solves the following convex optimization problem:

\[
\begin{align*}
\max_{\{d_{n,1} | n \in N_1\}} & \sum_{n \in N_1} (C_0 - C_1 d_{n,1}) d_{n,1} n \\
\text{s.t.} & \sum_{n \in N_1} d_{n,1} \leq Q_1 \\
& d_{n,1} \geq 0, \forall n \in N_1,
\end{align*}
\]

and reports to the coordinator the values of \(d_{l,1}\) (\(d_{l,1}\) is from the optimal solution of the above problem) and \(|N_1|\) (the leasing duration for the \(d_{l,1}\) spectrum bandwidth). On the other hand, seller 2 has the following constraints for \(d_{n,2}\):

\[
0 \leq d_{n,2} \leq Q^II_2, \forall n \in N_2 \backslash N_1,
\]

\[
\sum_{n \in N_2 \backslash N_1} d_{n,2} \leq Q^II_2.
\]

So seller 2 solves the following convex optimization problem:

\[
\begin{align*}
\max_{\{d_{n,2} | n \in N_2 \backslash N_1\}} & \sum_{n \in N_2 \backslash N_1} (C_0 - C_1 d_{n,2}) d_{n,2} (n - |N_{\text{III}}|) \\
\text{s.t.} & \sum_{n \in N_2 \backslash N_1} d_{n,2} \leq Q^II_2 \\
& d_{n,2} \geq 0, \forall n \in N_2 \backslash N_1,
\end{align*}
\]

and reports to the coordinator the values of \(d_{l,2}\) (\(d_{l,2}\) is from the optimal solution of the above problem) and \(|N_2| - |N_1|\)

\^[4] In Epoch 1, seller 2 does not know the value of \(N_{\text{II}}\). However, it knows the value of \((n - |N_{\text{III}}|)\) (the length from stage \(n\) until the end of seller 2’s leasing period). Thus, in Problem 1, we use notation \((n - |N_{\text{III}}|)\), for consistency of the formulated spectrum leasing problems in the three epochs.
when seller 2 would like to lease to secondary users spectrum bandwidth amount $z$ in Epoch II. The objective of seller 1 is to solve the following optimization problem

**Problem 2:**

$$
\max_{S_1} R_1 (S_1, S_2) \\
\text{s.t.} \quad S_1 \in F_1 (Q_1 - Q_1^{III}) ,
$$

and the objective of seller 2 is to solve the following optimization problem

**Problem 3:**

$$
\max_{S_2} R_2 (S_1, S_2) \\
\text{s.t.} \quad S_2 \in F_2 (Q_2^{II}) .
$$

For the non-cooperative game of the two sellers, a Nash equilibrium defines a strategy pair $(S_1, S_2)$ that a seller cannot earn more revenue by deviating from its strategy while keeping the other seller’s strategy unchanged. In other words, a Nash equilibrium should be a joint optimal solution of Problem 2 and Problem 3.

**C. Spectrum Leasing Problem in Epoch III**

In Epoch III, only seller 1 is active in the spectrum market, and thus, monopoly spectrum leasing is performed. Once an amount of spectrum bandwidth is leased to secondary users, the spectrum will be used by secondary users until the end of Epoch III.

Seller 1 has the following constraints for $d_{n,1}$ in its spectrum leasing in Epoch III:

$$
0 \leq d_{n,1} \leq Q_1^{III}, \forall n \in N_{III}, \\
\sum_{n \in N_{III}} d_{n,1} \leq Q_1^{III}.
$$

To maximize its revenue in Epoch III, seller 1 solves the following optimization problem.

**Problem 4:**

$$
V (Q_1^{III}) \triangleq \max_{(d_{n,1} | n \in N_{III})} \sum_{n \in N_{III}} (C_0 - C_1 d_{n,1}) d_{n,1} n \\
\text{s.t.} \quad \sum_{n \in N_{III}} d_{n,1} \leq Q_1^{III} \quad d_{n,1} \geq 0, \forall n \in N_{III}.
$$

Similar to Problem 1, Problem 4 is also a convex optimization problem, and thus, can be solved by existing numerical optimization methods.

**D. Overall Leasing Strategy of the Sellers**

In Epoch I, since seller 2 does not know when Epoch II will start, it assumes a monopoly market, and solves Problem 1 to get its leasing strategy in Epoch I.

At the beginning of Epoch II, some of seller 2’s spectrum has been leased to secondary users in preceding stages. In other words, seller 2 could not “go back” to revise its leasing strategy in Epoch I. Thus, seller 2’s leasing strategy decision for Epoch II is separate from that in Epoch I. In specific, in Epoch II, seller 2 follows a Nash equilibrium of the non-cooperative game.

For seller 1 at the beginning of Epoch II, it knows that both sellers will be present in Epoch II and it will be the only seller in Epoch III. Thus, seller 1 can use the amount of spectrum it reserves for Epoch III, i.e., $Q_1^{III}$, as an input for Epoch II’s non-cooperative game, and find the optimal value of $Q_1^{III}$ that maximizes its overall revenue. Thus, seller 1’s leasing strategies in Epoch II and III are jointly determined, as follows.

From the perspective of seller 1, it can adjust $Q_1^{III}$. For a specific $Q_1^{III}$, the two sellers need to follow a Nash equilibrium in the non-cooperative game in Epoch II. Thus, the strategy of seller 1 can be written as $Q_1^{III}$ and $S_1$, while the strategy of seller 2 can be written as $S_2$.

When seller 1 reserves spectrum bandwidth $Q_1^{III}$ for Epoch III, it means that seller 1 would like to lease spectrum bandwidth $(Q_1 - Q_1^{III})$ in Epoch II. Accordingly, we denote the revenue of seller 1 in Epoch II as $U (Q_1 - Q_1^{III})$, a function of $(Q_1 - Q_1^{III})$. Then for seller 1 to maximize its overall revenue, the following optimization problem should be solved

**Problem 5:**

$$
\max_{Q_1^{III}} U (Q_1 - Q_1^{III}) + V (Q_1^{III}) \\
\text{s.t.} \quad 0 \leq Q_1^{III} \leq Q_1
$$

in which $V (Q_1^{III})$ is defined in (8).

In the following, in Section IV we find out Nash equilibria in Epoch II for a specific $Q_1^{III}$, and in Section V we select the optimal value of $Q_1^{III}$ for seller 1.

**IV. NASH EQUILIBRIA IN THE NON-COOPERATIVE GAME IN EPOCH II WITH GIVEN $Q_1^{III}$**

**A. Existence and Uniqueness of Nash Equilibrium in the Non-Cooperative Game in Epoch II**

Recall that a Nash equilibrium of the non-cooperative game in Epoch II should be a joint optimal solution of Problem 2 and Problem 3. Since the objective functions of Problem 2 and Problem 3 are continuous and concave, and the feasible regions of the two sellers’ strategies are convex, closed, bounded, and uncoupled\(^7\), there exists at least one Nash equilibrium point for the non-cooperative game in Epoch II according to Theorem 1 of [19].

For uniqueness of Nash equilibrium, we have the following theorem.

**Theorem 1:** When $|N_{III}| \leq 12$, there is only one Nash equilibrium for the non-cooperative game in Epoch II.

**Proof:**

Define the vectorized strategy of seller 1 and seller 2 in Epoch II as $x_1 = [d_{1|N_{III}|+1}, d_{1|N_{III}|+1}^T, \ldots, d_{1|N_{III}|+1|N_{III}|+1}^T]$ and $x_2 = [d_{2|N_{III}|+1|N_{III}|+2}, d_{2|N_{III}|+1|N_{III}|+2}^T, \ldots, d_{2|N_{III}|+1|N_{III}|+2}^T]$, respectively, in which $[\cdot]^T$ means transpose operation. The payoff function of seller 1 and seller 2 can be rewritten as $R_1 (S_1, S_2) = R_1 (x_1, x_2)$ and $R_2 (S_1, S_2) = R_2 (x_1, x_2)$, respectively. Denote $x = (x_1, x_2)^T$ and define

$$
\sigma (x) = R_1 (x_1, x_2) + R_2 (x_1, x_2). \tag{10}
$$

\(^7\)When the two feasible regions are independent from each other, we say that the two feasible regions are uncoupled.
Then the pseudo-gradient of $\sigma(x)$ can be given as

$$k(x) = \begin{bmatrix} \nabla_1 R_1(x_1, x_2) \\ \nabla_2 R_2(x_1, x_2) \end{bmatrix} \quad (11)$$

where $|N_\| \times 1$ matrix $\nabla_1 R_1(x_1, x_2)$ is the gradient of $R_1(x_1, x_2)$ with respect to vector $x_1$, and $|N_\| \times 1$ matrix $\nabla_2 R_2(x_1, x_2)$ is the gradient of $R_2(x_1, x_2)$ with respect to vector $x_2$. According to Theorem 2 and Theorem 6 of [19], Nash equilibrium of the non-cooperative game in Epoch II is unique if the $2|N_\| \times 2|N_\|$ symmetric matrix $L(x) = \begin{bmatrix} K(x) + K^T(x) \end{bmatrix}$ is positive definite, where $K(x)$ is the Jacobian matrix of $k(x)$ with respect to $x$. After some math manipulation, the matrix $L(x)$ can be written as the following form

$$L(x) = \begin{bmatrix} L_{11}(x) & L_{12}(x) \\ L_{21}(x) & L_{22}(x) \end{bmatrix} \quad (12)$$

where $L_{11}(x) = \text{Diag}(4C_1(|N_\| + |N_\|), 4C_1(|N_\| + |N_\| - 1), ..., 4C_1(|N_\| - 1))$, $L_{12}(x) = L_{21}(x) = \text{Diag}(C_1(|N_\| + 2|N_\|), C_1(|N_\| - 2) \text{...}, C_1(|N_\| - 2))$, and $L_{22}(x) = \text{Diag}(4C_1(|N_\|), 4C_1(|N_\| - 1) \text{...}, 4C_1)$. Here $\text{Diag}(\cdots)$ means a diagonal matrix with all diagonal elements listed in $(\cdots)$. The matrix $L(x)$ can be guaranteed to be positive definite, if the leading principal minors are all positive [20], i.e., the determinant of $m \times m$ upper-left submatrix of $L(x)$ is larger than 0 for $m = 1, 2, \ldots, 2|N_\|$. Since there is

$$\det \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \det(A) \det(D - CA^{-1}B)$$

when matrix $A$ is invertible [21], the determinant of $m \times m$ upper-left submatrix of $L(x)$ is larger than 0 for $m = 1, 2, \ldots, 2|N_\|$ when the following inequalities hold

$$12(|N_\| - k)^2 + 12|N_\|(|N_\| - k) - |N_\| > 0, \quad \forall k = 0, 1, \ldots, (|N_\| - 1), \quad (13)$$

i.e., when

$$\frac{|N_\| - k}{|N_\|} > \left( -\frac{1}{2} + \frac{1}{\sqrt{3}} \right), \forall k = 0, 1, \ldots, (|N_\| - 1). \quad (14)$$

The inequalities in (14) hold if

$$|N_\| < \frac{1}{\frac{1}{2} + \frac{1}{\sqrt{3}}} = 12.9282. \quad (15)$$

This completes the proof. ■

From the proof, it can be seen that the derivation of value 12 largely depends on the fact that there are two sellers in the spectrum market.

As the number of stages in Epoch III is normally limited, it is very likely that the value of $|N_\|$ is bounded by 12, and thus, Nash equilibrium of the non-cooperative game in Epoch II is unique. Nevertheless, in next subsection, we show how to find Nash equilibria in the non-cooperative game in Epoch II without constraint $|N_\| \leq 12$ (i.e., Nash equilibrium may or may not be unique).

B. Finding Nash Equilibria in the Non-Cooperative Game in Epoch II

As aforementioned, a Nash equilibrium of the non-cooperative game in Epoch II is a joint optimal solution of Problem 2 and Problem 3. As both Problem 2 and Problem 3 are convex problems and satisfy the Slater’s condition, KKT condition is a sufficient and necessary condition for optimal solution for each problem [22]–[24].

For the ease of presentation, we denote $Q^n_{1}^\| = Q_1 - Q_1^\|$ as the spectrum bandwidth amount that seller 1 would like to lease to secondary users in Epoch II. For Problem 2, the KKT condition is

$$2C_1nd_{n,1} - (C_0 - C_1d_{n,2}) n + \lambda - \mu_n = 0, \forall n \in N_\| \quad (16a)$$

$$\lambda \left( \sum_{n \in N_\|} d_{n,1} - Q_1^\| \right) = 0 \quad (16b)$$

$$\mu_n d_{n,1} = 0, \forall n \in N_\| \quad (16c)$$

$$\sum_{n \in N_\|} d_{n,1} \leq Q_1^\| \quad (16d)$$

$$d_{n,1} \geq 0, \forall n \in N_\| \quad (16e)$$

$$\lambda \geq 0; \mu_n \geq 0, \forall n \in N_\| \quad (16f)$$

where $\lambda$ and $\mu_n$ are Lagrange multipliers associated with the constraints $\sum_{n \in N_\|} d_{n,1} \leq Q_1^\|$ and $d_{n,1} \geq 0$, respectively.

For Problem 3, the KKT condition is

$$2C_1 (n - |N_\|) d_{n,2} - (C_0 - C_1d_{n,1})(n - |N_\|) n + \zeta - \nu_n = 0, \forall n \in N_\| \quad (17a)$$

$$\zeta \left( \sum_{n \in N_\|} d_{n,2} - Q_2^\| \right) = 0 \quad (17b)$$

$$\nu_n d_{n,2} = 0, \forall n \in N_\| \quad (17c)$$

$$\sum_{n \in N_\|} d_{n,2} \leq Q_2^\| \quad (17d)$$

$$d_{n,2} \geq 0, \forall n \in N_\| \quad (17e)$$

$$\zeta \geq 0; \nu_n \geq 0, \forall n \in N_\| \quad (17f)$$

where $\zeta$ and $\nu_n$ are Lagrange multipliers associated with the constraints $\sum_{n \in N_\|} d_{n,2} \leq Q_2^\|$ and $d_{n,2} \geq 0$, respectively.

To get Nash equilibrium of the non-cooperative game in Epoch II, the equations (16) and (17) should be solved jointly.

We have two properties for the joint optimal solution:

- **Property 1**: Equality should hold in (16d) and (17d) (in other words, we have $\sum_{n \in N_\|} d_{n,1} = Q_1^\|$ and $\sum_{n \in N_\|} d_{n,2} = Q_2^\$).

- **Property 2**: If $d_{n,1} > 0 (n \in N_\|)$, then we have $\mu_n = 0$; if $d_{n,2} > 0$, then we have $\nu_n = 0$.

Property 1 is due to the facts that the objective function of Problem 2 is a monotonically increasing function of $d_{n,1} (n \in N_\|)$ and that the objective function of Problem 3 is a monotonically increasing function of $d_{n,2} (n \in N_\|)$. Property 2 can be obtained directly from the equalities (16c) and (17c).

Next, we try to find the expressions of $d_{n,1}$ and $d_{n,2}$ by solving (16) and (17).
From the equalities (16a) and (17a), $d_{n,1}$ and $d_{n,2}$ for $n \in \mathcal{N}_I$ can be expressed as
\[
d_{n,1} = \frac{(C_0 - C_1 d_{n,2}) n - \lambda + \mu_n}{2C_1 n},
\]
\[
d_{n,2} = \frac{(C_0 - C_1 d_{n,1}) (n - |N_{II}|) - \zeta + \nu_n}{2C_1 (n - |N_{II}|)},
\]
from which we have
\[
d_{n,1} = \frac{2(C_0 n - \lambda + \mu_n)}{3C_1 n} - \frac{C_0 (n - |N_{II}|)}{3C_1 (n - |N_{II}|)} - \zeta + \nu_n, \\
d_{n,2} = -\frac{C_0 n - \lambda + \mu_n}{3C_1 n} + \frac{2(C_0 (n - |N_{II}|) - \zeta + \nu_n)}{3C_1 (n - |N_{II}|)}.
\]

Define $Z_1 = \{n|d_{n,1} > 0, d_{n,2} > 0, n \in \mathcal{N}_I\}, Z_2 = \{n|d_{n,1} > 0, d_{n,2} = 0, n \in \mathcal{N}_I\}, Z_3 = \{n|d_{n,1} = 0, d_{n,2} > 0, n \in \mathcal{N}_I\}$ and $Z_4 = \{n|d_{n,1} = 0, d_{n,2} = 0, n \in \mathcal{N}_I\}$. Then $\{Z_1, Z_2, Z_3, Z_4\}$ constitutes a decomposition of the set $\mathcal{N}_I$, which means that $Z_1 \cup Z_2 \cup Z_3 \cup Z_4 = \mathcal{N}_I$ and $Z_i \cap Z_j = \emptyset$ ($\emptyset$ being a null set) for $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$. Totally there are $2^{|\mathcal{N}_I|}$ decompositions.

Next we find out the expressions of $d_{n,1}$ and $d_{n,2}$ for a specific decomposition $\{Z_1, Z_2, Z_3, Z_4\}$.

From Property 1, we have
\[
\sum_{n \in Z_1} d_{n,1} + \sum_{n \in Z_2} d_{n,1} = Q_1^{II}, \\
\sum_{n \in Z_1} d_{n,2} + \sum_{n \in Z_2} d_{n,2} = Q_2^{II}.
\]

In the two equations, substituting the expressions of $d_{n,1}$ and $d_{n,2}$ in (20) and (21) for $n \in Z_1$, substituting the expressions of $d_{n,1}$ and $d_{n,2}$ in (18) and (19) for $n \in Z_2$ and $n \in Z_3$, and using Property 2, we have the following equations:
\[
-A_{11} \lambda + A_{12} \zeta = Q_1^{II} - \sum_{n \in Z_1} \frac{C_0}{3C_1} - \sum_{n \in Z_3} \frac{C_0}{3C_1}, \\
A_{21} \lambda - A_{22} \zeta = Q_2^{II} - \sum_{n \in Z_1} \frac{C_0}{3C_1} - \sum_{n \in Z_3} \frac{C_0}{3C_1},
\]
where
\[
A_{11} = \sum_{n \in Z_1} \frac{2}{3C_1 n} + \sum_{n \in Z_2} \frac{1}{2C_1 n}, \\
A_{12} = \sum_{n \in Z_1} \frac{1}{3C_1 (n - |N_{II}|)}, \\
A_{21} = \sum_{n \in Z_1} \frac{1}{3C_1 n}, \\
A_{22} = \sum_{n \in Z_1} \frac{2}{3C_1 (n - |N_{II}|)} + \sum_{n \in Z_3} \frac{1}{2C_1 (n - |N_{II}|)}.
\]

Note that $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ are all nonnegative. According to the equations in (22), the Lagrange multipliers $\lambda$ and $\zeta$ can be expressed as
\[
\lambda = -\frac{A_{22}}{A_{11} A_{22} - A_{21} A_{12}} \left( Q_1^{II} - \sum_{n \in Z_1} \frac{C_0}{3C_1} - \sum_{n \in Z_3} \frac{C_0}{3C_1} \right), \\
\zeta = -\frac{A_{11} A_{22} - A_{21} A_{12}}{A_{11} A_{22} - A_{21} A_{12}} \left( (Q_1^{II} - \sum_{n \in Z_1} \frac{C_0}{3C_1} - \sum_{n \in Z_3} \frac{C_0}{3C_1}) - \sum_{n \in Z_2} \frac{C_0}{3C_1} \right).
\]
This completes the proof.

Remark: Theorem 2 shows that in a feasible decomposition, if \( Z_4 \) is not empty, then it contains consecutive stages until the end of Epoch II. Therefore, in the exhaustive search of all possible decompositions, we can skip those decompositions in which \( Z_4 \) contains non-consecutive stages or does not last until the end of Epoch II. Thus, the number of decompositions that we should check reduces from \( 2^{3|N_{III}|} \) to \( \sum_{i=0}^{3} 3^i \). The term \( \sum_{i=0}^{3} 3^i \) is calculated as follows. We only need to check decompositions in which \( Z_4 \) contains consecutive stages until the end of Epoch II. When \( Z_4 \) contains the last \( i \) \((i \in \{0,1,2,\ldots,|N_{III}|\})\) stages in Epoch II, there are \( 3^{|N_{III}|−i} \) decompositions since each of the first \((|N_{III}|−i)\) stages in Epoch II can be in \( Z_1, Z_2, \) or \( Z_3 \). Thus, the number of decompositions that we should check is \( \sum_{i=0}^{3} 3^{|N_{III}|−i} \), which is mathematically equal to \( \sum_{i=0}^{3} 3^i \).

So far all Nash equilibria of the non-cooperative game in Epoch II have been found. If there exists only one unique Nash equilibrium (e.g., when \( |N_{III}| \leq 12 \)), then both sellers follow the unique Nash equilibrium. If there are two or more Nash equilibria, the two sellers need to select one Nash equilibrium to follow. Here the two sellers agree to follow the max-min Nash equilibrium, defined as the Nash equilibrium that maximizes the minimum unit-bandwidth revenue of the two sellers. Here for seller 1, its unit-bandwidth revenue is the ratio of its total revenue in Epoch II to \( Q_{1}^{II} \); for seller 2, its unit-bandwidth revenue is the ratio of its total revenue in Epoch II to \( Q_{2}^{II} \).

V. TOTAL REVENUE MAXIMIZATION FOR SELLER 1

In the previous section, we have found the strategies of the two sellers in Epoch II with a specific \( Q_{1}^{III} \) (the bandwidth that seller 1 reserves for Epoch III). Now, we try to solve Problem 5, i.e., find out the optimal value of \( Q_{1}^{III} \) that maximizes seller 1’s total revenue. A method by exhaustive search could be: 1) for each possible value of \( Q_{1}^{III} \), search all possible Nash equilibria, find the max-min Nash equilibrium, and calculate the revenue that seller 1 can earn during its leasing period with the max-min Nash equilibrium; 2) for different \( Q_{1}^{III} \), compare the revenue values that seller 1 can earn during its leasing period, and select the optimal \( Q_{1}^{III} \) that makes seller 1 earn the most revenue. However, the complexity of the exhaustive search method is huge, due to the infinite number of values of \( Q_{1}^{III} \in [0, Q_{1}] \). Thus, we target at an approximation method to select \( Q_{1}^{III} \).

When \( Q_{1}^{III} = x \), \( U(Q_{1} − x) \) and \( V(x) \) given in (8) are the revenue of seller 1 in Epoch II and Epoch III, respectively. To select \( x \) (i.e., \( Q_{1}^{III} \)), we need to evaluate how \( V(x) \) and \( U(Q_{1} − x) \) change when \( x \) varies.

Lemma 1: The function \( V(x) \) is an increasing and concave function with \( x \).

Proof: The proof follows a similar procedure to the proof of Lemma 6 of [25].

Now we evaluate function \( U(Q_{1} − x) \) when \( x \) varies. To evaluate \( U(Q_{1} − x) \) for a specific decomposition \( \{Z_1, Z_2, Z_3, Z_4\} \), we need to know \( d_{n,1} \) and \( d_{n,2} \) \((n \in N_{III})\) in the Nash equilibrium corresponding to the decomposition.

Therefore, next we show how \( d_{n,1} \) and \( d_{n,2} \) change when \( x \) varies.

Consider a decomposition \( \{Z_1, Z_2, Z_3, Z_4\} \). Consider two \( Q_{1}^{III} \) values (recalling that \( Q_{1}^{III} = Q_{1} − Q_{1}^{II} \)): \( Q_{1} \) and \( Q_{1}^{II} \), with \( Q_{1}^{II} \leq Q_{1} \). Assume the decomposition \( \{Z_1, Z_2, Z_3, Z_4\} \) is feasible for both \( Q_{1}^{III} \) values. For the decomposition, denote the corresponding Nash equilibrium when \( Q_{1}^{III} = Q_{1}^{II} \) as

\[
(S_{1}, S_{2}) \triangleq \left( \{d_{n,1} | n \in N_{III}\}, \{d_{n,2} | n \in N_{III}\} \right),
\]

and the corresponding Nash equilibrium when \( Q_{1}^{III} = Q_{1} \) as

\[
(S_{1}′, S_{2}′) \triangleq \left( \{d_{n,1} | n \in N_{III}\}, \{d_{n,2} | n \in N_{III}\} \right).
\]

Then the following lemmas can be expected.

Lemma 2: For seller 1, \( d_{n,1}^{III} \leq d_{n,1}^{II} \) for \( n \in Z_2 \), and \( d_{n,1}^{III} = d_{n,1}^{II} = 0 \) for \( n \in Z_3 \cup Z_4 \).

Proof: By the definitions of set \( Z_3 \) and \( Z_4 \), seller 1 does not offer spectrum bandwidth to be leased in stages in \( Z_3 \) and \( Z_4 \), and thus, \( d_{n,1}^{III} = d_{n,1}^{II} = 0 \) for \( n \in Z_3 \cup Z_4 \).

From (23), (25), and the fact that \( Z_1 \cup Z_2 \neq \emptyset \), we have \( A_{11} > A_{21} \geq 0 \). From (24), (26), and the fact that \( Z_1 \cup Z_3 \neq \emptyset \), we have \( A_{22} > A_{12} \geq 0 \). Thus, we have \( A_{11}A_{22} > A_{21}A_{12} > 0 \).

For \( n \in Z_2 \), with the aid of (29) and (27), we have

\[
d_{n,1}^{III} − d_{n,1}^{II} \leq \frac{C_{n,1}A_{22}}{2C_{n,1}A_{12} + A_{22} − A_{21}} (Q_{1} − Q_{1}^{II}) \tag{31}
\]

in which the inequality comes from the fact that \( A_{22} \geq 0 \), \( Q_{1}^{II} \leq Q_{1} \), and \( (A_{11}A_{22} − A_{21}A_{12}) > 0 \).

This completes the proof.

Lemma 3: For seller 2, \( d_{n,2}^{III} \leq d_{n,2}^{II} \) for \( n \in Z_3 \), and \( d_{n,2}^{III} = d_{n,2}^{II} = 0 \) for \( n \in Z_2 \cup Z_4 \).

Proof: The proof is similar to the proof for Lemma 2, and thus, is omitted here.

Theorem 3: If a decomposition \( \{Z_1, Z_2, Z_3, Z_4\} \) is feasible when \( Q_{1}^{III} = x \in \mathcal{I} \) where \( \mathcal{I} \subseteq [0, Q_{1}] \) is an interval, then when the Nash equilibrium corresponding to the decomposition is followed by the two sellers in Epoch II, seller 1’s revenue \( U(Q_{1} − x) \) in Epoch II can be written as \( U(Q_{1} − x) = G(x) − H(x) \) where \( G(x) \) and \( H(x) \) are monotonically increasing functions with respect to \( x \in \mathcal{I} \).

Proof: Suppose the Nash equilibrium corresponding to the decomposition \( \{Z_1, Z_2, Z_3, Z_4\} \) is \( \{d_{n,1} | n \in N_{III}\} \),
\{d_{n_2}|n \in \mathbb{N}_1\}). Then \(U(Q_1 - x)\) can be written as
\[
U(Q_1 - x) = \sum_{n \in \mathbb{N}_1} (C_0 - C_1 (d_{n_1} + d_{n_2})) d_{n_1} n
\]
\((a)\) obtained by substituting \(d_{n_1}\) and \(d_{n_2}\) according to (29) and (30).

As the decomposition \(\{Z_1, Z_2, Z_3, Z_4\}\) is feasible, \(\lambda\) and \(\zeta\) are non-negative. Additionally, from (27) and (28), it can be seen that \(\lambda\) and \(\zeta\) are monotonically decreasing with \(Q_1^{\text{II}},\) i.e., \((Q_1 - x)\). So in the expression (32), both the term
\[
\sum_{n \in Z_1} \left( \frac{\zeta^2}{9C_1 (n - |\mathbb{N}_1|)^2} + \frac{2C_0 \zeta}{9C_1 (n - |\mathbb{N}_1|)} + \frac{C_0^2}{9C_1} \right) n
\]
and the term
\[
\sum_{n \in Z_1} \left( \frac{\zeta \lambda}{9C_1 (n - |\mathbb{N}_1|)} + \frac{2\lambda^2}{9C_1 n^2} + \frac{C_0 \lambda}{9C_1 n} \right) n
\]
are monotonically decreasing with \(Q_1^{\text{II}},\) and thus, are monotonically increasing with \(x\) (as \(Q_1^{\text{II}} = Q_1 - x\)). It can be also checked that the term \(\sum_{n \in Z_2} (C_0 - C_1 d_{n_1}) d_{n_1} n\) in (32) is a monotonically increasing function with respect to \(Q_1^{\text{II}}\) (since the function \((C_0 - C_1 y) y\) is monotonically increasing with \(y\) and \(d_{n_1}\) grows with \(Q_1^{\text{II}}\) from Lemma 2), and thus, is a monotonically decreasing function with respect to \(x\).

Define
\[
G(x) = \sum_{n \in Z_1} \left( \frac{\zeta^2}{9C_1 (n - |\mathbb{N}_1|)^2} + \frac{2C_0 \zeta}{9C_1 (n - |\mathbb{N}_1|)} + \frac{C_0^2}{9C_1} \right) n
\]
and
\[
H(x) = \sum_{n \in Z_1} \left( \frac{\zeta \lambda}{9C_1 n (n - |\mathbb{N}_1|)} + \frac{2\lambda^2}{9C_1 n^2} + \frac{C_0 \lambda}{9C_1 n} \right) n
\]
\[\text{ and } \sum_{n \in Z_2} (C_0 - C_1 d_{n_1}) d_{n_1} n. \quad (34)\]

It can be seen that \(U(Q_1 - x) = G(x) - H(x),\) and both \(G(x)\) and \(H(x)\) monotonically increase with \(x.\)

This completes the proof.

Remark: Theorem 4 shows that if a decomposition \(\{Z_1, Z_2, Z_3, Z_4\}\) is feasible for \(x = x_0,\) then there exists an interval of \(x\) containing \(x_0\) such that the decomposition is always feasible inside the interval, and infeasible outside the interval.

Based on Lemma 1, Theorem 3, and Theorem 4, we propose that seller 1 uses the following Algorithm 1 to select \(x\) (i.e., \(Q_1^{\text{III}}\)).
Algorithm 1 Searching procedure for \( x \) (i.e., \( Q_{1}^{III} \)).

1. Set \( x^* = 0 \), and \( R^* = 0 \).
2. Set \( x^1 = 0 \).
3. For \( x = x^1 \), find all feasible Nash equilibria, and pick up the max-min Nash equilibrium and corresponding decomposition \( \{ Z_1, Z_2, Z_3, Z_4 \} \).
4. Find (using bisection search) a point denoted \( x_1 \) such that the decomposition picked in Step 3 is feasible for \( x \in [x^1, x_1] \), and infeasible for \( x \in (x_1, Q_1^I] \).
5. Set \( x^2 = x_1 \).
6. For complexity reduction, approximately seller 1 considers that the decomposition picked in Step 3 always leads to the max-min Nash equilibrium of the non-cooperative game with any \( x \in [x^1, x^2] \). In other words, for any \( x \in [x^1, x^2] \), both sellers always follow the Nash equilibrium corresponding to the decomposition picked in Step 3. Then the revenue of seller 1 can be written as \( U(Q_1 - x) + V(x) \). Here \( U(Q_1 - x) \) is the difference of two monotonically increasing functions of \( x \) (from Theorem 3), while \( V(x) \) is an increasing function of \( x \) (from Lemma 1). Thus, \( U(Q_1 - x) + V(x) \) can be viewed as the difference of two monotonically increasing functions of \( x \in [x^1, x^2] \). To maximize the difference of two monotonically increasing functions, a polyblock method can be used to find the global optimal solution (please refer to [26]-[28] for details). Denote the optimal point is \( \hat{x} \) and the corresponding revenue \( U(Q_1 - \hat{x}) + V(\hat{x}) \) of seller 1 as \( R \).
7. If \( R > R^* \), then set \( x^* = \hat{x} \) and \( R^* = R \).
8. If \( x^1 = Q_1^I \), then terminate the algorithm, and output \( x^* \).
9. Set \( x^{i+1} = x^i \), and proceed to Step 3.

In the algorithm, \( x^* \) denotes the optimal selection of seller 1 for \( x \), and \( R^* \) denotes the corresponding overall revenue of seller 1. For \( x = x^1 = 0 \), in Step 3 we first select the max-min Nash equilibrium and corresponding decomposition \( \{ Z_1, Z_2, Z_3, Z_4 \} \). In Steps 4 and 5, we find the interval of \( x \), denoted \( [x^1, x^2] \), such that the previously picked decomposition is feasible inside the interval and infeasible when \( x > x^2 \).

In Step 6, we approximately consider that for \( x \in [x^1, x^2] \), both sellers always follow the Nash equilibrium corresponding to the decomposition picked in Step 3. Then for \( x \in [x^1, x^2] \), seller 1’s revenue \( U(Q_1 - x) + V(x) \) can be shown as the difference of two monotonically increasing functions of \( x \). A polyblock algorithm can be used to find the global optimal value of \( x \in [x^1, x^2] \), denoted \( \hat{x} \), such that the overall revenue of seller 1 is maximized. Then the \( \hat{x} \) is a candidate for seller 1’s selection of \( x \) (Step 7). Since interval \( [x^1, x^2] \) has been dealt with in Step 6, we proceed to the next interval starting from \( x^2 \) in Step 9, to repeat the procedure and find other candidates for seller 1’s selection of \( x \). Among all the candidates, the one that has the maximal overall revenue of seller 1 is eventually selected by seller 1.

Overall, the strategies of the two sellers are as follows. In Epoch I, seller 2 derives its optimal strategy by solving Problem 1. At the beginning of Epoch II, seller 1 uses Algorithm 1 to find the value of \( x \), denoted \( x^* \). Then in the non-cooperative game in Epoch II with \( Q_{1}^{III} = x^* \), both sellers follow the max-min Nash equilibrium. In Epoch III, seller 1 can derive its optimal strategy by solving Problem 4 with \( Q_{1}^{III} = x^* \).
TABLE I
THE NUMBER OF DECOMPOSITIONS WITHOUT AND WITH THE AID OF THEOREM 2.

| $|N_n|$ | 2 | 4 | 6 | 8 | 10 | 15 | 20 |
|-------|---|---|---|---|----|----|----|
| Total number of decompositions | 16 | 256 | 4096 | $6.6 \times 10^9$ | $1.0 \times 10^{10}$ | $1.1 \times 10^{10}$ | $1.1 \times 10^{10}$ |
| Checked decompositions (with Theorem 2) | 13 | 121 | 1093 | 9841 | $8.9 \times 10^{4}$ | $2.2 \times 10^{4}$ | $5.2 \times 10^{4}$ |

TABLE II
THE DECOMPOSITIONS USED WHEN VERIFYING THEOREM 3

<table>
<thead>
<tr>
<th>1st decomposition</th>
<th>$Z_1$</th>
<th>$Z_2$</th>
<th>$Z_3$</th>
<th>$Z_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd decomposition</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

3) Verification of Lemma 2 and Lemma 3: In this subsection, Lemma 2 and Lemma 3 are verified. Consider $N_{i1} = \{4, 5, 6, 7, 8\}$. Fig. 3 plots $d_{n,1}$ and $d_{n,2}$ versus the stage index $n$ for a feasible decomposition in which $Z_1 = \{7, 8\}$, $Z_2 = \{6\}$, $Z_3 = \emptyset$, and $Z_4 = \{4, 5\}$ when $Q_{11}^* = 70$ and 80. Fig. 4 plots $d_{n,1}$ and $d_{n,2}$ versus the stage index for a feasible decomposition in which $Z_1 = \{8\}$, $Z_2 = \emptyset$, $Z_3 = \{7\}$, and $Z_4 = \{4, 5, 6\}$ when $Q_{11}^* = 2, 5,$ and 8. From Fig. 3 and Fig. 4, it can be seen that, when $Q_{11}^*$ changes, $d_{n,1}$ and $d_{n,2}$ vary in the same way as Lemma 2 and Lemma 3 describe.

4) Verification of Theorem 3: In this subsection, the characteristic of $U(Q_1 - x)$ described in Theorem 3 is verified. Still consider $N_{i1} = \{4, 5, 6, 7, 8\}$. Two decompositions are investigated, which are listed in Table II. Consider two intervals of $x$: $[0, 30]$ and $[40, 70]$, in which the two decompositions are feasible, respectively. Fig. 5 and Fig. 6 plot the function $U(Q_1 - x)$ as well as $G(x)$ and $H(x)$ (from Theorem 3) for the two decompositions over the two corresponding intervals, respectively. It can be seen that both the functions $G(x)$ and $H(x)$ are monotonically increasing for each decomposition in the corresponding interval of $x$, which is consistent with Theorem 3.

B. Performance of Algorithm 1

1) Computation complexity of Algorithm 1: Computation complexity of Algorithm 1 largely depends on the number of iterations (i.e., how many times are Steps 3–7 repeated?). Table III shows the iteration number for different settings of $|N_{i1}|$ and $|Q_{11}^*|$. It can be seen that in all the cases, the number of iterations is 2 or 3, which shows that Algorithm 1 is efficient in reducing computation complexity.

2) Suboptimality of Algorithm 1: In Algorithm 1, to search optimal $x$ (i.e., $Q_{11}^*$), an approximation is used to reduce the computation complexity, as follows. For the non-cooperative game with $x_1$, in Step 3 of Algorithm 1, we first find the max-min Nash equilibrium and the corresponding decomposition $\{Z_1, Z_2, Z_3, Z_4\}$. We then find an interval of $x$, denoted $[x^1, x^2]$, such that the decomposition picked in Step 3 is a feasible decomposition of the non-cooperative game with any $x \in [x^1, x^2]$. Then we approximately consider that the decomposition, which leads to the max-min Nash equilibrium...
of the non-cooperative game with $x^\dagger$, also leads to the max-min Nash equilibrium of the non-cooperative game with any $x \in [x^l, x^r]$. In other words, for any $x \in [x^l, x^r]$, both sellers always follow the Nash equilibrium corresponding to the decomposition picked in Step 3. Due to this approximation, Algorithm 1 in general is suboptimal in finding the optimal $x$.

From Theorem 1, if $|N_{III}| \leq 12$, then the non-cooperative game in Epoch II always has a unique Nash equilibrium, which means that the decomposition picked in Step 3 of Algorithm 1 is the only feasible decomposition for any $x \in [x^l, x^r]$. In other words, the decomposition indeed always leads to the max-min Nash equilibrium of the non-cooperative game with any $x \in [x^l, x^r]$. Thus, Algorithm 1 is optimal if $|N_{III}| \leq 12$.

We have tried a large number of scenarios with $|N_{III}| \in \{13, 14, ..., 30\}$, and have found that the associated non-cooperative games in Epoch II all have unique Nash equilibrium, which means our Algorithm 1 is optimal in those considered scenarios. However, we find that it is hard to prove theoretically that Nash equilibrium of the non-cooperative game in Epoch II is unique or not unique when $|N_{III}| \geq 13$. This is an interesting problem, and we leave it for future investigation.

### C. Comparison with a Cooperative Scheme

Now we compare with a cooperative scheme. The difference of the cooperative scheme from our proposed scheme is as follows. When the two sellers know the existence of each other (i.e., at the beginning of Epoch II), the two sellers cooperate to jointly maximize the total revenue of them over Epoch II and III, by solving the following optimization problem.

$$
\max_{\{d_{n,1} \mid n \in N_I\}, \{d_{n,2} \mid n \in N_{II}\}} \sum_{n \in N_0} (C_0 - C_1(d_{n,1} + d_{n,2})) d_{n,1}n \\
+ \sum_{n \in N_{III}} (C_0 - C_1d_{n,1}) d_{n,1}n \\
+ \sum_{n \in N_{III}} (C_0 - C_1(d_{n,1} + d_{n,2})) d_{n,2}(n - |N_{III}|)
$$

s.t. $d_{n,1} \leq Q_1$

$$
\sum_{n \in N_{III}} d_{n,2} \leq Q_2^n
$$

$$
d_{n,1} \geq 0, \forall n \in N_{III} \cup N_{III}
$$

$$
d_{n,2} \geq 0, \forall n \in N_{II}.
$$

(35)

For performance comparison, the simulation is set up as follows. Since the cooperative scheme and our proposed scheme perform the same in Epoch I, we set $N_I = \emptyset$. And $N_{III} = \{6, 5, 4, 3\}$, $N_{III} = \{2, 1\}$. We fix the sum of $Q_1$ and $Q_2$ to be 200, and consider three configurations of $(Q_1, Q_2)$: $(50, 150)$, $(100, 100)$, and $(150, 50)$. Fig. 7 shows the achieved

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<table>
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<th>Table III: Number of Iterations in Algorithm 1</th>
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<tr>
<td>2</td>
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respectively. Here \( \lambda \) denotes two sellers, we have a number of zero-seller, one-seller, and two-seller epochs.

When the state of the seller changes from OFF to ON, we call it an ON state begins, the amount of available spectrum to lease at the new state is \( \mu \). The time duration of one stage is set as 1 day. We vary \( \mu_1 \) and \( \mu_2 \) exponentially distributed with mean value being \( 1/\mu \). For seller \( i \in \{1, 2\} \), the ON duration and OFF duration are exponential distributed with mean value being \( 1/\lambda_i \), respectively. Here \( \lambda_i \) and \( \mu_i \) are called the arrival and departure rate, respectively. The time duration of one stage is set as 1 day. We vary \( \mu_1 \), \( \mu_2 \), \( \lambda_1 \), and \( \lambda_2 \). For seller 1 and seller 2, when an ON state begins, the amount of available spectrum to lease is \( Q_1 = 100 \) and \( Q_2 = 60 \), respectively.

Following the random arrival and departure processes of the two sellers, we have a number of zero-seller, one-seller, and two-seller epochs.

- A zero-seller epoch means a number of consecutive stages in which neither seller has spectrum to lease.
- A one-seller epoch means a number of consecutive stages in which only one seller has spectrum to lease. The seller can get its leasing strategy by solving an optimization problem similar to Problem 1 in Section III-A.
- A two-seller epoch means a number of consecutive stages in which both sellers have spectrum to lease. If a two-seller epoch is followed by a one-seller epoch, the leasing strategy in the two-seller epoch can be obtained similar to our treatment for our Epoch II in Section IV with \( Q_1^\text{III} = 0 \).

Without loss of generality, the average revenue of seller 1 per leased stage (i.e., the total revenue of seller 1 divided by the number of total stages in which seller 1 has spectrum to lease) is plotted in Fig. 8, for varying \( \mu_1 \), \( \mu_2 \), \( \lambda_1 \), and \( \lambda_2 \). We have the following observations.

- When \( \mu_1 \) increases, seller 1’s average revenue decreases. This is because for a higher \( \mu_1 \), seller 1’s leasing period has less duration. A shorter leasing period reduces the flexibility in seller 1’s spectrum leasing, thus reducing seller 1’s average revenue.
- When \( \mu_2 \) increases, seller 1’s average revenue increases. This is because higher \( \mu_2 \) leads to shorter leasing period of seller 2, which means less competition of seller 2 to seller 1. Thus, seller 1’s average revenue increases.
- When \( \lambda_1 \) increases, seller 1’s average revenue almost keeps the same. The reason is as follows. When \( \lambda_1 \) increases, seller 1’s average OFF state duration (expressed as \( 1/\lambda_1 \)) decreases. However, when seller 1 is ON, its chance to overlap with seller 2’s leasing period is not affected by seller 1’s average OFF duration. In other words, seller 1’s average OFF duration length does not affect the competition that seller 1 receives from seller 2 when seller 1 is ON. Thus, seller 1’s average revenue almost keeps the same.
- When \( \lambda_2 \) increases, seller 1’s average revenue decreases. The reason is as follows. When \( \lambda_2 \) increases, seller 2’s average OFF state duration (expressed as \( 1/\lambda_2 \)) decreases. Thus, when seller 1 is ON, its chance to overlap with seller 2’s leasing period is higher. In other words, seller 1 receives more competition from seller 2, and thus, its average revenue decreases.

D. Performance with Random Leasing Periods

We consider that each seller has alternating ON and OFF states. Here an ON and OFF state mean that the seller has and does not have spectrum to lease, respectively. When the state of the seller changes from OFF to ON, we call it an arrival, and when the state of the seller changes from ON to OFF, we call it a departure. Thus, for the seller, the duration from an arrival to the following departure is its leasing period. For seller \( i \in \{1, 2\} \), the ON duration and OFF duration are exponential distributed with mean value being \( 1/\mu_i \) and \( 1/\lambda_i \), respectively. Here \( \lambda_i \) and \( \mu_i \) are called the arrival and departure rate, respectively. The time duration of one stage is set as 1 day. We vary \( \mu_1 \), \( \mu_2 \), \( \lambda_1 \), and \( \lambda_2 \). For seller 1 and seller 2, when an ON state begins, the amount of available spectrum to lease is \( Q_1 = 100 \) and \( Q_2 = 60 \), respectively.

Following the random arrival and departure processes of the two sellers, we have a number of zero-seller, one-seller, and two-seller epochs.

- A zero-seller epoch means a number of consecutive stages in which neither seller has spectrum to lease.
- A one-seller epoch means a number of consecutive stages in which only one seller has spectrum to lease. The seller can get its leasing strategy by solving an optimization problem similar to Problem 1 in Section III-A.
- A two-seller epoch means a number of consecutive stages in which both sellers have spectrum to lease. If a two-seller epoch is followed by a one-seller epoch, the leasing strategy in the two-seller epoch can be obtained similar to our treatment for our Epoch II in Section IV with \( Q_1^\text{III} = 0 \).

Without loss of generality, the average revenue of seller 1 per leased stage (i.e., the total revenue of seller 1 divided by the number of total stages in which seller 1 has spectrum to lease) is plotted in Fig. 8, for varying \( \mu_1 \), \( \mu_2 \), \( \lambda_1 \), and \( \lambda_2 \). We have the following observations.

- When \( \mu_1 \) increases, seller 1’s average revenue decreases. This is because for a higher \( \mu_1 \), seller 1’s leasing period has less duration. A shorter leasing period reduces the flexibility in seller 1’s spectrum leasing, thus reducing seller 1’s average revenue.
- When \( \mu_2 \) increases, seller 1’s average revenue increases. This is because higher \( \mu_2 \) leads to shorter leasing period of seller 2, which means less competition of seller 2 to seller 1. Thus, seller 1’s average revenue increases.
- When \( \lambda_1 \) increases, seller 1’s average revenue almost keeps the same. The reason is as follows. When \( \lambda_1 \) increases, seller 1’s average OFF state duration (expressed as \( 1/\lambda_1 \)) decreases. However, when seller 1 is ON, its chance to overlap with seller 2’s leasing period is not affected by seller 1’s average OFF duration. In other words, seller 1’s average OFF duration length does not affect the competition that seller 1 receives from seller 2 when seller 1 is ON. Thus, seller 1’s average revenue almost keeps the same.
- When \( \lambda_2 \) increases, seller 1’s average revenue decreases. The reason is as follows. When \( \lambda_2 \) increases, seller 2’s average OFF state duration (expressed as \( 1/\lambda_2 \)) decreases. Thus, when seller 1 is ON, its chance to overlap with seller 2’s leasing period is higher. In other words, seller 1 receives more competition from seller 2, and thus, its average revenue decreases.

VII. CONCLUSIONS

In this paper, we investigate spectrum leasing with two sellers, in which seller 1 leases spectrum in Epoch II and Epoch III, and seller 2 leases spectrum in Epoch I and Epoch II, as shown in Fig. 1. In Epoch I, only seller 2 has spectrum to lease, and its strategy is derived by solving a convex problem. In Epoch II, since the two sellers both have spectrum to lease, competition between the two sellers exists. Thus, the spectrum leasing in Epoch II is formulated as a non-cooperative game. Nash equilibria of the game are derived in closed form by jointly solving two optimization problems. By analyzing the choice of seller 1 for Epoch III, seller 1’s strategy in Epochs II and III and seller 2’s strategy in Epoch II are developed.

In this paper, we consider a duopoly market. When there are three or more sellers, similarly the union of their leasing periods can be divided into a number of epochs. For an epoch with one seller, the seller can derive its leasing strategy by solving a convex optimization problem similar to Problem 1 in this paper. For an epoch with two or more sellers, a game model can be set up, and all sellers follow the Nash equilibrium that maximizes the minimum unit-bandwidth revenue of the sellers. If a seller’s leasing period also continues
Fig. 8. Average revenue of seller 1.

into subsequent epochs, the seller’s leasing strategies over the multiple epochs can be derived jointly. In other words, similar to the method in Section V in this paper, the seller can use the amount of the spectrum reserved for later epochs as an input to the game model, and find the optimal amount of the reserved spectrum that maximizes its overall revenue.

REFERENCES


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