7. Lumped-Element System Dynamics

Introduction

- The spring-mass-dashpot system we encountered in the position-control system of Section 2.3 was linear and was described by the following set of state equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}F$$

- Where x_1 is the position of the object, x_2 is the velocity of the object, m is the mass, k the spring constant, and b the damping constant.
- We have also encountered a *nonlinear* system, the voltage-controlled parallel-plate electrostatic actuator of Section 6.4. The three state equations were:

$$\dot{x}_1 = \frac{1}{R}\left(V_{in} - \frac{x_1 x_2}{\varepsilon A}\right)$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -\frac{1}{m}\left(\frac{x_1^2}{2\varepsilon A} + k(x_2 - g_0) + bx_3\right)$$

- Where x_1 is the capacitor charge, x_2 is the capacitor gap, x_3 is the velocity of the moveable plate, ε is the permittivity of the air, A is the plate area, g_0 is the at-rest gap, and R is the source resistance of the (possibly time-dependent) voltage source V_{in}.

Introduction

- If a system is linear, a host of powerful and quite general analytical techniques are available: Laplace transform, s-plane analysis with poles and zeros, Fourier transform, convolution, superposition, and eigenfunction analysis, to name several.
- If the system is nonlinear, the approach becomes much more problem-specific, and, in general, much more difficult to bring to rigorous closure.
- In both types of systems, we shall see that direct numerical integration of the state equations is a very useful simulation method.
- An additional issue is whether or not the system involves feedback.
- Nonlinear systems with feedback can exhibit new and complex behavior, including sustained oscillations called *limit cycles.*

Linear System Dynamics

- A typical set of linear state equations can be written in compact matrix form,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- where x is a column vector of state variables, u is a column vector of system inputs, y is a column vector of outputs, and *A*, *B*, *C*, and *D* are the time-independent matrices that constitute the system.
- For the spring-mass-dashpot system already examined, there is one scalar input F, and the output is equal to x, The matrices for this case are, for the state equations,

$$A = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}$$

- And, for the output equations,

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linear System Dynamics

- If we want to have both the position, and the velocity available as system outputs, then the C-matrix becomes

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Equations in this form can always be developed from the equivalent circuit for a linear system.
- We will now examine a variety of ways of analyzing system behavior.

System Functions

- We can compute the single-sided Laplace transform of the state equations to obtain:

$$sX(s) - x(0) = AX(s) + BU(s)$$

- Where $X(s)$ is the Laplace transform of the vector $x(t)$, $x(0)$ is the initial state for the system, and $U(s)$ is the Laplace transform of the input. This leads directly to the solution for

$$X(s) = (sI - A)^{-1}[x(0) + BU(s)]$$

- We can split this system response in two parts. The first is the ***zero-input response*** $X_{zir}(s)$, whose source is the initial state with zero applied input, (also called the "natural response" or the "transient response", although these usages have a risk of imprecision). It is the first term in the expression above:

$$X_{zir}(s) = (sI - A)^{-1} x(0)$$

- The second term is the ***zero-state response*** $X_{zsr}(s)$ whose source is the applied inputs assuming one started in the zero state, $x(0) = 0$.

$$X_{zsr}(s) = (sI - A)^{-1} BU(s)$$

- Both parts of the response require that the matrix $(sI - A)$ be an invertible matrix; that is, it cannot be singular.

System Functions (ctnd.)

- However, as is well-known, there are isolated values of ***s*** which matrix will have a vanishing determinant, and hence will be singular.
- The s-values for which this occurs are said to be the ***natural frequencies*** of the system. The s-values are also called the ***poles*** of the system.
- A partial-fraction expansion of $X_{zir}(s)$ leads, via the inverse Laplace transform, to a sum of terms dependent on the initial state, each term having a characteristic time dependence of the form

$$e^{s_i t}$$

- Where s_i is one of the natural frequencies.
- We can write the Laplace transform of the system output as:

$$Y(s) = C[X_{zir}(s) + X_{zsr}(s)] + DU(s)$$

- In most systems we shall encounter, the D matrix is zero, which is assumed in the following.

System Functions (ctnd.)

- We can rewrite the zero-state response part of the output in the form

$$Y_{zsr}(s) = H(s)U(s)$$

- where

$$H(s) = C(sI - A)^{-1} B$$

- $H(s)$ is called the ***system function*** or ***transfer function*** for the system.
- A general component of $H(s)$ has the from:

$$H_{ij}(s) = \frac{Num_{ij}(s)}{Den(s)}$$

- Where $Num_{ij}(s)$ is a numerator polynomial and $Den(s)$ is the denominator polynomial.
- Once the poles and zeros of a transfer function are specified, the only remaining feature of $H(s)$ is an overall multiplicative constant for each entry that sets the correct units and magnitude.
- Thus, systems with the same pole-zero combinations have the same basic dynamic behavior.

System Functions (ctnd.)

- We now continue with our spring-mass-dashpot example. In this case, the outputs $Y(s)$ are the state variables themselves. The $H(s)$ matrix becomes:

$$H(s)=\begin{pmatrix}\dfrac{1}{ms^2+bs+k}\\ \dfrac{s}{ms^2+bs+k}\end{pmatrix}$$

- Where the first entry is the input-output response with F as the input and position $X_1(s)$ as the output, and the second entry is the input-output response with F as the input and velocity $X_2(s)$ as the response.

System Functions (ctnd.)

- We can also write $H(s)$ in a factored form that emphasizes its pole-zero structure:

$$H(s)=\begin{pmatrix}\dfrac{1}{m(s-s_1)(s-s_2)}\\ \dfrac{s}{m(s-s_1)(s-s_2)}\end{pmatrix}$$

- Where the two poles, s_1 and s_2 are given by:

$$s_{1,2}=-\frac{b}{2m}\pm\sqrt{\left(\frac{b}{2m}\right)^2-\frac{k}{m}}$$

- We see that this system has two poles, and the velocity response has a zero at $s=0$. The location of the poles depends on the *undamped resonant frequency* ω_0 , given by

$$\omega_0=\sqrt{\frac{k}{m}}$$

- And on the *damping constant* α , given by:

$$\alpha=\frac{b}{2m}$$

System Functions (ctnd.)

- We define the *quality factor* or Q of the system as :

$$Q=\frac{\omega_0}{2\alpha}=\frac{m\omega_0}{b}$$

- For this example, $Q=2$. This gives rise to a moderately *underdamped* step response which exhibits a few cycles of overshoot and ringing after the step
- When $\alpha=\omega_0$, the response is *critically damped*, with both poles occurring at $s_{1,2}=-\alpha$. The Q value of a critically damped system is ½.
- When a system is driven with an input of the form e^{st}, then the response consist of a sum of terms, one which is $H(s)e^{st}$, the remaining terms having time dependences determined by the poles of the system.
- The term that has the same e^{st} time dependence as the input is called the *forced response,* and, provided that the transient pole-related terms die out faster than the e^{st} input, the forced response is the persistent response
- We can use this fact to gain insight into the interpretation of a zero.
- The complex frequency $s=0$ correspond to a constant applied force, which will clearly outlast any transient.
- In contrast, the position forced response is finite at $s=0$ and has value F/K, just has one would expect from the mechanics of the problem.

Sinusoidal Steady State

- An important signal domain is the *sinusoidal steady state,* in which all inputs are sinusoids and all transients are presumed to have died out.
- The forced response must itself be sinusoidal, consisting of a sum of terms (or, as with the Fourier transform, a continuous superposition) at the various frequencies of the inputs. Specifically, for a sinusoidal input $u(t)$ given by:

$$u(t)=U_0\cos(\omega t)$$

- The sinusoidal-ready-state output $y_{sss}(t)$ is given by :

$$y_{sss}(t)=Y_0\cos(\omega t+\theta)$$

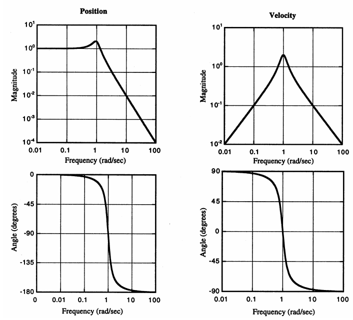
- where:

$$Y_0=|H(j\omega)|U_0$$

- And where the phase angle θ is given by

$$\tan\theta=\frac{\mathrm{Im}\{H(j\omega)\}}{\mathrm{Re}\{H(j\omega)\}}$$

Sinusoidal Steady State (ctnd)



- Graphs of Y_0 and θ versus ω are frequently used to show the full ***frequency response*** of a system in the sinusoidal steady state.
- When plotted on logarithmic axes for H and θ and a linear axis for ω, these plots are referred to as ***Bode plots.***
- Figure above shows the position and velocity Bode plots for the spring-mass-dashpot system with the values ***m***=1, ***k***=1, and ***b***=0.5.

Non-Linear Dynamics

- When the system is nonlinear, the most general form of the state and output equations is :

$$\dot{x} = f(x,u)$$

- and :

$$y = g(x,u)$$

- Where f and g are nonlinear functions of the state and inputs.
- The behavior of nonlinear system can be very complex, especially when system have nonlinearities that lead to so-called ***jump-phenomena*** between different states.
- We will examine only three specific issues using the parallel-plate electrostatic actuator as a common example:
  - Fixed-point analysis
  - Linearization about an operating point
  - Numerical integration of the state equations

Fixed Points of Nonlinear Systems

- The ***fixed points*** of a nonlinear systems are the solutions of :

$$f(x,u) = 0$$

- A ***global fixed-point*** is a fixed point of the system when all inputs are zero. It correspond to the system at rest.
- An ***operating point*** is a fixed point that is established by non-zero but constant inputs.
- When we refer to an operating point, we also imply a ***stable*** fixed point set up by constant inputs.
- We already performed fixed-point analysis for a nonlinear system in Section 6.4 when we examined the equilibrium points of the parallel-plate electrostatic actuator, whether charge-driven or voltage-driven.
- For the charge-driven case, we found a well-behaved single fixed point for the gap at each value of charge.
- For the voltage-driven case, we found two fixed points for voltages below pull-in, one which was stable, the other unstable, and no fixed points for voltages above pull-in.
- In the following section, we examine a general method that can, among other things, assist in the evaluation of fixed-point stability.

Linearization About an Operating Point

- In many system examples, we are interested in a small domain of state space near an operating point.
- Suppose that:

$$x(t) = X_0 + \delta x(t)$$

- And:

$$u(t) = U_0 + \delta u(t)$$

- Where X_0 and U_0 are constants (the operating point), and where, on an appropriate scale, both δx and δu are small. Substitution into the state equations followed by the use of Taylor's theorem leads to:

$$\delta\dot{x}(t) = \left(\frac{\partial f}{\partial x}\right)\bigg|_{X_0,U_0} \delta x(t) + \left(\frac{\partial f}{\partial u}\right)\bigg|_{X_0,U_0} \delta u(t)$$

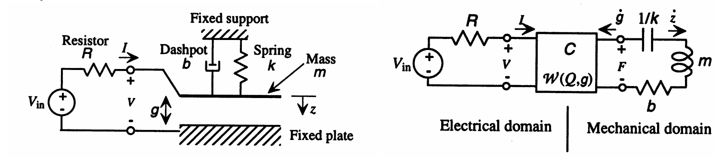
Linearization About an Operating Point

- The two matrices in this equation are the ***Jacobian's*** of the original function $f(x,u)$. In full matrix form, these equations becomes:

$$\begin{pmatrix} \delta\dot{x}_1 \\ : \\ \delta\dot{x}_n \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ : & & : \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{X_0,U_0} \begin{pmatrix} \delta x_1 \\ : \\ \delta x_n \end{pmatrix} + \left.\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ : & & : \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}\right|_{X_0,U_0} \begin{pmatrix} \delta u_1 \\ : \\ \delta u_n \end{pmatrix}$$

- Where the first Jacobian is an n X n matrix, where n is the number of state variables, and the second Jacobian is an m X m matrix, where m is the number of inputs.
- This is now a linear problem, which, for small amplitudes of input and output, can be analyzed with all the powerful tools available for linear system
- One side-benefit of linearization is that it makes the assessment of fixed-point stability quite easy. An example follows.

Linearization of the Electrostatic Actuator



- For the electrical domain:

$$\dot{Q} = I = \frac{1}{R}\left(V_{in} - \frac{Qg}{\varepsilon A}\right)$$

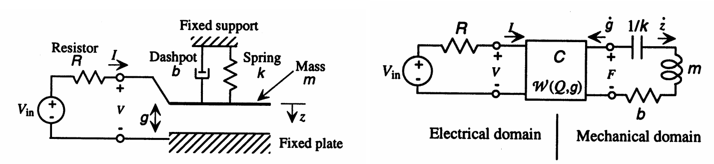
- And for the mechanical domain, using the force as:

$$F = \frac{Q^2}{2\varepsilon A}$$

- We obtain:

$$\frac{Q^2}{2\varepsilon A} + b\dot{g} + m\ddot{g} + k(g - g_0) = 0$$

Linearization of the Electrostatic Actuator (ctnd.)



- To get into a state form, we must identify three state variables. If we select as state variables:

$$x_1 = Q$$
$$x_2 = g$$
$$x_3 = \dot{g}$$

- The state equations become:

$$\dot{x}_1 = \frac{1}{R}\left(V_{in} - \frac{x_1 x_2}{\varepsilon A}\right)$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -\frac{1}{m}\left(\frac{x_1^2}{2\varepsilon A} + k(x_2 - g_0) + bx_3\right)$$

Linearization of the Electrostatic Actuator (ctnd.)

- The Jacobian J of the set of three state equations for this parallel-plate electrostatic actuator is therefore given by:

$$J = \begin{pmatrix} -\frac{X_2}{R\varepsilon A} & -\frac{X_1}{R\varepsilon A} & 0 \\ 0 & 0 & 1 \\ \frac{X_1}{m\varepsilon A} & -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

- Where operating point values for the charge X_1 and position X_2 have been substituted into the Jacobian.
- If we use some numerical values for the parameters, things simplify quite a lot.
- With the component values in Table 7.1, the state equations reduce to

Linearization of the Electrostatic Actuator (ctnd.)

- If we use some numerical values for the parameters, things simplify quite a lot.
- With the component values in Table 7.1, the state equations reduce to

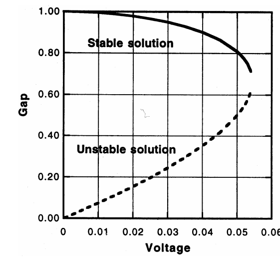
$$\dot{x}_1 = 1000V_{in} - 10x_1x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -\frac{1}{200}x_1^2 - x_2 - 0.5x_3 + 1$$

- The operating-point solutions are found by setting the three equations to zero, and solving.
- Mathematically, there are three possible solutions for the equilibrium charge X_1, the three roots of:
- The corresponding value of the equilibrium gap is then

$$X_1^3 - 200X_1 + 20000V_0 = 0$$

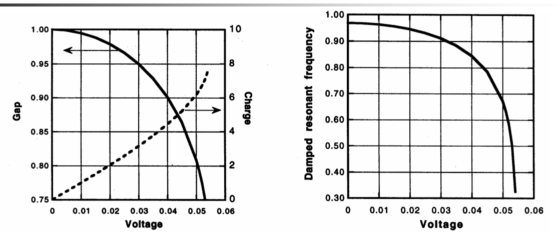
$$X_2 = 1 - \frac{1}{200}X_1^2$$

Linearization of the Electrostatic Actuator (ctnd.)



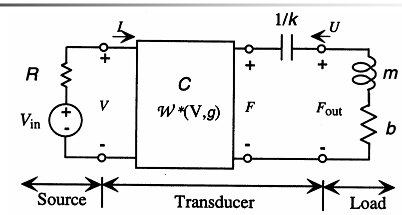
- One of the three roots turns out to be negative for positive values of V_0 and is discarded. The other two roots have positive values of both charge and gap. The locations of these three roots in the voltage-gap plane are plotted in above Figure.
- We recognize a pair of solutions whereas, with the values above $2g_0/3$ being stable and the values below $2g_0/3$ being unstable (corresponding to pull-in).
- Every point on the dashed curve, when substituted into the Jacobian, yields a right-half-plane natural frequency, indicating an unstable operating point.
- Every point on the solid curve yields a left-half-plane natural frequency, meaning that the set of operating points along the solid curve are locally stable.

Linearization of the Electrostatic Actuator (ctnd.)



- We can also use the linearized system to study the small-amplitude behavior about the operating point. In particular, it is very interesting to examine the small-amplitude damped resonant frequency for this system as a function of an applied operating-point DC voltage.
- Figure above shows, on the left, the equilibrium position and charge as a function of DC operating-point voltage and, on the right, the damped natural frequency for small amplitude vibration about the operating point.
- The damped resonance frequency is found as the imaginary part of the oscillatory natural frequencies obtained from the Jacobian evaluated at the operating point.

Transducer Model for Linearized Actuator

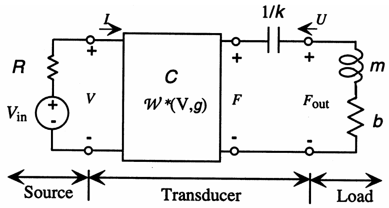


- With the concept of linearization in hand, we can now apply the linear transducer model of Section 6.6 to a linearized model of the electrostatic actuator.
- Figure 7.6 shows a redrawn version of the electrostatic actuator circuit of Fig. 6.9.
- For purposes of finding the operating point, this is a voltage-controlled system because, at the operating point, $I = 0$, and V must equal V_{in}.
- We recall from a previous section that for a voltage-controlled representation of the transducer

$$F = \frac{\varepsilon A V^2}{2g^2} \quad \text{and} \quad g = g_0 - \frac{\varepsilon A V^2}{2kg^2}$$

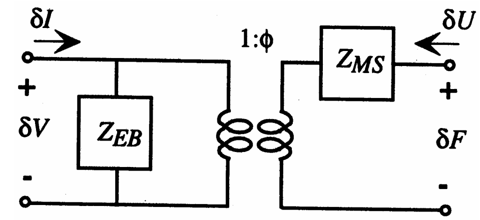
- where g_0 is the gap with no applied voltage.

Transducer Model for Linearized Actuator



- To linearize these equations, we can assume that V_{in} has a DC value $V_{in,0}$ plus some variation δV_{in} .
- The DC value sets the operating point of the nonlinear system while the effects of the variation in V_{in} are what will be captured in the linearized model.
- Using V equal to $V_{in,0}$, we can find the operating-point force, which we denote F_0 , and the operating gap, which we denote $\hat{g}_0$ to distinguish it from the gap with zero applied voltage.
- The operating-point output force $F_{out,0}$ is zero, since U must be zero.
- We now assume that we have solved this problem, and have the operating point values in hand.

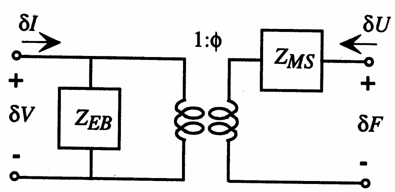
Transducer Model for Linearized Actuator (ctnd.)



- To linearize the model so it can be matched to the form of Fig. 7.7, we need expressions for δV and δF .
- It turns out to be easier to work with the energy formulation rather than the co-energy formulation.
- Therefore, we need one additional operating point value, the charge on the capacitor at the operating point, denoted Q_0 .
- It has the value:

$$Q_0 = \frac{\varepsilon A}{\hat{g}_0} V_{in,0}$$

Transducer Model for Linearized Actuator (ctnd.)



- The equations governing the charge-controlled representation of the transducer, now including both the electrostatic force and the force on the spring, are :

$$V = \frac{Qg}{\varepsilon A}$$

- And

$$F_{out} = \frac{Q^2}{2\varepsilon A} - k(g - g_0)$$

- To linearize these equations about the operating point, we form the Jacobian, and find:

$$\begin{pmatrix} \delta V \\ \delta F \end{pmatrix} = \begin{pmatrix} \frac{\hat{g}_0}{\varepsilon A} & \frac{Q_0}{\varepsilon A} \\ \frac{Q_0}{\varepsilon A} & k \end{pmatrix} \begin{pmatrix} \delta Q \\ \delta g \end{pmatrix}$$

Transducer Model for Linearized Actuator (ctnd.)

- But we have a problem. The linearized variables on the right-hand side are δQ and δg , whereas the variables we will need to form the linear transducer model are δI and δU .
- The relation between δQ and δI is simply integration. That is,

$$\delta Q = \int \delta I dt$$

- Since the linearized equations are now *linear*, we can use the Laplace transform to obtain

$$\delta Q = \frac{\delta I}{s}$$

- Similarly,

$$\delta g = \frac{\delta U}{s}$$

- Therefore, after substituting, the linearized model becomes:

$$\begin{pmatrix} \delta V \\ \delta F \end{pmatrix} = \begin{pmatrix} \frac{\hat{g}_0}{s\varepsilon A} & \frac{Q_0}{s\varepsilon A} \\ \frac{Q_0}{s\varepsilon A} & \frac{k}{s} \end{pmatrix} \begin{pmatrix} \delta I \\ \delta U \end{pmatrix}$$

- From these equations, we can identify the elements of the transducer model:

$$Z_{EB} = \frac{\hat{g}_0}{s\varepsilon A}$$

- This impedance is a capacitor whose value is that of the parallel-plate capacitance at the operating-point gap.

Transducer Model for Linearized Actuator (ctnd.)

- The other impedance in the model, Z_{MO} is given by:

$$Z_{MO} = \frac{k}{s}$$

- This is simply the capacitance that represents the spring. The coupling impedances are equal to each other, and have the value

$$T_{EM} = T_{ME} = \frac{Q_0}{s\varepsilon A}$$

- To form the equivalent of Fig. 7.7 we need to compute φ, which is given by

$$\varphi = \frac{T_{EM}}{Z_{EB}} = \frac{Q_0}{\hat{g}_0}$$

- The apparent units of φ are Coulombs/meter. These units are the same as the Newtons/Volt we would expect on the basis of a transformer that couples voltage to force.

Transducer Model for Linearized Actuator (ctnd.)

- We also need the electromechanical coupling constant k_e given by:

$$k_e^2 = \frac{T_{EM}^2}{Z_{EB} Z_{MO}} = \frac{Q_0^2}{\varepsilon A k \hat{g}_0}$$

- With the result that the final element in the model Z_{MS} can be found:

$$Z_{MS} = \frac{k}{s}\left(1 - \frac{Q_0^2}{\varepsilon A k \hat{g}_0}\right)$$

- We see that Z_{MS} represents a spring that at zero applied voltage (*i.e.* $Q_0 = 0$) has value k , but as voltage is applied, shifting the operating point, the value of the spring constant decreases.

Direct Integration of Steady State Equations

- When the state variables of a system undergo large variations, too large to represented accurately by a linearized model, it is necessary to use the full set of state equations to examine dynamic behavior.
- Nonlinear state equations can be integrated numerically, using, for example, the ode command in MATLAB.
- Alternatively, a system block-diagram can be constructed in SIMULINK, and simulations can be performed in that environment.
- We recall the three state equations for the parallel-plate electrostatic actuator:

$$\dot{x}_1 = \frac{1}{R}\left(V_{in} - \frac{x_1 x_2}{\varepsilon A}\right)$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -\frac{1}{m}\left(\frac{x_1^2}{2\varepsilon A} + k\left(x_2 - g_0\right) + b x_3\right)$$

- Where x_1 is the capacitor charge, x_2 is the capacitor gap, x_3 is the velocity of the moveable plate, ε is the permittivity of the air, A is the plate area, g_0 is the at-rest gap, and R is the source resistance of the time-dependent V_{in} voltage source. A SIMULINK model implementing these equations appears on next slide.

Direct Integration of Steady State Equations (ctnd.)

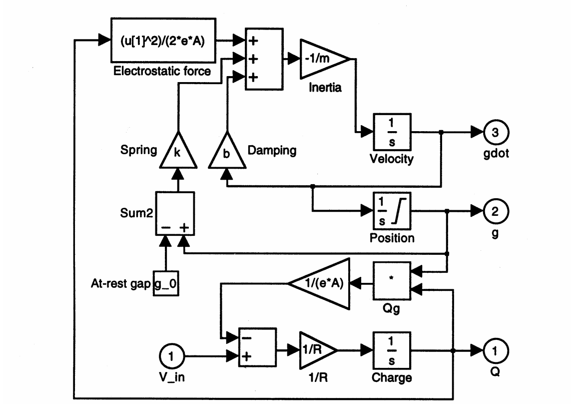


Figure 7.8. SIMULINK implementation of state equations for the parallel-plate electrostatic actuator. The labeling of the various blocks, including the use *g_0* to represent g_0, is characteristic of SIMULINK block diagrams.

Direct Integration of Steady State Equations (ctnd.)

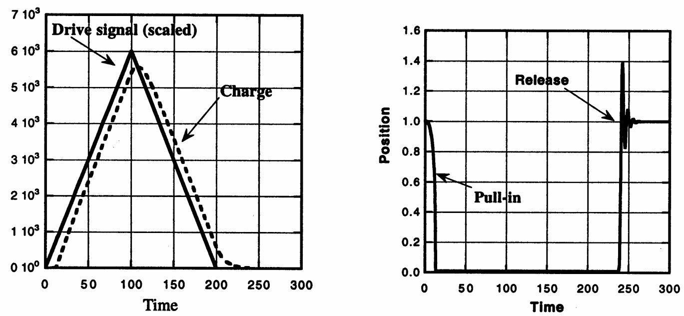


Figure 7.10. Dynamics of the electrostatic actuator with a single triangular pulse.

Direct Integration of Steady State Equations (ctnd.)

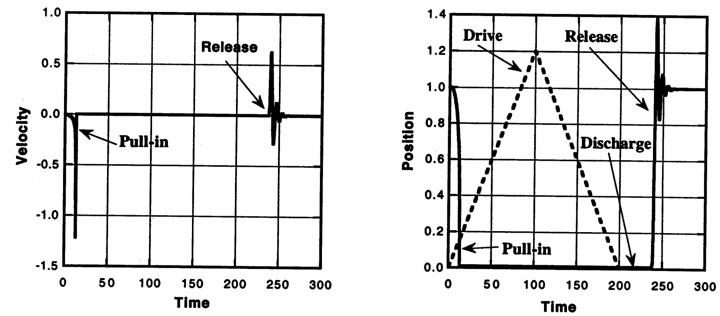


Figure 7.11. Velocity of the plate during the transient of Fig. 7.10, together with the gap and drive signal superimposed. Note the delay between the drive signal going to zero and the release of the plate due to the need to discharge the pulled-in capacitor through the source resistor.

Direct Integration of Steady State Equations (ctnd.)

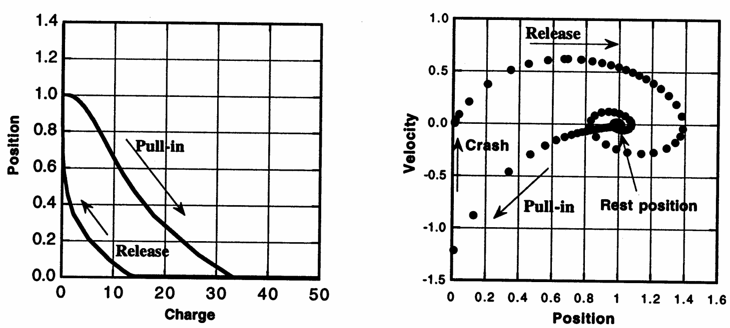


Figure 7.12. Phase-plane plots for the transient of Fig. 7.10.