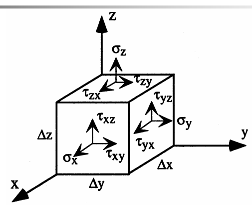
8. Elasticity

Introduction

- What is an elastic material? It's a material that possess the ability to deform in response to applied forces and recover their original shape when removed.
- This section will introduce concepts and definitions related to elastic behavior, while chapter 9 will focus on analysis of specific structures such as beams, plates and membranes.

Constitutive Equations of Linear Elasticity

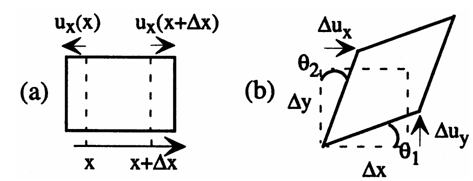


Stress

- Definition: Stress is defined as the force per unit area acting on a differential volume element of a solid body.
- Stresses perpendicular to a differential face are called "normal" stresses and are noted by σ_i.
- Stresses parallel to a differential face are called "shear" stresses and are noted by τ_{ij}.
- Under static equilibrium we must have no net torque, thus:

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{xz} = \tau_{zx}$$
$$\tau_{yz} = \tau_{zy}$$

Constitutive Equations of Linear Elasticity (ctnd.)



Strain

- Definition: Strain is the change of length per unit length that results from the applied stresses.
- Two types of strain: ***uniaxial strain*** (left) and ***shear strain (right)***.

Uniaxial Strain (above left)

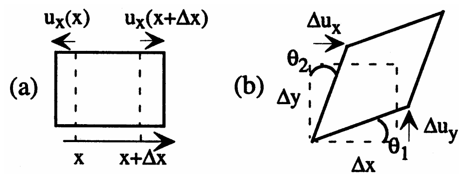
- The change of length is:

$$\Delta L_x = u_x(x+\Delta x) - u_x(x)$$

- The axial strain at point x in this example is: (note that u(x) in figure above is actually negative)

$$\varepsilon_x = \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} = \frac{\partial u_x}{\partial x}$$

Constitutive Equations of Linear Elasticity (ctnd.)



Shear Strain (above right)

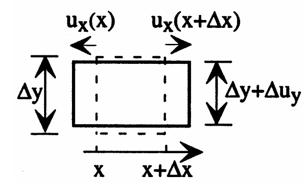
- **Shear strain is defined as:**

$$\gamma_{xy} = \left(\frac{\Delta u_x}{\Delta y} + \frac{\Delta u_y}{\Delta x}\right) = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

- **In the above example, it is equal to (for small angles) :**

$$\gamma_{xy} \approx \theta_1 + \theta_2$$

Elastic Constants for Isotropic Materials (ctnd.)



- **Young's Modulus**

$$\sigma_x = E\varepsilon_x$$

(units are Pascals)

- **Poisson Ratio**

$$\varepsilon_x = \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} = -\upsilon\varepsilon_y$$

$$\varepsilon_y = \frac{u_y(y+\Delta y) - u_y(y)}{\Delta y} = \frac{\Delta u_y}{\Delta y}$$

Elastic Constants for Isotropic Materials (ctnd.)

- **Now consider a three dimensional object subjected to a uniaxial stress σ_x.**
- **In this case, the dimensions of the deformed volume element are:**

$$\Delta x \rightarrow \Delta x(1+\varepsilon_x)$$
$$\Delta y \rightarrow \Delta y(1-\upsilon\varepsilon_x)$$
$$\Delta z \rightarrow \Delta z(1-\upsilon\varepsilon_x)$$

- **Thus the change in volume ΔV is:**

$$\Delta V = \Delta x\Delta y\Delta z(1-2\upsilon)\varepsilon_x$$

- **For small strain (i.e. neglecting terms in ε_x^2), we get**

$$\Delta V = \Delta x\Delta y\Delta z(1+\varepsilon_x)(1-\upsilon\varepsilon_x)^2 - \Delta x\Delta y\Delta z$$

- **Thus for υ = 0.5, there is no change in volume taking place.**

Elastic Constants for Isotropic Materials (ctnd.)

- **Shear Modulus**

- **There is also a linear relation between shear stress and resulting shear strain written in terms of the shear modulus noted by *G*:**

$$\tau_{xy} = G_{xy}\gamma_{xy}$$

- **It can be shown that this quantity is related to Young's modulus and Poisson ration through:**

$$G = \frac{E}{2(1+\upsilon)}$$

Isotropic Elasticity in Three Dimensions

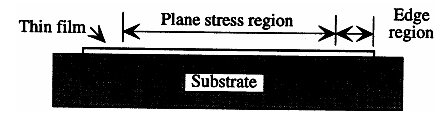
- **Combining previous definitions we get the following set of relationships between stress, strain as the elastic constants:**

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon(\sigma_y + \sigma_z)\right] \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon(\sigma_z + \sigma_x)\right] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon(\sigma_x + \sigma_y)\right] \qquad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

Plane Stress



- **Residual stress may exist in the thin film as a result of deposition process and thermal mismatch.**
- **The stresses are usually in-plane ($\sigma_z = 0$) and expressed through**

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \upsilon\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \upsilon\sigma_x)$$

- **In the special case where the in-plane stress is equal in all directions ($\sigma_x=\sigma_y=\sigma$, $\varepsilon_x=\varepsilon_y=\varepsilon$)**

$$\varepsilon = \frac{\sigma}{E}(1-\upsilon)$$

$$\sigma = \left(\frac{E}{1-\upsilon}\right)\varepsilon$$

- **The quantity $E/(1-\upsilon)$ is defined as "biaxial modulus".**

Elastic Constants for Anisotropic Materials

- **Until now we have assumed an isotropic material where elastic constants are equal regardless of directions**
- **In anisotropic materials (such as crystals) these moduli are direction dependent and express in matrix form**

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}$$

Elastic Constants for Anisotropic Materials (ctnd)

- **In the case of cubic crystals this matrix only contains three independent quantities and reduces to**

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}$$

- **where C_{ij} are stiffness coefficients. For silicon:**

$$C_{11} = 166\,GPa$$
$$C_{12} = 64\,GPa$$
$$C_{44} = 80\,GPa$$

- **In a more compact notation:**

$$\sigma_I = \sum_J C_{IJ}\varepsilon_J$$

$$\varepsilon_I = \sum_J S_{IJ}\sigma_J$$

Thermal Expansion and Thin-Film Stress

- **Thermal expansion refers to the tendency of free bodies to expand when heated.**
- **The linear thermal expansion coefficient is defined as:**

$$\alpha_T = \frac{d\varepsilon_x}{dT}$$

- **where T is temperature.**
- **The units for α_T are K^{-1} and tends to be in the 10^{-6} to 10^{-7} range.**
- **We write:**

$$\varepsilon_x(T) = \varepsilon_x(T_0) + \alpha_T \Delta T \quad \text{where} \quad \Delta T = T - T_0$$

- **Consider a thin film deposited onto a substrate at a deposition temperature T_d Assume the film is stress free at that temperature. The sample is then cooled down to room temperature T_r Since substrate is much thicker than film it will contract according to its own thermal expansion coefficient and force the thin film to contract with it.**

Thermal Expansion and Thin-Film Stress (ctnd.)

- **The thermal strain in the substrate (in one in-plane dimension) is:**

$$\varepsilon_s = -\alpha_{Ts} \Delta T$$

where:

$$\Delta T = T_d - T_r$$

- **If the thin film were not attached to the substrate it would experience a thermal strain given by:**

$$\varepsilon_{f,free} = -\alpha_{Tf} \Delta T$$

- **However, because it is attached the actual strain in the film must be equal to that of the substrate:**

$$\varepsilon_{f,attached} = -\alpha_{Ts} \Delta T$$

- **The thermal mismatch strain is given by:**

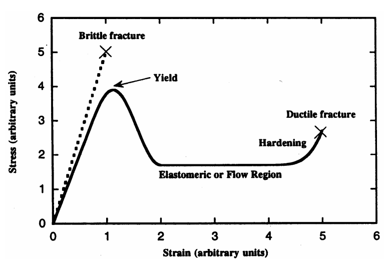
$$\varepsilon_{f,mismatch} = \varepsilon_{f,attached} - \varepsilon_{f,free} = (\alpha_{Tf} - \alpha_{Ts}) \Delta T$$

the film will therefore develop a bi-axial residual stress of:

$$\sigma_{f,residual} = \left(\frac{E}{1-\upsilon}\right) \varepsilon_{f,mismatch}$$

Material Under Large Strain

Ductile versus Brittle Behavior



- **Brittle material (dashed line) ≡ elastic deformation until material fails**
- **Ductile material (solid line) ≡ - elastic deformation until yield and flow of the material**
  - ductile fracture
- **The amount of energy required to break is greater in ductile materials. Such materials are therefore "tougher" than brittle materials**

Plastic Deformation

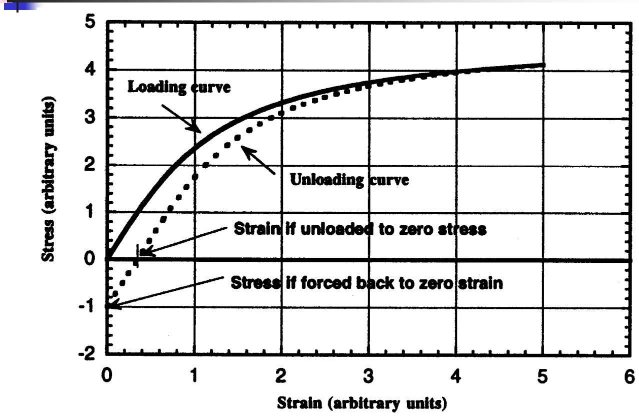


Figure 8.8. Illustrating plastic deformation.