Dynamic Pricing over Multiple Rounds of Spectrum Leasing in Cognitive Radio

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***Abstract*—In this paper, the problem of dynamic pricing over multiple rounds of spectrum leasing is investigated. One primary network is considered, which is the spectrum seller and would like to lease its unused channels to secondary users. To accommodate different arrival instants of secondary users' spectrum requests, spectrum leasing is performed in multiple rounds (stages), and in each stage, a separate spectrum price is set. First we consider the case that, for each specific price value, the spectrum demand (the number of channels requested by secondary users) is a random variable. An optimization problem is formulated to set up the spectrum prices in the multiple stages, with the purpose of maximizing the total revenue of the primary network. The solving method of the formulated optimization problem is presented. Additionally, some interesting properties of the optimal solution are also presented, such as monotonicity and convexity of the maximal total revenue with respect to stage index, and lower/upper bounds of the maximal total revenue. Further, we consider the case that, for a specific price value, the spectrum demand is non-random, and can be solely determined by the price. An incremental algorithm is given to find out the optimal price values at the stages. We also demonstrate the monotonicity of the optimal price value with respect to the stage index. Numerical results are provided to verify the research findings and compare with existing work.**

***Index Terms*—Cognitive radio, spectrum leasing, dynamic pricing.**

I. Introduction

We are currently experiencing spectrum shortage since almost all wireless spectrum has been allocated to existing wireless applications for the exclusive use of the licensed users. On the other hand, it has been shown by measurements of the wireless spectrum usage [1], [2] that licensed spectrum is actually not fully utilized by licensed users (referred to as *primary users*) for a large portion of time. Accordingly, cognitive radio has the potential to solve the above two problems, by allowing idle spectrum to be accessed by unlicensed users (referred to as *secondary users*) [3]. Secondary users pay the primary network a certain amount of payment for spectrum usage, referred to as *spectrum leasing*. In specific, for a primary network, if its primary users do not use the spectrum for a while, the primary network will announce its spectrum price, and secondary users can decide whether or not to lease the spectrum.

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In spectrum leasing, the spectrum price is the most important design parameter [4], which directly affects the primary network's revenue as well as the willingness of secondary users to lease the spectrum. Spectrum leasing has been investigated in the literature, with two major settings: monopoly spectrum leasing and oligopoly spectrum leasing.

In monopoly spectrum leasing, there is one single primary network (or broker), targeting at revenue maximization of the primary network (or broker) [5]–[7]. A broker is considered in [5], which first decides on the spectrum amount that will be purchased from primary networks, and then sets spectrum leasing price for secondary users to purchase. The research problem, i.e., to maximize the revenue of the broker, is formulated as a Stackelberg game. Authors of [6] take a similar model, but consider that secondary users' spectrum demand is random. In [7], a primary licence holder sets the spectrum price to achieve the optimal balance between the earned revenue and the cost due to extra interference (received from secondary transmissions) and reduced coverage area (by letting secondary users access the spectrum).

In oligopoly spectrum leasing, there are multiple primary networks (or brokers) that lease spectrum to secondary users. So the spectrum price is also affected by the competition among primary networks (or brokers), and one major research focus in the literature is to achieve equilibrium among primary networks (or brokers) [8]–[12]. The work in [8] considers two brokers, and uses a three-stage game. In Stage one, the two brokers purchase spectrum from primary networks; In Stage two, the two brokers set and announce their spectrum prices; In Stage three, secondary users decide on their spectrum demand from one broker. The work in [9] also considers two brokers. Each broker has a common spectrum band to be shared by secondary users. So multiple secondary users that lease spectrum from the same broker will generate interference to each other. Potential interference is considered in secondary users' strategies. In [10], there are multiple primary networks and multiple secondary users. When secondary users make purchase, they are unaware of the spectrum price or spectrum bandwidth that will be allocated. The purchasing process of secondary users is formulated as an evolutionary game. In [11], there are multiple primary networks, one broker and multiple secondary users. The utility function of a primary network reflects both the revenue earned and quality-of-service loss due to leasing some spectrum to secondary users. In [12], multiple primary networks compete with each other by price setting, while each secondary user may have a unique criterion on whether or not to lease the spectrum. In all these works, Nash equilibrium among primary networks (or brokers) is achieved.

All the research efforts mentioned above focus on the static decision making (i.e., the price of a primary network or broker is fixed, and secondary users have spectrum requests at the same time). However, for spectrum leasing in a long term, secondary users may have spectrum demand at different time moments, and thus, the stock of available spectrum should vary with time. In [13], a pricing strategy for dynamic cognitive networks in monopoly spectrum leasing is investigated. The primary network decides on spectrum price dynamically to maximize the average revenue over an infinite time duration.

In this paper, we investigate dynamic pricing for monopoly spectrum leasing with one primary network that leases spectrum to secondary users. We consider spectrum leasing for a finite time duration, in which a number of channels can be leased to secondary users. The time duration for spectrum leasing is equally divided into multiple stages, and in each stage a spectrum price is set. The spectrum demand in each stage depends on the spectrum price. Our target is to set spectrum prices in the stages dynamically such that the total revenue of the primary network over all stages is maximized. The contributions of this paper are:

- When the spectrum demand is random for a given spectrum price, we formulate a revenue maximization problem, and give the method to solve it. We also demonstrate interesting properties of the optimal solution including monotonicity and convexity of the maximal total revenue with respect to stage index as well as lower/upper bounds of the maximal total revenue.
- When the spectrum demand is non-random for a given spectrum price, we provide an incremental algorithm to find the optimal price in each stage, and we also demonstrate the monotonicity of the optimal price with respect to the stage index.

The rest of this paper is organized as follows. Section II gives the system model. Section III formulates the revenue maximization problem, gives the method to solve the problem, and demonstrates properties of the optimal solution. Section IV discusses the case when the spectrum demand is non-random. Section V presents numerical results, and after that, we have conclusions in Section VI.

II. System Model

Consider one primary network with a number of channels. In the primary network, the primary users do not use all the channels at all time, and thus, the primary network can lease unused channels to secondary users. The primary network may allow secondary users to opportunistically or exclusively use the leased channels.

- Opportunistic channel access of secondary users: the primary users have priority to use the channels. Secondary users can access the channels only when primary users do not use them. So secondary users need to sense the channels first, and access the channels if the channels are sensed free.
- Exclusive channel access of secondary users: Primary users do not use the channels that are leased to secondary users. So if a secondary user leases a channel, it can exclusively use the channel during the leasing period.

Compared to exclusive channel access, the opportunistic channel access has the following drawbacks: there may be possible interference between primary users and secondary users (due to imperfect spectrum sensing); sensing equipment is required at secondary users; there is no guarantee for quality-of-service of secondary users; and the primary network needs to have realtime monitoring of the channel usage of secondary users. Therefore, in this paper, the primary network allows exclusive channel access of secondary users over the leased channels. A similar setting is also adopted in [10], [11].

Accordingly, the primary network can partition its channels into two sets: the *primary set* of channels that can be used by primary users only, and the *secondary set* of channels to be leased to secondary users. To maximize its own revenue, the primary network should use as few channels as possible in the primary set, conditioned on that primary users can be served with quality-of-service. To achieve this, the primary network should use dynamic channel assignment for primary users (i.e., the primary network assigns channels to a primary user when the primary user has packets to transmit, and takes back the channels when the transmission is complete). The primary network should also estimate the number of channels in the primary set that are sufficient to serve primary users with quality-of-service guarantee. Although the estimation method is out of scope of this paper, some factors that should be considered in the estimation are listed below:

- Population of primary users and call arrival rate of primary users: Higher population and higher call arrival rate result in a larger size of the primary set;
- Durations and data rates of primary users' calls: Longer call duration and higher data rates lead to a larger size of the primary set;
- Channel quality of primary users: Better channel quality means that higher transmission rates can be achieved, and thus, a smaller primary set is needed.

Once the estimation of the primary set size is done, the primary network can determine the size of the secondary set. Note that since the primary user population, primary call arrival rate, primary call duration, and primary data rate may vary with time, a new estimation should be done after an interval T (defined as a duration that the above factors do not have big changes). The interval T is called a leasing period. So for each leasing period T, the size of the secondary set is fixed, and the size may change in the next leasing period. This paper targets at the pricing strategy of the primary network for a leasing period with a given secondary set of M channels.

As the secondary users may have spectrum access requests at different time instants, it is reasonable for the primary network to perform spectrum leasing once for a while. For simplicity of presentation, the whole duration T is equally divided into N stages, indexed as Stages $N, N-1, ..., 2, 1$, respectively (in other words, the first stage is called Stage N, and the last stage is called Stage 1). At the beginning of each stage, the primary network first announces a spectrum leasing price. Then the secondary users who can accept the announced price make a contract with the primary network. Once a channel is leased to a secondary user, the lease will last until the end of the spectrum leasing period T. This rule is easier for the primary network to manage its spectrum leasing. If a secondary user leases a channel and finishes all its transmissions before the end of the spectrum leasing period T, it can rent out the channel until the end of the spectrum leasing period T in a secondary market.

III. Problem Formulation, Solving Method, and Properties

Denote $p_{n,m}$ as the price at Stage n when there are m available channels (i.e., m channels remain un-leased). Here we assume that the price is taken from a finite set of discrete values. For a given price value x, the spectrum demand (i.e., the number of requested channels by secondary users) is a random variable (with integer values), and we denote $g(y;x)$ as the probability mass function of the demand value y (i.e., some secondary users agree on the price x and request y channels)[1]. If the number of requested channels is more than the number of available channels (m), the primary network only accepts totally m channels' requests. Therefore, for a given price value x, the probability mass function of the accepted demand value y when there are m available channels is given by: $f_m(y;x) = g(y;x)$ if $y < m$; $f_m(y;x) = \sum_{i=m}^{\infty} g(i;x)$ if $y = m$; $f_m(y;x) = 0$ if $y > m$.

Denote $V(n,m)$ as the maximum attainable revenue of the primary network from Stage n with m available channels to the last stage (i.e., Stage 1). Then, we have

$$V(n,m) = \max_{p_{n,m}} \sum_{m'=0}^{m} \Big(f_m(m';p_{n,m}) \times [p_{n,m} \cdot n \cdot m' + V(n-1, m-m')] \Big) \tag{1}$$

in which the term $p_{n,m} \cdot n \cdot m'$ represents the revenue the primary network can collect from Stage n until Stage 1 by leasing m' channels to secondary users at Stage n at price $p_{n,m}$; $V(0,m) = 0, \forall m \in \mathcal{M} \triangleq \{1,2,...,M\}$; and $V(n,0) = 0, \forall n \in \mathcal{N} \triangleq \{1,2,...,N\}$.

Using the formula in Eqn. (1), $V(n,m)$ can be calculated iteratively starting from $V(n,0), \forall n \in \mathcal{N}$ and $V(0,m), \forall m \in \mathcal{M}$, by using dynamic programming, and the optimal price $p_{n,m}$ for $n \in \mathcal{N}, m \in \mathcal{M}$, denoted as $p^*_{n,m}$, can be determined accordingly.

Next we present some properties of the dynamic pricing problem.

Property 1: The function $V(n,m)$ is an increasing function with respect to n, $\forall m \in \mathcal{M}$.

Proof: We use mathematical induction for proving. The proof consists of two steps.

In the first step, we should prove $V(1,m) - V(0,m) \geq 0, \forall m \in \mathcal{M}$. Since $V(1,m) \geq 0$ and $V(0,m) = 0$, apparently we have $V(1,m) - V(0,m) \geq 0$.

In the second step, we should prove that if $V(n,m) - V(n-1,m) \geq 0, \forall m \in \mathcal{M}$, then $V(n+1,m) - V(n,m) \geq 0, \forall m \in \mathcal{M}$. Suppose $V(n,m) - V(n-1,m) \geq 0, \forall m \in \mathcal{M}$ holds, then based on (1), we have

$$\begin{aligned}
& V(n+1,m) - V(n,m) \\
= & \sum_{m'=0}^{m} f_m(m';p^*_{n+1,m})\big[p^*_{n+1,m} \cdot (n+1) \cdot m' \\
& \qquad + V(n, m-m')\big] \\
& - \sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m} \cdot n \cdot m' \\
& \qquad + V(n-1, m-m')\big] \\
\geq & \sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m} \cdot (n+1) \cdot m' \\
& \qquad + V(n, m-m')\big] \\
& - \sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m} \cdot n \cdot m' \\
& \qquad + V(n-1, m-m')\big] \\
= & \sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m} \cdot m' + V(n, m-m') \\
& \qquad - V(n-1, m-m')\big] \\
\geq & \; 0.
\end{aligned} \tag{2}$$

In (2), the last inequality comes from $V(n,l) - V(n-1,l) \geq 0, \forall l \in \mathcal{M}$, and the first inequality is due to the following fact: Recall that $p^*_{n+1,m}$ is the optimal price at Stage $n+1$ when there are m available channels. So at Stage $n+1$ with m available channels, if we change the optimal price $p^*_{n+1,m}$ to any other price value, such as $p^*_{n,m}$, then the revenue should not increase. Thus, we have

$$\begin{aligned}
& \sum_{m'=0}^{m} f_m(m';p^*_{n+1,m})\big[p^*_{n+1,m} \cdot (n+1) \cdot m' \\
& \qquad + V(n, m-m')\big] \\
& \geq \sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m} \cdot (n+1) \cdot m' \\
& \qquad + V(n, m-m')\big].
\end{aligned}$$

This completes the proof. ■

Property 2: The function $V(n,m)$ is an increasing function with respect to m, $\forall n \in \mathcal{N}$.

Proof: We can use mathematical induction for proving, which includes two steps.

In the first step, it is apparent that $V(0,m+1) - V(0,m) \geq 0, \forall m \in \{0,1,2,...,M-1\}$.

[1]Here whether or not a secondary user agrees on a price is independent of the channel quality of the secondary user. The following is how to also consider channel quality of the secondary user. The secondary user's channel may experience path loss, shadowing, and fading. Since the spectrum leasing period is normally much longer than the time scale of the shadowing and fading, the average channel quality of the secondary user over a channel during the spectrum leasing period can be approximated by only considering path loss. For a secondary user, if its average channel quality is above a threshold (or equivalently, its path loss attenuation is less than a threshold), its channel quality is considered as acceptable; otherwise, the channel quality is considered as unacceptable and the secondary user does not accept any price. Based on this, from the location distribution of secondary users, we can calculate the probability of having acceptable channel quality, denoted as η. Therefore, if $g(y;x)$ is the probability that y channels are requested by secondary users with a given price x without considering secondary channel quality, then when secondary channel quality is considered, $g(y;x)$ is the probability that ηy channels are requested by secondary users with a given price x. So, if secondary channel quality is also considered, we only need to replace $g(y;x)$ with $g'(y;x)$ in our paper, where the values of $g'(y;x)$ are given as: $g'(y;x) = g(y/\eta;x)$.

In the second step, we should prove that if $V(n,m+1)-V(n,m)\geq 0,\forall m\in\{0,1,2,...,M-1\}$, then we have $V(n+1,m+1)-V(n+1,m)\geq 0,\forall m\in\{0,1,2,...,M-1\}$. We have

$$\begin{aligned}
&V(n+1,m+1)-V(n+1,m)\\
&=\sum_{m'=0}^{m+1} f_{m+1}(m';p^*_{n+1,m+1})\big[p^*_{n+1,m+1}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m+1-m')\big]\\
&\quad-\sum_{m'=0}^{m} f_m(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m-m')\big]\\
&\geq\sum_{m'=0}^{m+1} f_{m+1}(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m+1-m')\big]\\
&\quad-\sum_{m'=0}^{m} f_m(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m-m')\big]\\
&\overset{(a)}{=}\sum_{m'=0}^{m} g(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m+1-m')\big]\\
&\quad+\sum_{m'=m+1}^{\infty} g(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\\
&\qquad\qquad \times(m+1)+V(n,0)\big]\\
&\quad-\sum_{m'=0}^{m} g(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m'\\
&\qquad\qquad +V(n,m-m')\big]\\
&\quad-\sum_{m'=m+1}^{\infty} g(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m\\
&\qquad\qquad +V(n,0)\big]\\
&=\sum_{m'=0}^{m} g(m';p^*_{n+1,m})\big[V(n,m+1-m')\\
&\qquad\qquad -V(n,m-m')\big]\\
&\quad+\Big[\sum_{m'=m+1}^{\infty} g(m';p^*_{n+1,m})\Big]\cdot p^*_{n+1,m}\cdot(n+1)\\
&\overset{(b)}{\geq} 0
\end{aligned}\tag{3}$$

in which the first inequality comes from the fact that $p^*_{n+1,m+1}$ is the optimal price at Stage $n+1$ when there are $m+1$ available channels (similar to the reasoning for the first inequality in (2)), (a) comes from expression of $f_m(y;x)$ given at the beginning of Section III, and (b) comes from the assumption that $V(n,l+1)-V(n,l)\geq 0,\forall l\in\{0,1,2,...,M-1\}$. This completes the proof. ■

Remark: From Property 1 and Property 2, it can be seen that $V(n,m)$ grows with the increase of n and m. In other words, when there is more time or channels left for spectrum leasing, the maximum attainable revenue of the primary network is larger.

Property 3: $nV(1,m)\leq V(n,m)\leq\big(n(n+1)/2\big)V(1,m),\ \forall m\in\mathcal{M},n\in\mathcal{N}$.

Proof: First, we prove the left-handside inequality in Property 3. According to (1), we have

$$\begin{aligned}
&V(n,m)\\
&=\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m}\cdot n\cdot m'\\
&\qquad\qquad +V(n-1,m-m')\big]\\
&\geq\sum_{m'=0}^{m} f_m(m';p^*_{1,m})\big[p^*_{1,m}\cdot n\cdot m'\\
&\qquad\qquad +V(n-1,m-m')\big]\\
&\geq\sum_{m'=0}^{m} f_m(m';p^*_{1,m})\cdot p^*_{1,m}\cdot n\cdot m'\\
&=nV(1,m)
\end{aligned}\tag{4}$$

where the first inequality comes from the fact that $p^*_{n,m}$ is the optimal price at Stage n when there are m available channels (similar to the reasoning for the first inequality in (2)), and the last equality comes from the fact that $V(1,m)=\sum_{m'=0}^{m} f_m(m';p^*_{1,m})\cdot p^*_{1,m}\cdot 1\cdot m'$ based on (1) and $V(0,l)=0$ for $l\in\{1,2,...,M\}$.

Next we prove the right-handside inequality in Property 3. Still according to (1), we have

$$\begin{aligned}
&V(n,m)\\
&=\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m}\cdot n\cdot m'\\
&\qquad\qquad +V(n-1,m-m')\big]\\
&\overset{(c)}{\leq}\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m}\cdot n\cdot m'+V(n-1,m)\big]\\
&\overset{(d)}{=}\Big[\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\cdot p^*_{n,m}\cdot n\cdot m'\Big]+V(n-1,m)\\
&\overset{(e)}{\leq} nV(1,m)+V(n-1,m)
\end{aligned}\tag{5}$$

in which (c) follows from Property 2, (d) comes from the fact $\sum_{m'=0}^{m} f_m(m';p^*_{n,m})=1$, and (e) is due to $V(1,m)=\max_{p_{1,m}}\sum_{m'=0}^{m} f_m(m';p_{1,m})\cdot p_{1,m}\cdot 1\cdot m'$. Accordingly,

$$\begin{aligned}
V(n,m)&=V(1,m)+\sum_{n'=2}^{n}\big(V(n',m)-V(n'-1,m)\big)\\
&\leq V(1,m)+\sum_{n'=2}^{n} n'V(1,m)\\
&=\tfrac{n(n+1)}{2}V(1,m)
\end{aligned}\tag{6}$$

where the inequality comes from (5).

This completes the proof. ■

Remark: Property 3 gives the performance limit, i.e., a lower bound and an upper bound, for the maximal attainable revenue $V(n,m)$. Specifically, when $n=N$ and $m=M$, the inequality in Property 3 becomes $NV(1,M)\leq V(N,M)\leq\big((N+1)/2\big)NV(1,M)$. In this inequality, the term $V(1,M)$ is the maximal attainable revenue if the spectrum leasing is performed over one stage. So $NV(1,M)$ means the revenue for N stages if the spectrum leasing is performed only in the first stage and all leased channels continue to be used by secondary users in the subsequent $N-1$ stages (in other words, the price is "non-dynamic" over stages). Inequality $NV(1,M)\leq V(N,M)$ means that the revenue achieved by setting up dynamic pricing values over the stages is larger than the revenue achieved in the non-dynamic pricing case. On the other hand, the inequality $V(N,M) \le \big((N+1)/2\big) \cdot NV(1,M)$ shows that the ratio of $V(N,M)$ to $NV(1,M)$, which represents the benefit of dynamic pricing over the non-dynamic pricing, is upper bounded by $(N+1)/2$.

Property 4: $V(n,m) - V(n-1,m) \le V(n+1,m) - V(n,m), \forall m \in \mathcal{M}, n \in \mathcal{N}\backslash\{N\}$.

Proof: We use mathematical induction for proving. The proof consists of two steps. In the first step, it should be proved that the property holds for $n=1$, i.e.,

$$V(1,m) - V(0,m) \le V(2,m) - V(1,m), \forall m \in \mathcal{M}. \quad (7)$$

From Property 3, we have $2V(1,m) \le V(2,m), \forall m \in \mathcal{M}$. Together with the fact that $V(0,m)=0$, it can be seen that inequality (7) holds.

In the second step, we need to prove that

$$V(n,m) - V(n-1,m) \le V(n+1,m) - V(n,m), \forall m \in \mathcal{M} \quad (8)$$

holds if

$$V(n-1,m) - V(n-2,m) \le V(n,m) - V(n-1,m), \forall m \in \mathcal{M}. \quad (9)$$

We have (10) on top of the next page. In (10), the inequality in (10b) is obtained by moving $V(n-2,m)$ to the right-handside and moving $V(n-1,m)$ to the left-handside; then inequality in (10c) holds since $2V(n-1,m-m') \le \big[V(n,m-m') + V(n-2,m-m')\big]$ from (10b); inequality (10d) is achieved by adding term $\sum_{m'=0}^{m} f_m(m';p^*_{n,m}) \cdot 2p^*_{n,m} \cdot n \cdot m'$ on both sides of (10c); inequality (10e) comes from splitting the right-handside of (10d) to two terms; inequality (10f) comes from two facts (similar to the reasoning for the first inequality in (2)):

$$\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m}\cdot(n+1)\cdot m' + V(n,m-m')\big] \le \sum_{m'=0}^{m} f_m(m';p^*_{n+1,m})\big[p^*_{n+1,m}\cdot(n+1)\cdot m' + V(n,m-m')\big]$$

and

$$\sum_{m'=0}^{m} f_m(m';p^*_{n,m})\big[p^*_{n,m}\cdot(n-1)\cdot m' + V(n-2,m-m')\big] \le \sum_{m'=0}^{m} f_m(m';p^*_{n-1,m})\big[p^*_{n-1,m}\cdot(n-1)\cdot m' + V(n-2,m-m')\big];$$

inequality in (10g) holds by following the definition in (1).

This completes the proof. ■

Remark: Recall that Property 1 shows that $V(n,m)$ increases if n increases (i.e., the primary network has more time for leasing the channels). In Property 4, $V(n,m) - V(n-1,m)$ is the revenue increase when the number of stages increases from $n-1$ to n, and $V(n+1,m) - V(n,m)$ is the revenue increase when the number of stages increases from n to $n+1$. So Property 4 implies that, when the number of stages increases, the primary network has a higher revenue increase rate. In other words, $V(n,m)$ is an increasing convex-shaped function with respect to n.

IV. When Spectrum Demand is Non-random

In Section III, given a price value, the spectrum demand (i.e., the number of requested channels by secondary users) is a random variable. In this section, we consider the case that the spectrum demand for a given price value is non-random, and can be solely determined by the price value. Before we investigate the case, we first give an example for the case. In Section III, the primary network needs to know the probability mass function of the spectrum demand for a given price. However, it is also likely that for a given price, the primary network does not know the probability mass function of the spectrum demand, but only knows the mean value of the spectrum demand. So for a given price value, the primary network has to estimate the spectrum demand by using the known mean value. Thus, for the primary network to find the optimal pricing strategy, the spectrum demand is viewed as non-random, and takes the mean value of the actually random spectrum demand for a given price.

Next we investigate the non-random spectrum demand case.

For price p, the spectrum demand d (i.e., the number of requested channels) can be expressed as $d = D(p)$.[2] As $d \in \mathcal{I}$ where $\mathcal{I}$ means the set of non-negative integers, $D(p)$ is a piecewise function mapping intervals of price into non-negative integers. In Stage n, denote p_n as the price, and $d_n = D(p_n)$ as the demand for the number of channels. The achieved revenue of the leased channels in Stage n can be expressed as $p_n d_n n$. So the total revenue over all the stages is $\sum_{n=1}^{N} p_n d_n n$, i.e., $\sum_{n=1}^{N} p_n D(p_n) n$. Then the revenue maximization problem can be formulated as

Problem 1:

$$\begin{aligned} \max_{p_1,p_2,\ldots,p_N} \quad & \sum_{n=1}^{N} p_n D(p_n) n \\ s.t. \quad & \sum_{n=1}^{N} D(p_n) \le M; \\ & p_n \ge 0, n \in \mathcal{N}; \\ & D(p_n) \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

For the ease of analysis, by defining $P(d) = \max_{D(p)=d} p$,[3] Problem 1 can be reformulated as

Problem 2:

$$\begin{aligned} \max_{d_1,d_2,\ldots,d_N} \quad & \sum_{n=1}^{N} d_n P(d_n) n \\ s.t. \quad & \sum_{n=1}^{N} d_n \le M; \quad d_n \ge 0, d_n \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

For the price function with respect to demand, $P(d)$, three characteristics are assumed and justified in the following.

- $P(d)$ is a decreasing function with respect to demand d. This assumption is in concordance with the fact that when the announced price is higher, the spectrum for

[2] Here we assume that, for any demand value $d \in \{0,1,2,\ldots,M\}$, there exists at least a price p such that $D(p)=d$.

[3] The conceptional meaning of $P(\cdot)$ is the inverse function of $D(\cdot)$. Since $D(\cdot)$ is a piecewise function, its inverse function does not exist mathematically. Therefore, $P(d)$ is defined as the maximal price such that $D(P(d)) = d$, rather than $D^{-1}(d)$.

$$
\begin{aligned}
& V(n-1,m) - V(n-2,m) \le V(n,m) - V(n-1,m),\ \forall m \in \mathcal{M} && \text{(10a)}\\
\Leftrightarrow\quad & 2V(n-1,m) \le V(n,m) + V(n-2,m),\ \forall m \in \mathcal{M} && \text{(10b)}\\
\Rightarrow\quad & \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[2V(n-1,m-m')\big] \\
& \le \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[V(n,m-m') + V(n-2,m-m')\big] && \text{(10c)}\\
\Rightarrow\quad & \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[2p^*_{n,m}\cdot n\cdot m' + 2V(n-1,m-m')\big] \\
& \le \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[2p^*_{n,m}\cdot n\cdot m' + V(n,m-m') + V(n-2,m-m')\big] && \text{(10d)}\\
\Leftrightarrow\quad & \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[2p^*_{n,m}\cdot n\cdot m' + 2V(n-1,m-m')\big] \\
& \le \Big(\sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[p^*_{n,m}\cdot (n+1)\cdot m' + V(n,m-m')\big] \\
& \quad + \sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[p^*_{n,m}\cdot (n-1)\cdot m' + V(n-2,m-m')\big]\Big) && \text{(10e)}\\
\Rightarrow\quad & 2\sum_{m'=0}^{m} f_m(m'; p^*_{n,m})\big[p^*_{n,m}\cdot n\cdot m' + V(n-1,m-m')\big] \\
& \le \Big(\sum_{m'=0}^{m} f_m(m'; p^*_{n+1,m})\big[p^*_{n+1,m}\cdot (n+1)\cdot m' + V(n,m-m')\big] \\
& \quad + \sum_{m'=0}^{m} f_m(m'; p^*_{n-1,m})\big[p^*_{n-1,m}\cdot (n-1)\cdot m' + V(n-2,m-m')\big]\Big) && \text{(10f)}\\
\Leftrightarrow\quad & 2V(n,m) \le V(n+1,m) + V(n-1,m) && \text{(10g)}\\
\Leftrightarrow\quad & V(n,m) - V(n-1,m) \le V(n+1,m) - V(n,m). && \text{(10h)}
\end{aligned}
$$

leasing is less attractive to secondary users, and there is less demand.

- $d \cdot P(d)$ is an increasing function with respect to demand d. This assumption is reasonable as the total revenue of the primary network should be more if more channels are leased.
- $d \cdot P(d)$ is "concave", which means $\big[(d+1)P(d+1) - dP(d)\big] \le \big[dP(d) - (d-1)P(d-1)\big], \forall d > 0, d \in \mathcal{I}$. This assumption conforms to the *law of diminishing returns* [14] in economics: the increase of revenue slows down as the sale volume grows.

For Problem 2, the following lemma is in order.

Lemma 1: When the maximal value of the objective function $\sum_{n=1}^{N} d_n P(d_n) n$ is achieved, we should have $\sum_{n=1}^{N} d_n = M$.

Proof: We use proof by contradiction. According to the second assumption on $P(d)$, the objective function $\sum_{n=1}^{N} d_n P(d_n) n$ is an increasing function with respect to $d_n, n \in \mathcal{N}$. Define the optimal d_n as $d^*_n, n \in \mathcal{N}$. Suppose $\sum_{n=1}^{N} d^*_n = M' < M$, then the objective function in Problem 2 can be further increased by increasing d^*_1 to $d^*_1 + M - M'$, which contradicts the assumption that $d^*_n, n \in \mathcal{N}$ is the optimal solution.

This completes the proof. ■

After substituting the constraint $\sum_{n=1}^{N} d_n \le M$ with $\sum_{n=1}^{N} d_n = M$, Problem 2 has the following features: the objective function is separable and "concave", all the constraints are linear, and all the variable coefficients in the constraints are 1's. Thus, Problem 2 can be solved by using an incremental algorithm [15]. The procedure is given by the following Algorithm 1.

Algorithm 1 Incremental Algorithm solving Problem 2.

1: Set $d_n = 0, n \in \mathcal{N}$.
2: If $\sum_{n=1}^{N} d_n < M$, find $n^* = \arg\max_{n \in \mathcal{N}} \big((d_n+1)P(d_n+1)n - d_n P(d_n) n\big)$, and proceed to Step 3; Otherwise, proceed to Step 4.
3: $d_{n^*} = d_{n^*} + 1$, proceed to Step 2.
4: Output $\{d_n, n \in \mathcal{N}\}$, which is the optimal solution of Problem 2.

Based on the procedure of Algorithm 1, the following theorem can be proved.

Theorem 1: d^*_n increases when n increases, and p^*_n decreases when n increases, where d^*_n and p^*_n are optimal d_n and p_n, respectively, for Problem 2.

Proof: To prove that d^*_n increases when n increases, we use proof by contradiction.

Suppose there exist $n_1, n_2 \in \mathcal{N}$ such that $n_1 > n_2$ and $d^*_{n_1} < d^*_{n_2}$. According to Algorithm 1, there are M rounds of search. In each round, the $n^* = \arg\max_{n \in \mathcal{N}} \big((d_n+1)P(d_n+1)n - d_n P(d_n) n\big)$ is found. Before the first round, $d_{n_1} =$

$d_{n_2} = 0$. After the last round, $d^*_{n_1} < d^*_{n_2}$. Then there should exist a round such that: before the round we have $d_{n_1} = d_{n_2}$, and after the round, d_{n_2} is increased by 1. In other words, before this round we have

$$\begin{aligned}&\big[(d_{n_1}+1)P(d_{n_1}+1) - d_{n_1}P(d_{n_1})\big]n_1\\ \le &\big[(d_{n_2}+1)P(d_{n_2}+1) - d_{n_2}P(d_{n_2})\big]n_2.\end{aligned} \tag{11}$$

Since we have $d_{n_1} = d_{n_2}$ before this round, the inequality in (11) is equivalent to $n_1 \le n_2$, which contradicts the assumption that $n_1 > n_2$. So we have $d^*_{n_1} \ge d^*_{n_2}$, $\forall n_1 > n_2$, i.e., d^*_n increases with respect to n.

As $p^*_n = P(d^*_n)$, the function $P(\cdot)$ is a decreasing function, and d^*_n increases when n increases, it is easy to conclude that p^*_n decreases when n increases.

This completes the proof. ■

Remark: According to Theorem 1, as the time approaches the end of the spectrum leasing period T, the primary network should set the price higher, while in early stages, the primary network should set lower prices to attract more spectrum demand. This leads to a suggestion to secondary users: If a secondary user wants to access the spectrum at a lower unit price, it is better to accept the announced price earlier. Therefore, based on our findings in Theorem 1, an interesting future research problem is to investigate the interaction between the primary network and secondary users. The result in Theorem 1 also shares some similarity with pricing strategy in flight ticket booking: long before the flight departure date, the flight ticket price is low which can attract more bookings, while as the flight departure date is approaching, the flight ticket price goes higher.

V. Numerical Results

A. Verification of Properties 1–4 and Theorem 1

We first verify Properties 1–4. In the numerical example, N is set as 10 and M is set as 50. Set $g(y;x)$ as a discrete uniform distribution over $\{m_0, m_0+1, ..., m_0+4\}$ where $m_0 \triangleq \lfloor 1/x^2 \rfloor$ (here $\lfloor\cdot\rfloor$ is the floor function), i.e.,

$$g(y;x) = \{0.2, 0.2, 0.2, 0.2, 0.2\} \tag{12}$$

for $y = \{m_0, m_0+1, m_0+2, m_0+3, m_0+4\}$. The spectrum price in a stage is selected from a discrete set of 100 values that are evenly spaced between 0.1474 and 1.001. Here 0.1474 is the minimum selectable price because among its five demand values $\{46, 47, 48, 49, 50\}$, the largest demand value is $M = 50$; and 1.001 is the maximum selectable price because among its five demand values $\{0, 1, 2, 3, 4\}$, the smallest demand value is 0.

Fig. 1, Fig. 2, and Fig. 3 show $V(n,m)$ and its lower and upper bounds, $nV(1,m)$ and $(n(n+1)/2)V(1,m)$, as n grows from 1 to N when $m = 10$, $m = 20$, and $m = 30$, respectively. It can be seen that $V(n,m)$ increases when n or m increases, as indicated in Property 1 and Property 2. In addition, $V(n,m)$ lies between the lower bound and upper bound, which is consistent with the conclusion in Property 3. And $V(n,m)$ is convex-shaped with n, as indicated by Property 4.

Next we verify Theorem 1. For non-random spectrum demand, N is set as 10, the function $P(d)$ is set to be $1/\sqrt{d}$,

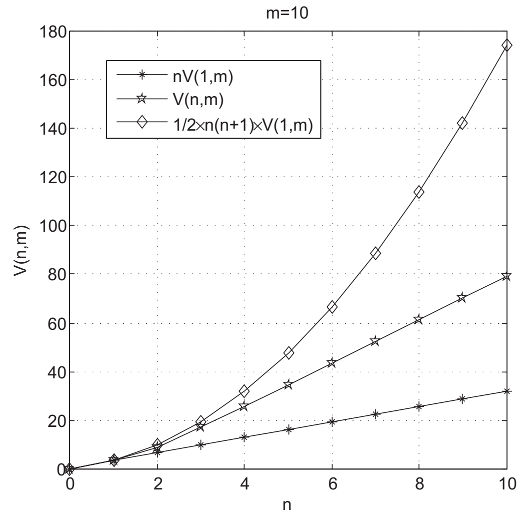

Fig. 1. $V(n,m)$ versus n with $m = 10$.

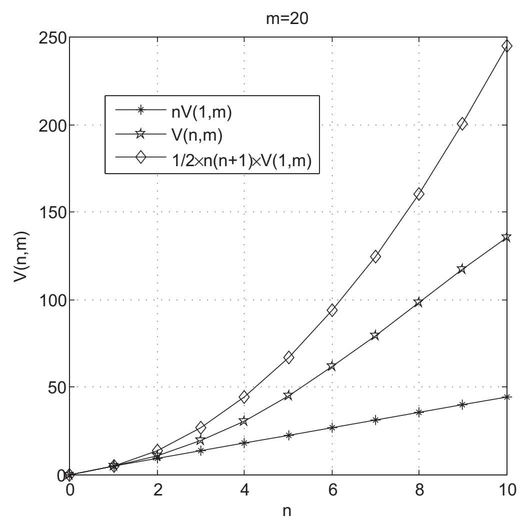

Fig. 2. $V(n,m)$ versus n with $m = 20$.

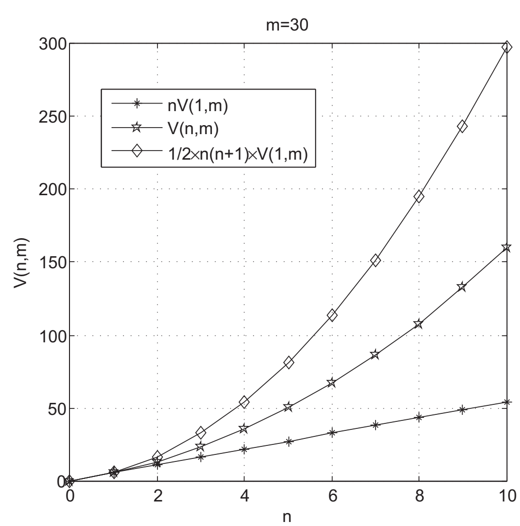

Fig. 3. $V(n,m)$ versus n with $m = 30$.

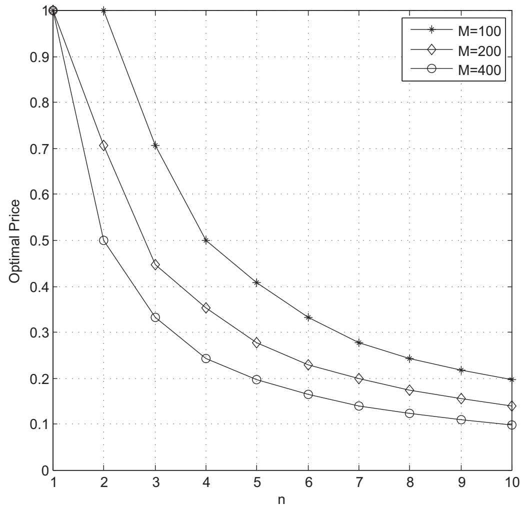

Fig. 4. The optimal price.

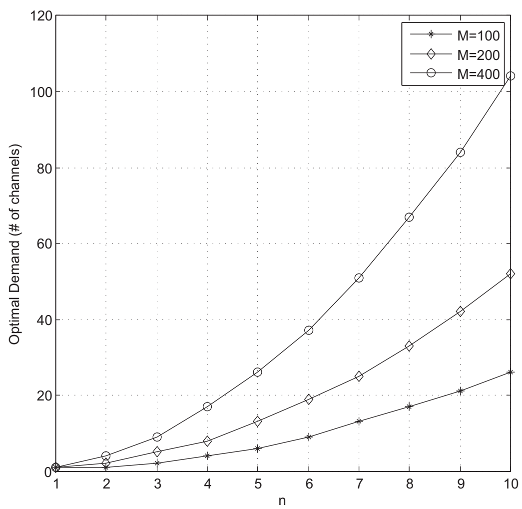

Fig. 5. The optimal demand.

which satisfies the three assumptions on $P(d)$ in Section IV. In Fig. 4 and Fig. 5, the optimal price p_n^* and optimal demand d_n^* are plotted, respectively, when $M = 100$, $M = 200$, and $M = 400$. It can be seen that the optimal price p_n^* decreases with n, and the optimal demand d_n^* grows with n. This result matches Theorem 1.

B. Comparison of Random Spectrum Demand Case and Non-Random Spectrum Demand Case

Recall that in random spectrum demand case considered in Section III, the primary network knows the probability mass function of spectrum demand for a given price, while an example for the non-random spectrum demand case considered in Section IV is that the primary network knows only the mean value of spectrum demand for a given price. Accordingly, a comparison of the random spectrum demand case and the non-random spectrum demand case is interesting.

We take the same spectrum demand distribution as used in Figs. 1-3. Consider $M = 30$. The achieved revenue of the random spectrum demand case (when the primary network knows

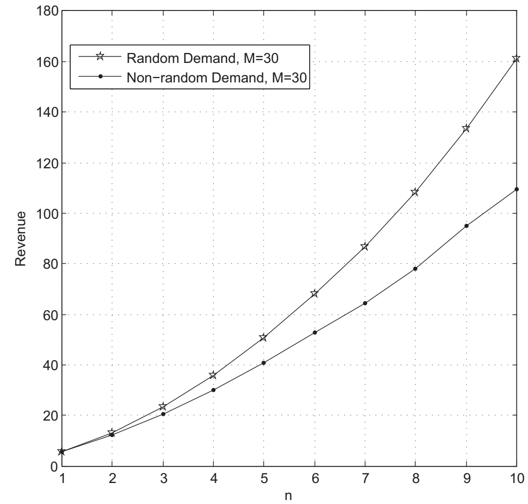

Fig. 6. Comparison of the achieved revenue in random spectrum demand case (when the probability mass function of spectrum demand for a given price is known) and in non-random spectrum demand case (when only the mean value of the spectrum demand for a given price is known).

probability mass function of the spectrum demand, given as in (12)) is shown in Fig. 3. Now suppose the primary network knows only mean value of the spectrum demand for a given price, i.e., the expected demand for price value x is given as $\lfloor 1/x^2 \rfloor$ + 2. Based on this demand function, we use Algorithm 1 to find out the optimal price for each stage. Then using those price values at the stages, we carry out computer simulations to find out the achieved revenue. In the simulations, we use the real random spectrum demand at each stage for the calculated optimal price. In other words, for a stage, if the optimal price at Stage i calculated from Algorithm 1 is p_i^*, then in the stage, the simulated spectrum demand is a random variable, taking values from $\{u^*, u^*+1, u^*+2, u^*+3, u^*+4\}$ with probability $\{0.2, 0.2, 0.2, 0.2, 0.2\}$. Here $u^* = \lfloor 1/(p_i^*)^2 \rfloor$. The simulated revenue (averaged over 500 simulation runs) is shown in Fig. 6. As a comparison, the achieved revenue of the random spectrum demand case is also shown in this figure. From the figure, it is clear that the non-random spectrum demand case achieves less revenue than the random spectrum demand case. This is reasonable, due to the lack of distribution information of the real spectrum demand in the non-random spectrum demand case.

C. Comparison with Existing Work

In the following, we compare our dynamic pricing strategy with the dynamic pricing strategy in [13], which also considers one seller and multiple buyers, but for infinite duration. We use the same spectrum demand distribution as used in Figs. 1-3. We simulate the dynamic pricing strategy in [13] with some modifications to fit with our considered multiple-stage pricing problem[4]. The simulation result is shown in Fig. 7.

[4]The price in our work is for the unit of a channel, while the price in [13] is for the unit of a packet transmission. Therefore, when implementing the dynamic pricing strategy of [13] in our simulations with multiple stages, the channels are treated as packets as in [13], and in each stage, we use the pricing strategy proposed in [13] to determine the price. In the simulations, the parameter V in [13] is set to be 200.

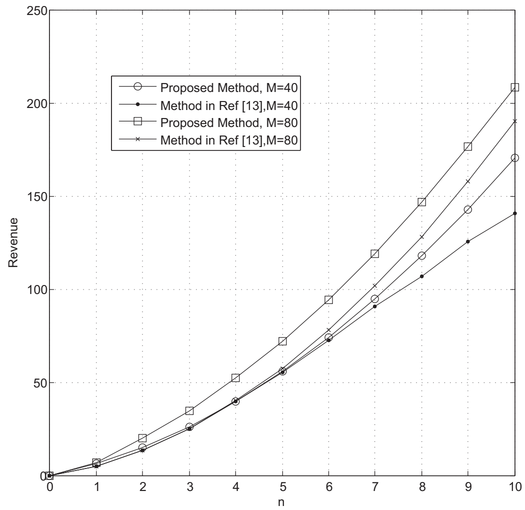

Fig. 7. Comparison between the proposed dynamic pricing strategy and the dynamic pricing strategy in [13].

As a comparison, the achieved revenue of our method is also shown. It can be seen that, when implemented in our pricing problem with a finite duration, the achieved revenue by using the dynamic pricing strategy of [13] is less than the revenue by our proposed dynamic pricing strategy.

VI. Conclusions

In this paper, we have investigated the problem of dynamic pricing over multiple stages of spectrum leasing. We have formulated optimization problems that find the optimal price in each stage so as to maximize the total revenue of the primary network. We have presented the solving methods for the optimization problems, as well as properties of the optimal solutions, such as monotonicity and convexity of the maximal total revenue with respect to stage index, and lower/upper bounds of the maximal total revenue in the random spectrum demand case, and monotonicity of the optimal price with respect to stage index in the non-random spectrum demand case. This research should provide helpful insights for pricing strategy design in spectrum leasing.

In this research, there is only one primary network that leases channels to secondary users. Another interesting research topic is to investigate the case when there are multiple primary networks performing the dynamic pricing. Thus, a primary network should consider the dynamic pricing with time (stage) as well as the price competition with other primary networks in each stage. To investigate the research problem, the first step could be to study dynamic pricing with two primary networks. This problem can be formulated as a differential game, which usually includes two players having conflicting goals in a dynamic system with multiple stages. It would be interesting to investigate the existence and uniqueness of Nash equilibrium between the two primary networks.

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