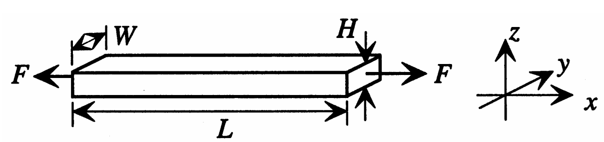
9. Structures

Overview

- Microelectromechanical systems include a large number of devices spanning various physical phenomena for the acquisition, transduction, and communication of information.
- One class of these devices employs moving micromechanical structures such as plates, resonating beams etc. in their design
- This chapter presents a brief overview of the physics and dynamics of mechanical structures.
- More specifically, we will cover beams, cantilevers, plates, and membranes.

Axially Loaded Beams



Beam with Uniform Cross-Section

- W, H, L are width, height, and length and F is applied uniaxial load.
- The axial force is uniformly applied across the cross-section.
- The resulting tensile stress is:

$$\sigma = \frac{F}{A} = \frac{F}{WH}$$

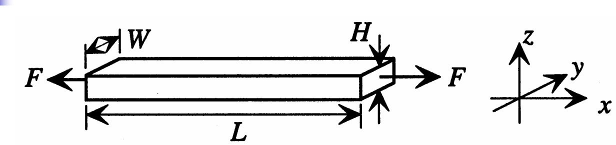
- The strain in the beam is defined as:

$$\varepsilon \equiv \frac{\delta L}{L}$$

- and is related to the stress through Young's modulus *E*:

$$\sigma = E\varepsilon$$

Axially Loaded Beams (ctnd.)



- combining:

$$\delta L = \frac{FL}{EWH}$$

- Defining a spring constant :

$$F = \kappa \delta L$$

- we get:

$$\kappa_{\text{axial beam}} = \frac{EWH}{L}$$

- Example: calculate spring constant for an axially loaded silicon beam of length L = 100 μm and square cross section of 2 μm on the side: use E = 160 GPa

$$\kappa = \frac{EWH}{L} = \frac{(160\times10^{9})(2\times6^{-6})^{2}}{100\times10^{-6}} = 6400\,\frac{N}{m}$$

Axially Loaded Beams (ctnd.)

Beam of Varying Cross-Section

- **The element dx is considered small enough to have a uniform cross section along its length**
- **Thus:**

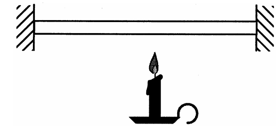
$$\Delta(dx) = \frac{F\,dx}{EA(x)}$$

- **Where $A(x)$ is the cross-sectional area at the position of the element**
- **The total length change is given by:**

$$\delta L = \int_0^L \Delta(dx) = \int_0^L \frac{F}{EA(x)}dx$$

$$\kappa = \frac{F}{\delta L} = \left[\int_0^L \frac{dx}{EA(x)}\right]^{-1}$$

Axially Loaded Beams (ctnd.)



Statically Indeterminate Beams

- **Consider a fixed beam subjected to a thermal strain**

$$\varepsilon_{thermal} = +\alpha_T \Delta T$$

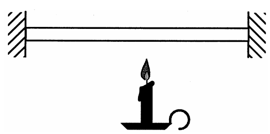
$$0 = \alpha_T \Delta T + \frac{\sigma_{thermal}}{E}$$

- **The source of heat would generate thermal expansion of the material and would generate a thermal stress**

$$\sigma_{thermal} = -E\alpha_T \Delta T = -(160\times10^9)(2.8\times10^{-6})(100) = -44.8MPa$$

- **where α_T is coefficient of thermal expansion**

Axially Loaded Beams (ctnd.)

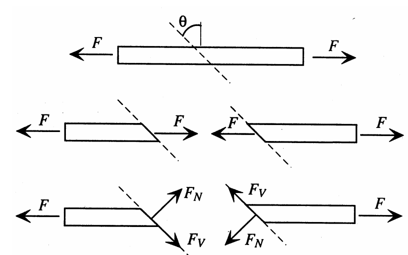


Statically Indeterminate Beams

- **However, since the beam is clamped the total strain must be zero. The clamping points are therefore applying an axial stress σ thermal in such a way to make the total strain equal to zero**

- **Example: Suppose the beam of previous example is fixed at both ends and heated by 100°C. Calculate resulting compressive stress. Use α_T = 2.8 × 10^{-6} K^{-1}**

Axially Loaded Beams (ctnd.)

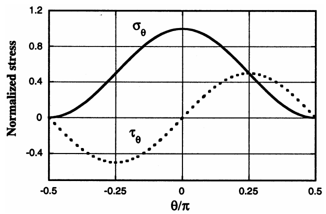


Stress on Inclined Sections

- **Consider a uniform beam with axial force F cut at an angle θ with respect to its axis**
- **We separate the axial force along two components normal and parallel to the force.**
- **F_N is the normal force and F_V is the shear force, where:**

$$F_N = F\cos\theta \qquad F_V = F\sin\theta$$

Axially Loaded Beams (ctnd.)



- we define the normal and shear stresses as:

$$\sigma_\theta = \frac{F_N}{A_\theta} \qquad \tau_\theta = \frac{F_V}{A_\theta}$$

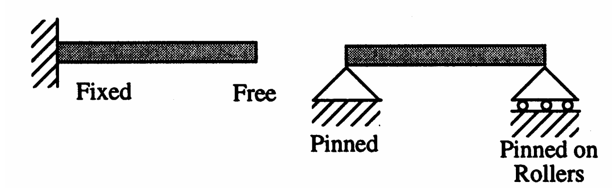
- where A_θ is the area of the inclined cross-section and is given by:

$$A_\theta = \frac{A}{\cos\theta}$$

- combining we get:

normal stress $\sigma_\theta = \frac{F}{A}\cos^2\theta$ and shear stress $\tau_\theta = \frac{F}{A}\cos\theta\sin\theta$

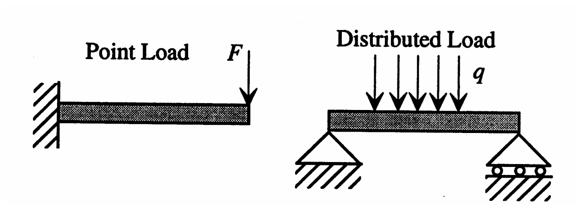
Bending of Beams



Types of Support

- Free: no constraints applied to extremity
- Fixed: fully constrained in position and angle
- Pinned: constrained in position not in angle
- Pinned on rollers: constrained in one direction, free in the other one and in angle

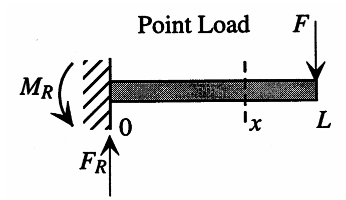
Bending of Beams (ctnd.)



Types of Load

- Point load
  - *F* = total point force [N]
  - *F'* = Force per unit width such as *F* = *F'* × *W* [N/m]
- Distributed load
  - *q* = force per unit length [N/m]
  - *P* = pressure
  - *q* = *P* × *W*

Bending of Beams (ctnd.)



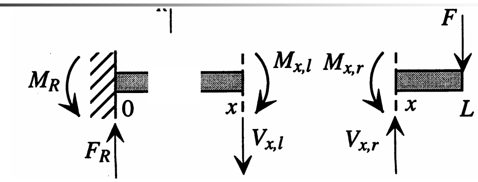
Reaction Forces and Moments

- The reaction forces and moments are the force and moment provided by support under external load in order to fulfill the constraints of the support
- For example: a point load on a clamped free cantilever
- Under static equilibrium $\sum m = 0$ at any point of the beam
- For instance, at clamping point :

$$\sum m = M_R - FL = 0$$

- thus *MR = FL*

Bending of Beams



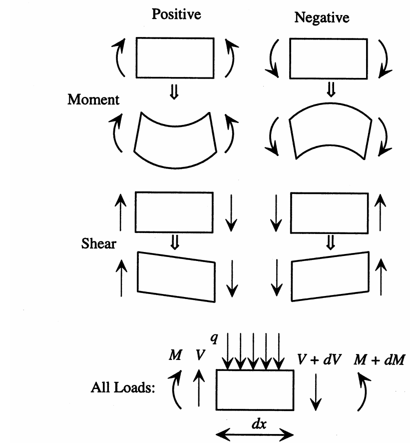
- **Lets now consider hypothetical situation of splitting the beam into two parts at point *x***
- **where** $M_{x,l} = M_{x,r} = F(L-x)$ **and** $V_{x,l} = V_{x,r} = F$
- **The total force acting on left-hand part is:**

$$F_R - V_{x,l} = 0 \quad (\text{since } F = F_R)$$
$$V_{x,l} = F$$

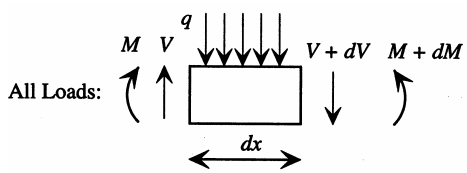
- **The moment with respect to fixed support is:**
$$\sum m = M_R - M_{x,l} - xV_{x,l} = FL - F(L-x) - xF = 0$$
- **Conclusion: in static conditions** $\sum F = 0$ **and** $\sum m = 0$ **is applicable at any part in the beam.**

Bending of Beams

Deformation due to moments and shear forces



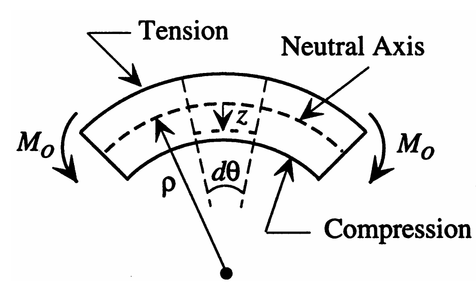
Bending of Beams (ctnd.)



- **Consider all possible loads on a differential element of length *dx*.**
- **The total force F_T acting on this element is:**
$$F_T = q\,dx + (V + dV) - V$$
- **in order to be zero we get:**
$$\frac{dV}{dx} = -q$$
- **The total moment applied on element with respect to left-hand edge is:**
$$M_T = (M + dM) - M - (V + dV)dx - \frac{q\,dx}{2}dx$$
- **neglecting the product of differentials, in order to be zero we get:**
$$\boxed{\frac{dM}{dx} = V}$$

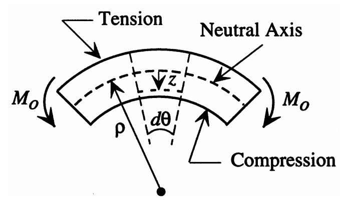
Bending of Beams (ctnd.)

Pure bending of a transversely loaded beam



- **Note:** **z is positive downward.**
  M_o is externally applied moment
  ρ is radius of curvature
  $d\theta$ is presumed to have a length dx when the element is not bent

Bending of Beams (ctnd.)



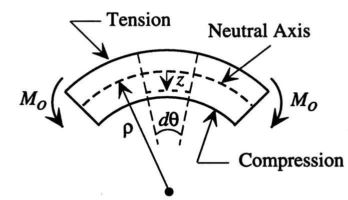
- Because stress and strain are proportional, the uniaxial stress is given by:

$$\sigma_x = \frac{-zE}{\rho}$$

- Note: for $z > 0$ σ_x is negative (compression)
  for $z < 0$ σ_x is positive (tension)
- The total moment about $z = 0$ in this segment is given by:

$$M = \int_{-H/2}^{H/2} Wz\sigma_x dz = \int_{-H/2}^{H/2} \frac{EWz^2}{\rho} dz = -\left(\frac{1}{12}WH^3\right)\frac{E}{\rho}$$

Bending of Beams (ctnd.)



- Thus

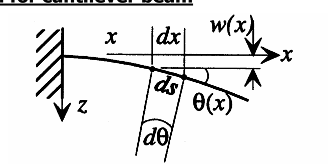
$$\frac{1}{\rho} = \frac{-M}{EI}$$

- under static conditions M_0 = -M thus :

$$\boxed{\frac{1}{\rho} = \frac{M_0}{EI}}$$

Bending of Beams (ctnd.)

Differential equation for cantilever beam



- The increment of beam length ds along neutral axis is related to dx by:

$$ds = \frac{dx}{\cos\theta} \approx dx$$

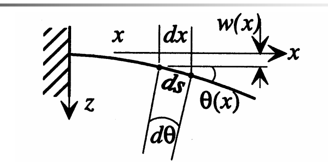
- and the slope of the beam at any point is:

$$\frac{dw}{dx} = \tan\theta \approx \theta$$

- For any given radius of curvature ρ at position x the relation between ds and the incremental subtended angle $d\theta$ is:

$$ds = \rho d\theta$$

Bending of Beams (ctnd.)



- For small angles cosθ ≈1, tanθ ≈θ, thus ds ≈ dx and

$$\frac{d\theta}{dx} \approx \frac{1}{\rho}$$

and:

$$\theta \approx \frac{dw}{dx}$$

- Combining:

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}$$

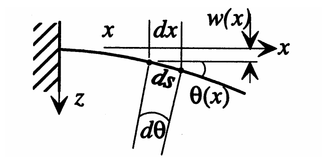
- From previous section:

$$\left(\frac{1}{\rho} = -\frac{M}{EI}\right)$$

$$\frac{d^2w}{dx^2} = -\frac{M}{EI}$$

Where M is the internal moment

Bending of Beams (ctnd.)



- Further deriving we get:

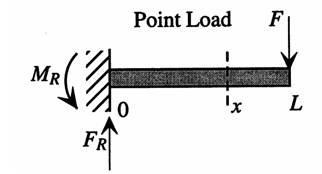
$$\boxed{\frac{d^3w}{dx^3} = -\frac{V}{EI}} \quad \text{since} \quad V = \frac{dM}{dx}$$

and:

$$\boxed{\frac{d^4w}{dx^4} = \frac{q}{EI}} \quad \text{since} \quad q = -\frac{dV}{dx}$$

Bending of Beams (ctnd.)

Cantilever beam with point load at extremity



- The internal moment at any point in the beam is given by:

$$M = -F(L-x)$$

- Thus

$$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

- The boundary conditions are:

$$w(0) = 0 \qquad \left.\frac{dw}{dx}\right|_{x=0} = 0$$

Bending of Beams (ctnd.)

- Trial solution: $w = A + Bx + Cx^2 + Dx^3$
- After application of trial solution to differential equation and boundary conditions we get:

$$C = \frac{FL}{2EI} \quad D = -\frac{F}{6EI}$$

- Thus:

$$w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)$$

- The deflection at end of cantilever is:

$$w(L) = \left(\frac{L^3}{3EI}\right)F$$

- Assigning a spring constant:

$$\kappa_{cantilever} = \frac{3EI}{L^3} = \frac{EWH^3}{4L^3}$$

- Example: calculate spring constant of the bent cantilever to that of the axially loaded beam situation (section 9.2)

$$\kappa = \frac{(160\times10^9)(2\times10^{-6})^4}{4(100\times10^{-6})^3} = 0.64\frac{N}{m}$$

- 10 000 times smaller !!!

Bending of Beams (ctnd.)

- What about the stresses in the cantilever
- The curvature is given by:

$$\frac{1}{\rho} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

- The stress inside the cantilever was given by:

$$\sigma_x = -\frac{zE}{\rho}$$

- Where *z* was the distance along the thickness of cantilever
- The stress is therefore maximum when $1/\rho$ is maximum which occurs at $x = 0$

$$\left.\frac{1}{\rho}\right|_{max} = \frac{L}{EI}F$$

- The stress is also maximum at the surfaces of the cantilever (i.e. $z = \pm H/2$ where H is thickness of cantilever). We get

$$\sigma_{max} = \frac{LH}{2I}F$$

- Given that for a square cross-section: $I = \frac{1}{12}WH^3$
- We get

$$\sigma_{max} = \frac{6L}{H^2W}F$$

Bending of Beams (ctnd.)

- For our cantilever of $L = 100\ \mu m$ and square cross-section $H = W = 2\mu m$ we get

$$\sigma_{max} = \frac{6(100\times10^{-6})}{(2\times10^{-6})^3}F = 7.5\times10^{13}\left(\frac{Pa}{N}\right)\times F$$

- A few questions on this cantilever:

a) what force is required at extremity to produce a deflection of 1 μm?

sol'n:

$$F = \kappa x = 0.64(1\times10^{-6}) = 0.64\mu N$$

b) what is maximum stress produced by this deflection?

sol'n:

$$\sigma_{max} = 7.5\times10^{13}(F) = 7.5\times10^{13}(0.64\times10^{-6}) = 72.7MPa$$

Plate Stiffness Modulus

Plate stiffness modulus

- Until now we have focused on thin beams with transverse dimensions small than their length
- Lets now consider more general situation where thickness is not being neglected
- Consider a plate under external stress σ_x

Plate Stiffness Modulus (ctnd.)

- The plate will be sufficiently wide to build up a transverse stress σ_y to offset the Poisson contraction in the transverse direction
- in other words the plate is thick enough that it will generate a strain $\varepsilon_y = \Delta H/H$ we assume $\varepsilon_y = 0$
- thus we have:

$$\varepsilon_x = \frac{\sigma_x - \upsilon\sigma_y}{E}$$

- where υ is the Poisson ratio of the material and σ_y is the lateral strain that responds to offset and stress in lateral direction
- we also have:

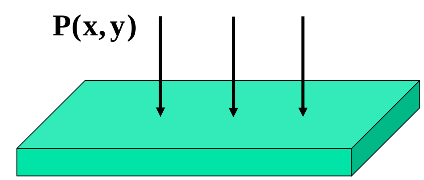
$$\varepsilon_y = \frac{\sigma_y - \upsilon\sigma_x}{E} = 0$$

- combining we get:

$$\sigma_x = \left(\frac{E}{1-\upsilon^2}\right)\varepsilon_x$$

- this quantity is the plate modulus, and is generally 10% greater than E

Plate in Pure Bending



- Extending approach employed for thin beams we would derive:

$$D\left(\frac{d^4w}{dx^4} + 2\frac{d^4w}{dx^2dy^2} + \frac{d^4w}{dy^4}\right) = P(x,y)$$

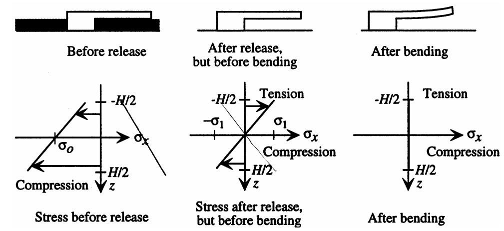
- Where P(x,y) is a two dimensional distributive load and:

$$D = \frac{1}{12}\left(\frac{EH^3}{1-\upsilon^2}\right)$$

Effects of residual stresses and stress gradients

- Deposited films can have built-in residual stresses that will affect the mechanical behavior of machined mechanical devices.
- For instance non-uniform residual stresses will cause cantilevers to curl.
- Residual stresses can also introduce non-linear response in the deflection of doubly-supported beams.
- Compressive residual stresses can also cause a doubly-supported beam or membrane to spontaneously buckle out of plane.

Stress gradients in cantilevers



Bending due to residual stress

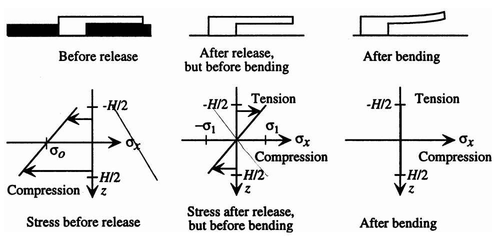
- Consider a cantilever machined out of a material containing residual compressive stresses
- The axial stress in the beam is approximated by:

$$\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$$

- where σ_1 is the stress gradient.
- The internal moment about the middle of the cantilever is:

$$M_x = \int_{-H/2}^{H/2} Wz\sigma\, dz = -\frac{1}{6} WH^2 \sigma_1$$

Stress gradients in cantilevers (ctnd.)



- <u>Immediately after release:</u> Once the sacrificial layer is removed, the cantilever is free to expand to relieve the compressive stress to bring its average to zero
- The gradient will however remain
- This will therefore create a net {word} stress at top surface and a net compression stress at bottom surface
- The cantilever will bend upwards to decrease both these stresses.

Stress gradients in cantilevers (ctnd.)

- Neglecting transverse Poisson effects we can calculate the resulting bending by treating the built-in moment as being externally applied and using the result for the cantilever under bending moment

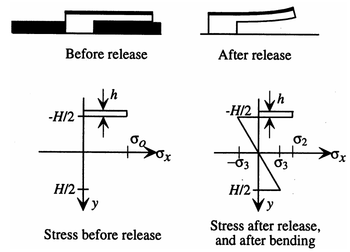
$$\frac{1}{\rho_x} = -\frac{M_x}{EI}$$

$$\rho_x = -\frac{EI}{M_x} = -\frac{1}{12}\frac{EWH^3}{M_x}$$

$$\rho_x = \frac{1}{2}\frac{EH}{\sigma_1}$$

- In order to take into account additional stiffness due to lateral Poisson effects one simply replaces $\frac{E}{1-\upsilon}$ in above results

Stress gradients in cantilevers (ctnd.)



Bending due to thin overlayer

- Imagine a cantilever with no residual stress upon which a thin stressed overlayer of thickness *h* is deposited
- Prior to release the thin overlayer possess a bi-axial tensile stress σ_0
- After release but prior to bending a biaxial relaxation would be expected to occur
- the average stress is zero
- However, given that the film is very thin compared to the beam we assume that even after release the stress in the films are approximated to remain the same

Stress gradients in cantilevers (ctnd.)

- Total moment about the centre of the beam (per unit volume)

$$M = \int_{\text{Total Thickness}} \sigma z\,dz$$

$$M = \int_{-H/2}^{H/2} \sigma_{beam} z\,dz + \int_{-\frac{H}{2}-h}^{-H/2} \sigma_{film} z\,dz$$

- with $\sigma_{beam} \approx 0$ and $\sigma_{film} = \sigma_0$ we obtain

$$M = \frac{\sigma_0}{2}\left[\frac{H^2}{4} - \frac{H^2}{4} + Hh - h^2\right]$$

$$M = \frac{\sigma_0 Hh}{2}$$

- Given the {word} nature of film we need to compute an effective EI product (also per unit width)

$$\left(\tilde{E}I\right)_{eff} = \int_{\text{Total Thickness}} \tilde{E} z^2 dz = \tilde{E}_1 \int_{-H/2}^{H/2} z^2 dz + \tilde{E}_0 \int_{-\frac{H}{2}-h}^{-H/2} z^2 dz$$

- where $\tilde{E}_1$ is biaxial modulus of beam and $\tilde{E}_0$ is biaxial modulus of film.

$$\left(\tilde{E}I\right)_{eff} = \frac{1}{12} H^2 \left(\tilde{E}_1 H + 3\tilde{E}_0 h\right)$$

- The radius of curvature is then fixed by using

$$\rho = \frac{\left(\tilde{E}I\right)_{eff}}{M}$$

Stress gradients in cantilevers (ctnd.)

- Numerical application:
- Consider a 100 μm long by 2 μm square silicon cantilever beam. A 10 nm film with tensile stress of 200 MPa and Young's modulus of 250 GPa is deposited on the cantilever. A Poisson ratio of $\upsilon = 0.3$ is assumed for both materials.
- Calculate the bending radius of curvature.

$$\tilde{E}_1 = \frac{E_{Si}}{1-\upsilon_{Si}} = \frac{160\times10^9}{1-0.3} = 228\times10^9\,\text{Pa}$$

$$\tilde{E}_0 = \frac{E_{film}}{1-\upsilon_{film}} = \frac{250\times10^9}{1-0.3} = 357\times10^9\,\text{Pa}$$

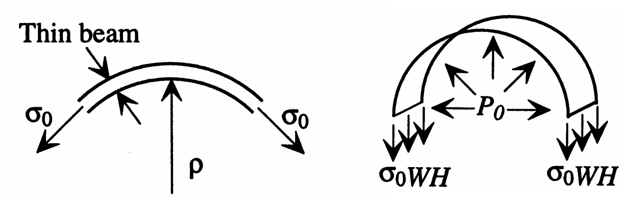
$$M = \frac{1}{2}\sigma_0 Hh = \frac{1}{2}(200\times10^6)(2\times10^{-6})(10\times10^{-9}) = 2\times10^{-6}$$

$$\left(\tilde{E}I\right)_{eff} = \frac{1}{12}H^2\left(\tilde{E}_1 H + 3\tilde{E}_0 h\right) = \frac{1}{12}(2\times10^{-6})^2[(228\times10^9)(2\times10^{-6}) + 3(357\times10^9)(10\times10^{-9})]$$

$$\left(\tilde{E}I\right)_{eff} = 1.55\times10^{-7}$$

$$\rho = \frac{\left(\tilde{E}I\right)_{eff}}{M} = \frac{1.55\times10^{-7}}{2\times10^{-6}} = 7.78\times10^{-2} = 78\,\text{mm}$$

Residual stresses in doubly supported beams



- Consider a doubly supported beam with axial stress σ_0 that has also been bent to a radius of curvature ρ (lefthand figure)
- Let's reduce this problem to extreme case where beam has been bent by 90° (righthand figure)

Residual stresses in doubly supported beams

- The geometry of problem allows to postulate that an effective uniform pressure load exist in this structure
- This pressure cancels out the downward force that induces axial tension
- The downward vertical force due to σ_0 is $2\sigma_0 WH$
- Under static equilibrium, the net force is zero:

$$F_{\sigma_0} + F_{P_0} = 0$$

- We therefore have:

$$F_{P_0} = -2\rho W P_0$$

- Since:

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}$$

- We get:

$$P_0 = \sigma_0 H \frac{d^2w}{dx^2}$$

- Also, since the distributed load associated to the pressure P_0 is given by $q_0 = P_0 W$ we get:

$$q_0 = \sigma_0 WH \frac{d^2w}{dx^2}$$

Residual stresses in doubly supported beams

- We substitute this into our differential equation

$$\frac{d^4w}{dx^4} = \frac{q_{Tot}}{EI}$$

- to get

$$EI\frac{d^4w}{dx^4} = q + q_0$$

- where $q_{Tot} = q + q_0$ and q is an externally applied load
- substituting q_0 above, we get

$$EI\frac{d^4w}{dx^4} - (\sigma_0 WH)\frac{d^2w}{dx^2} = q$$ The Euler Beam Equation

- Lets solve this equation <u>without</u> the uniaxial residual stress

$$EI\frac{d^4w}{dx^4} = q$$

- with

$$w(0) = 0 \qquad \left.\frac{dw}{dx}\right|_0 = 0 \qquad w(L) = 0 \qquad \left.\frac{dw}{dx}\right|_L = 0$$

- Trial solution is:

$$w = Cx^2 + Dx^3 + Fx^4$$

Residual stresses in doubly supported beams

- Substituting and applying boundary conditions we get

$$w = \frac{x^2(L^2 - 2Lx + x^2)q}{2EWH^3}$$

- Maximum deflection is at $x = L/2$:

$$w_{max} = \frac{L^4 q}{32EWH^3} \qquad \kappa_{stress\text{-}free} = \frac{qL}{w_{max}} = \frac{32EWH^3}{L^3}$$

- Lets now solve in presence of axial tension in beam $N = \sigma_0 WH$

$$EI\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = q$$

- Trial solution is:

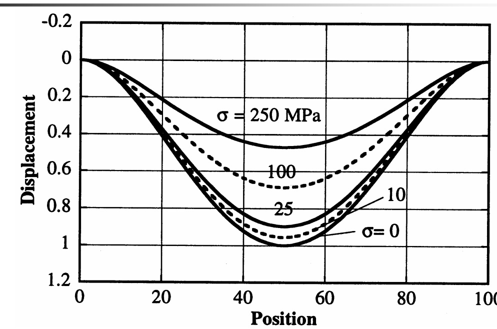
$$w = A + C\left(x - \frac{L}{2}\right)^2 + D\cosh\left[\kappa_0\left(x - \frac{L}{2}\right)\right]$$

- With:

$$\kappa_0 = \sqrt{\frac{12N}{EWH^3}} \qquad A = \frac{qL}{4N}\left[\frac{L}{2} - \frac{2}{\kappa_0}\coth(\kappa_0 L/2)\right]$$

$$C = -\frac{q}{2N} \qquad D = \frac{qL}{2\kappa_0 N\sinh(\kappa_0 L/2)}$$

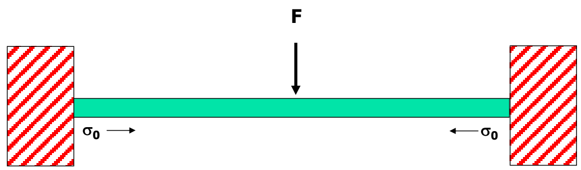
Residual stresses in doubly supported beams



- The maximum deflection in this case is

$$w_{max} = \frac{qL}{4N}\left(\frac{L}{2} - 2\frac{\cosh(\kappa_0 L/2) - 1}{\kappa_0\sinh(\kappa_0 L/2)}\right) \qquad \kappa_{with\text{-}stress} = \frac{qL}{w_{max}} = \frac{4N}{\frac{L}{2} - 2\frac{\cosh(\kappa_0 L/2) - 1}{\kappa_0\sinh(\kappa_0 L/2)}}$$

Buckling of beams



- We now consider the effects of compressive stress on doubly-clamped beams.
- We show that for a sufficiently large compressive stress the equilibrium position is no longer straight but buckled.
- The beam is subjected to a point load *F* in its middle.

Buckling of beams (ctnd.)

- Using Euler beam equation:

$$EI\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = F\delta(x)$$

- where $\delta(x)$ is a unit impulse function $\delta(0)=1$, $\delta(x)=0$ elsewhere.
- N is a tension and equal to $N = \sigma WH$, where σ is axial stress.
- We employ an eigenfunction approach to analyze this system.
- We define the eigenfunction $\psi_n(x)$ as the solutions to:

$$EI\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = \lambda_n\psi_n$$

- The eigen functions of this form are

$$\psi_n(x) = \sqrt{\frac{2}{L}}\cos(\kappa_n x)$$

- and

$$\lambda_n = EI\kappa_n^4 + N\kappa_n^2$$

- The complete trial solution consist of an eigen function {word} plus a solution of the homogeneous solution:

$$w(x) = A + Bx + \sum_{n=1}^{\infty} C_n\psi_n(x)$$

- Due to symmetry of problem B = 0 and above can be re-written as:

$$w(x) = \sum_{n=1}^{\infty}\left(A_n + C_n\psi_n(x)\right)$$

Buckling of beams (ctnd.)

- The boundary conditions impose:

$$A_n + C_n\psi_n(L/2) = 0$$

$$C_n\left.\frac{d\psi_n}{dx}\right|_{x=L/2} = 0$$

- First boundary condition is solved using

$$A_n = -C_n\sqrt{\frac{2}{L}}\cos(\kappa_n L/2)$$

- Second condition forces:

$$\left.\frac{d\psi_n}{dx}\right|_{x=L/2} = 0 \qquad \kappa_n\sin(\kappa_n L/2) = 0$$

- Thus

$$\frac{\kappa_n L}{2} = n\pi$$

- Thus

$$w(x) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}C_n\left[\cos\left(\frac{2n\pi x}{L}\right) - (-1)^n\right]$$

- To calculate C_n we substitute back into differential equation multiply each side by an arbitrary ψ_n integrate and employ expansion of eigenfuntion to find:

$$C_n = \frac{F}{(EI\kappa_n^4 + N\kappa_n^2)}$$

Buckling of beams (ctnd.)

- The solution becomes,

$$w(x) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{L}}\frac{F}{(EI\kappa_n^4 + N\kappa_n^2)}\left[\cos\left(\frac{2n\pi x}{L}\right) - (-1)^n\right]$$

- The maximum deflection occurs at *x* = 0

$$w_{max} = \sum_{n=1,odd}^{\infty}\sqrt{\frac{2}{L}}\frac{2F}{(EI\kappa_n^4 + N\kappa_n^2)}$$

- The related spring constant is:

$$\kappa_{beam} = \frac{1}{\sum_{n=1,odd}^{\infty}\sqrt{\frac{2}{L}}\frac{2}{(EI\kappa_n^4 + N\kappa_n^2)}}$$

- The denominator vanishes for first term (*n*=1)

$$N = -\frac{\pi^2}{3}\frac{EWH^3}{L^2}$$

- Corresponding to a critical value of stress called the Euler buckling limit:

$$\sigma_{euler} = -\frac{\pi^2}{3}\frac{EH^2}{L^2}$$