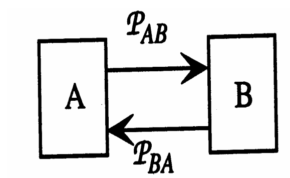
5. Lumped Modeling with Circuits Elements

Introduction

- MEMS devices exist in a three-dimensional physical continuum and their behavior is governed by the laws of physics, chemistry, and biology.
- Through analysis, we can extract simplified device representations that are readily expressible with equivalent electrics circuits.
- Circuit analogies also permit efficient modeling of the interaction between the electronic and the non-electronic components of a microsystem.
- Unlike 3D physical objects, which are bounded by surfaces, circuit elements are abstractions that have two or more discrete ***terminals*** to which potential difference (voltage) can be applied and into which electric currents can flow.
- Kirchhoff's Laws govern the relationships among the voltages and currents that must be satisfied when circuit elements are connected into complete circuits.

Conjugate Power Variables



- We define the ***power flow*** (***i.e.***, the energy flow per unit time) from ***A to B*** as P_{AB} and the reverse power flow from ***B*** to ***A*** as P_{BA} .

$$P_{AB} = r_1^2$$

$$P_{BA} = r_2^2$$

- The net power flow from ***A to B*** is the difference:

$$P_{Net} = P_{AB} - P_{BA}$$

- Inserting the appropriate squared term, we find:

$$P_{Net} = r_1^2 - r_2^2$$

- Which can be factored to yield:

$$P_{Net} = (r_1 + r_2)(r_1 - r_2)$$

Conjugate Power Variables (ctnd.)

- In the most general case, we define the two (time-dependent) conjugate power variables as an ***effort*** $e(t)$ and a flow $f(t)$.
- Associated with the flow is a time-dependent ***generalized displacement*** , $q(t)$, given by:

$$q(t) = \int_{t_0}^{t} f(t)dt + q(t_0)$$

- The dimension of the product $e \cdot f$ is power. Therefore, the dimension of the product $e \cdot q$ is energy.
- We can also define a time-dependent ***generalized momentum*** $p(t)$ given by:

$$p(t) = \int_{t_0}^{t} e(t)dt + p(t_0)$$

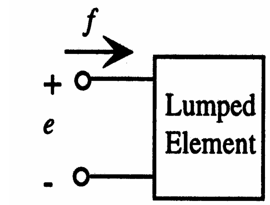
- The dimension of the product $p \cdot f$ is also energy.

Conjugate Power Variables (ctnd.)

Table 5.1. Examples of conjugate power variables.

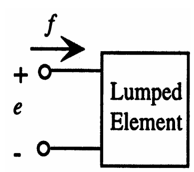
Energy Domain	Effort	Flow	Momentum	Displacement
Mechanical translation	Force F	Velocity $\dot{x}, v$	Momentum p	Position x
Fixed-axis rotation	Torque τ	Angular velocity ω	Angular momentum J	Angle θ
Electric circuits	Voltage V, v	Current I, i	...	Charge Q
Magnetic circuits	Magnetomotive force MMF	Flux rate $\dot{\phi}$	...	Flux ϕ
Incompressible fluid flow	Pressure P	Volumetric flow Q	Pressure momentum Γ	Volume V
Thermal	Temperature T	Entropy flow rate $\dot{S}$	...	Entropy S

One-Port Elements



- A *port* is a pair of terminals on a circuit element that must carry the same current *through* the element.
- In anticipation of generalizing this concept to other energy domains, we call the current the *through variable*.
- We call the voltage difference between the terminals the *across variable*.

One-Port Elements



- In electric circuits, the effort variable is the across variable (voltage), while the flow variable is the through variable (current).
- We will call this particular assignment the $e \rightarrow V$ convention because effort and voltage are linked.
- Standard sign conventions are useful to ensure that the algebraic sign of the power is calculated correctly.
- Once we assign a reference direction for positive flow into one of the terminals, we choose to use that same terminal as the positive terminal for defining effort.
- When we follow this convention, the product of e and f is the power *entering* the element.

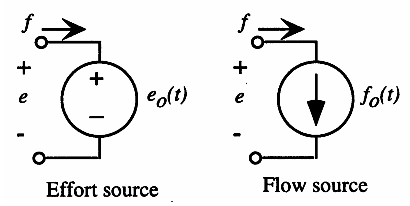
One-Port Elements

Table 5.2. Different conventions for assigning circuit variables.

Convention	Across Variable	Through Variable	Product	Principal Use
$e \rightarrow V$	e	f	power	electric circuit elements
$f \rightarrow V$	f	e	power	mechanical circuit elements
Thermal	T	$\dot{Q}$	Watt-Kelvin	thermal circuits
HDL	q	e	energy	HDL circuit representation of mechanical elements

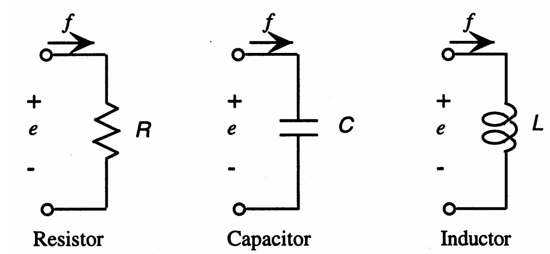
- We now wish to make an analogy to the other energy domains and, therefore, must decide which variable gets assigned as the through variable, and which as the across variable.
- As we will see, the $e \rightarrow V$ convention has the advantage that potential energy is *always* associated with energy storage in capacitors.
- In the mechanical energy domain, the $e \rightarrow V$ convention assigns force to voltage and velocity to current, making displacement analogous to electric charge.
- In the thermal energy domain, while temperature T is a perfectly good across variable, the entropy flow rate turns out not to be a good choice for the through variable. Instead, we use the heat current $\dot{Q}$, which has units of Watts.

One-Port Source Elements



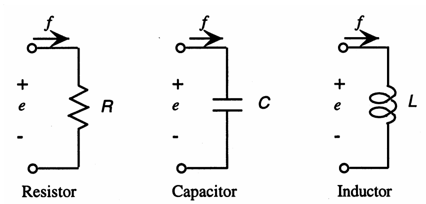
- Figure 5.3 shows two independent source elements, a ***flow source*** and an ***effort source*** (analogous to current and voltage sources in electrical circuits).
- The definition of the effort source is that the effort e equals the source value $e_0(t)$ for any flow f .
- The flow source (assuming it is connected to a network that provides a path for the flow) is defined as having a flow f equal to the source value $f_0(t)$ for any value of the effort e .
- These are clearly ***active*** elements, in that they supply power to the other elements whenever the product of f and e is negative.

One-Port Circuit Elements



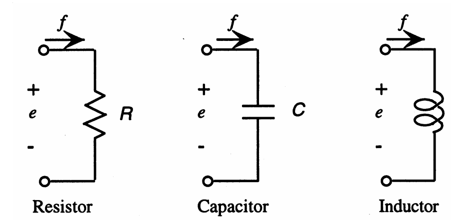
- There are three basic one-port circuit elements: the generalized resistor, which is a dissipative element, and two energy-storage elements: the generalized capacitor and the generalized inductor, also called an ***inertance.***

One-Port Circuit Elements



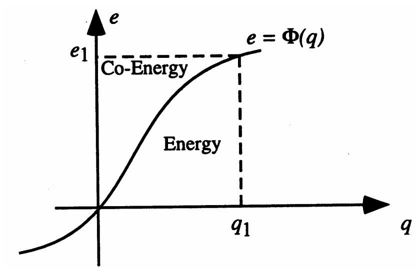
- The characteristics of the generalized resistor is defined directly in terms of f and e. That is, we write either e as a function of $[e = e(f)]$, or f as a function of $[f = f(e)]$.
- An important characteristic of these functions is that they go through the origin (if not, the element can be redefined as containing one of the source elements in a combination with a resistor whose characteristics does go through the origin).
- A second important characteristic is stated in terms of the quadrants of the plane the functions occupy.

One-Port Circuit Elements



- If the functions fall entirely in the first and third quadrants, so that the product ef is always positive, the element is a purely ***dissipative*** element.
- If the graph enters either the second or fourth quadrant at any point, the element can, over some portion of its characteristics, deliver net power to other elements. In this case, the element is considered an ***active*** element.

The Generalized Capacitor

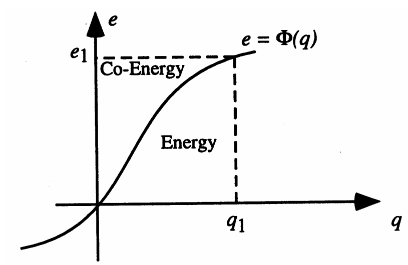


- The generalized capacitor is defined in terms of a relation between effort and displacement, that is, $e = \Phi(q)$ where Φ is a well-behaved function that go through the origin of the $e-q$ plane.
- Referring to Figure above, we can write

$$W(q_1) = \int_0^{q_1} e dq = \int_0^{q_1} \Phi(q) dp$$

- Where $W(q_1)$ is the ***stored potential energy*** in the capacitor defined by $\Phi(q)$ when it has displacement q_1 .

The Generalized Capacitor



- We will also use Figure 5.5 to define another energy quantity, the ***co-energy*** $W^*(e)$. The first definition we shall use is :

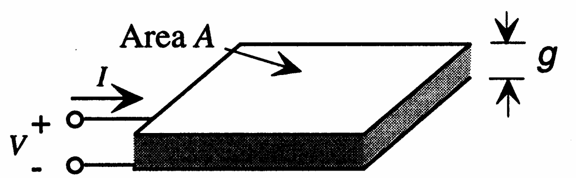
$$W^*(e) = eq - W(q)$$

- Based on this visualization, we can also write an integral definition:

$$W^*(e_1) = \int_0^{e_1} q de = \int_0^{e_1} \Phi^{-1}(e) de$$

- Where we have used the inverse of $\Phi(q)$ to represent the relation $q = \Phi^{-1}(e)$.

The Generalized Capacitor (ctnd.)



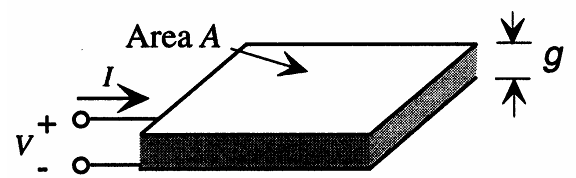
- An electrical example of a generalized capacitor is a parallel-plate capacitor, having area A and a medium of dielectric permittivity ε between the plates.
- In general, for a linear capacitor, we can write

$$Q = CV$$

- Where Q is the charge on the capacitor, V is the voltage across the capacitor, and C is the capacitance of the capacitor.
- The total charge on the plate is $\varepsilon AV/g$, and the capacitance is given by

$$W(Q) = \frac{Q^2}{2C} \qquad c = \frac{\varepsilon A}{g}$$

The Generalized Capacitor (ctnd.)



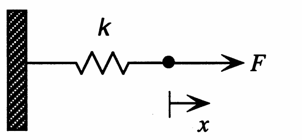
- The energy stored in the capacitor is :

$$W(Q) = \frac{Q^2}{2C}$$

- And the co-energy $W^*(V)$ is :

$$W^*(V) = \frac{CV^2}{2}$$

The Generalized Capacitor (ctnd.)



- We now examine a mechanical element, continuing to use the $e \to V$ convention.
- Figure above shows the schematic for a Hooke's Law spring, for which the force F is related to the displacement x by the linear spring constant k :

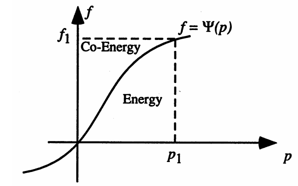
$$F = kx$$

- The stored energy in the spring when displaced by x_1 is :

$$W(x_1) = \int_0^{x_1} F(x)dx = \frac{1}{2}kx_1^2$$

- If we assume that the two generalized displacements, Q for the capacitor and X for the spring should lead to the same form of stored potential energy function, we are led to the conclusion that the circuit analog for a spring is a capacitor, with $C_{spring} = \frac{1}{k}$

The Generalized Inertance



- The generalized inertance (or generalized inductor), which is represented by the inductance symbol in Figure above, is defined by a functional relation between the flow f and the momentum p of the form $f = \Psi(p)$.
- The energy and co-energy are defined as :

$$W(p_1) = \int_0^{p_1} \Psi(p)dp$$

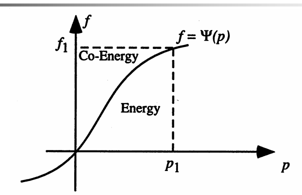
- And, in the general definition,

$$W^*(f) = pf - W(p)$$

- Leading to the integral definition for the co-energy :

$$W^*(f_1) = \int_0^{f_1} \Psi^1(f)df$$

The Generalized Inertance (ctnd.)



- In the $e \to V$ convention inertances are used for ***stored kinetic energy.***
- The most obvious example is an inertial mass, defined by the relation

$$p = mv$$

- Where p is the usual linear momentum, m is the mass, and v is the velocity. For this element, the function $\Psi(p)$ would be written

$$\Psi(p) = \frac{p}{m}$$

- The stored energy at a particular momentum p_1 would be :

$$W(p_1) = \frac{p_1^2}{2m}$$

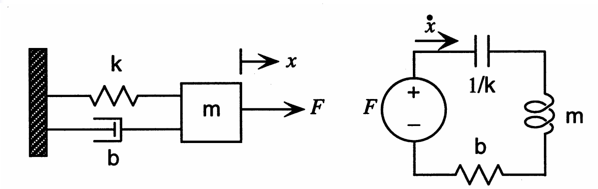
- While the co-energy $W^*(v_1)$ would be :

$$W^*(v_1) = \frac{mv_1^2}{2}$$

Circuit Connections in the e ->V Convention

- Perhaps the most difficult aspect of using lumped circuit elements to build device macro-models is to determine how to connect them together.
- We are all familiar with series and parallel connections for electrical elements.
- But how do these apply to mechanical or other elements?
- There are two basic concepts that hold for the $e \to V$ convention:
- **Shared Flow and Displacement**: Elements that share a common flow, and hence a common variation of displacement, are connected ***in series.***
- **Shared Effort:** Elements that share a common effort are connected ***in parallel.***

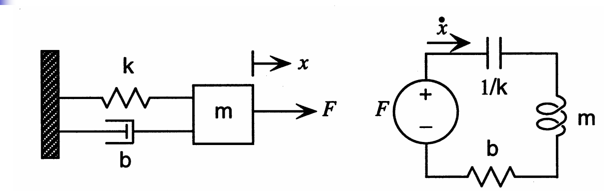
Circuit Connections… (ctnd.)



Spring-mass-dashpot system   Equivalent circuit

- Figure 5.9 shows a spring-mass-dashpot assembly, like that used to illustrate the object in a position-control system.
- The mass is connected via a spring to a fixed support, being pulled by a force F.
- Also shown is a *dashpot*, a mechanical damping element analogous to an electrical resistor.
- All three elements share the same displacement.
- Using the reasoning above, all three elements should be connected in series.

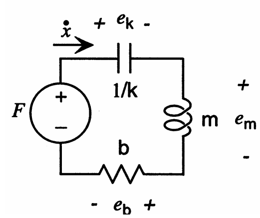
Kirchhoff's Laws



Spring-mass-dashpot system   Equivalent circuit

- The various circuit elements are connected at *nodes*, and are governed by generalizations of Kirchhoff's Laws.
- These are:
  - Kirchhoff's Current Law (KCL): The sum of all currents (flows) entering a node is zero.
  - Kirchhoff's Voltage Law (KVL): The oriented sum of all voltages (efforts) around any closed path is zero.

Kirchhoff's Laws (ctnd.)



- To use KVL, it is necessary to label the effort across each element, as in Figure 5.10.
- In this figure, the effort variable for each of the mechanical circuit elements is assigned the + sign at the terminal where the flow $\dot{x}$ enters the element.
- The KVL equation for this circuit then becomes:

$$-F + e_k + e_m + e_b = 0$$

- The physical interpretation of this equation is that the total applied force has three components: the part of the force that is accelerating the mass F, the part that is required to stretch the spring (e_m), and the part that is appears as a damping force in the dashpot (e_b).

Complex Impedances

- If complex impedances are used (the *s*-plane of the Laplace transform), each linear resistor is represented by itself, each linear capacitor by an impedance.

$$Z_C(s) = \frac{1}{sC}$$

- And each linear inductor by an impedance :

$$Z_L(s) = sL$$

- For each impedance, the relation between effort and flow is the same:

$$e(s) = Z(s)f(s)$$

- Thus, if $F(s)$ is the Laplace transform of the applied force, the KVL equation is written

$$F(s) = \left(sm + b + \frac{k}{s}\right)\dot{x}(s)$$

- Where $\dot{x}(s)$ is the Laplace transform of the velocity of the mass.

Complex Impedances

- This leads to the transfer function:

$$\frac{\dot{x}(s)}{F(s)} = \frac{s}{s^2 m + sb + k}$$

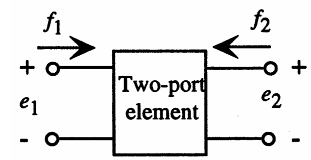
- When rationalized, this becomes:

$$\frac{\dot{x}(s)}{F(s)} = \frac{1}{sm + b + \frac{k}{s}}$$

- This transfer function has a pair of poles at:

$$s_1, s_2 = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

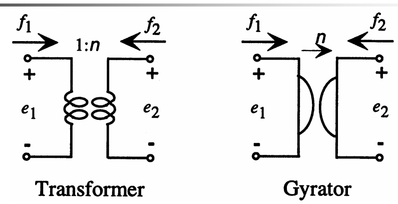
Transformers and Gyrators



- There are two important two-port elements that are used to aid in translating variables from one energy domain to another.
- These are the ***transformer*** and the ***gyrator.***
- In technical parlance, they are ***lossless*** and ***memoryless*** (***i.e.***, unlike capacitors and inductors, they contribute no state variables to the system).
- For a lossless two-port, the total power entering the element through both ports must be zero. Therefore, the element must satisfy the following equation:

$$e_1 f_1 + e_2 f_2 = 0$$

Transformers and Gyrators (ctnd.)



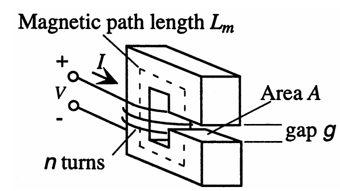
- There are two ways of constructing linear elements that satisfy this constraint. One is the ***transformer***; the second is the ***gyrator***.
- TRANSFORMER:

$$\begin{pmatrix} e_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & -\frac{1}{n} \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}$$

- GYRATOR:

$$\begin{pmatrix} e_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ -\frac{1}{n} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}$$

The Electrical Inductor



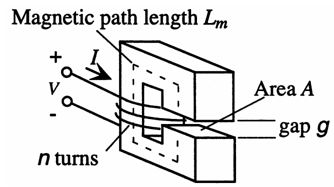
- Current in the coil establish a magnetic field which, because of the high permeability of the magnetic core, establishes flux ϕ in a ***magnetic circuit*** consisting of the core plus the air gap.
- The line integral of the magnetic field H around the magnetic circuit is the ***magnetomotive force,*** abbreviated MMF, and denoted by:

$$F_{MM} = H_\mu L_m + H_g g$$

- Where H_μ is the H-field in the permeable ring and H_g is the H-field in the gap.
- The flux through any cross-sectional surface S in the magnetic circuit is defined as:

$$\phi = \int_s B.dA$$

The Electrical Inductor (ctnd.)



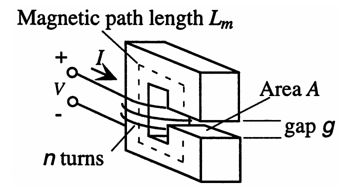
- Since the flux is continuous around the circuit, we conclude that $B_\mu = B_g$, or

$$H_\mu = \frac{\mu_0}{\mu} H_g$$

- Thus we can now solve for the flux density in the gap:

$$B_g = \mu_0 H_g = \left(\frac{\mu_0}{g + \frac{\mu_0}{\mu} L_m} \right) F_{MM}$$

The Electrical Inductor (ctnd.)

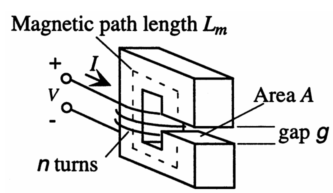


- From which we obtain the flux in the gap:

$$\phi = \left(\frac{\mu_0 A}{g + \frac{\mu_0}{\mu} L_m} \right) F_{MM}$$

- We recognize this as a relation between applied force F_{MM} and a generalized displacement, the flux ϕ . Thus, this element looks like a spring. It stores magnetic potential energy by establishing flux in response to an applied MMF.

The Electrical Inductor (ctnd.)



- The *reluctance* of a magnetic circuit is defined as the ratio of MMF to flux:

$$F_{MM} = \Re \phi$$

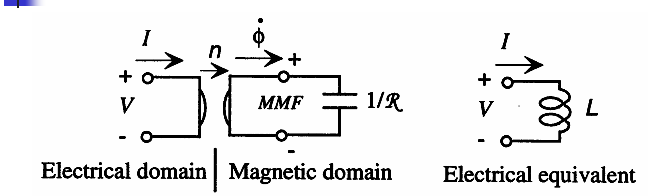
- Where $\Re$ is the reluctance, or, to put in the form analogous to the capacitor

$$\phi = \frac{1}{\Re} F_{MM}$$

- $\Re$ is directly analogous to the spring constant of a spring, and should be represented in the magnetic energy domain by a capacitor of value $1/\Re$, where

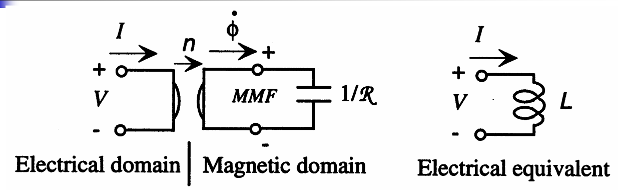
$$\frac{1}{\Re} = \frac{\mu_0 A}{g + \frac{\mu_0}{\mu} L_m}$$

The Electrical Inductor (ctnd.)



- So, *where is the inductor?* We thought this element was an inductor, not a capacitor. Yet it stores potential energy, like a capacitor.
- The answer lies in the choice of energy domain. We chose to model this inductor in the energy domain. But in a circuit, it must interact with real currents and voltages.
- The connection between the circuit domain and the magnetic domain is through a gyrator, as shown on the left-hand side of Fig. 5.16

The Electrical Inductor (ctnd.)



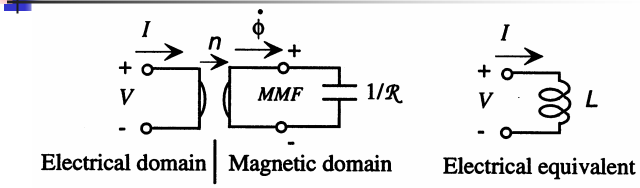
- The governing equations for the gyrator, taking note of the reference direction for $\dot{\phi}$ in the figure, are written:

$$\dot{\phi} = \frac{1}{n} V$$

$$F_{MM} = nI$$

- Where the gyrator parameter *n* is, in this case, the number of turns in the coil.
- The first gyrator equation is Faraday's Law of Induction, which says that the induced EMF in the coil is the time rate of change of the linked flux, while the second gyrator equation is just the definition of the MMF in terms of current derived in Appendix B of text.

The Electrical Inductor (ctnd.)



- It is readily shown from these two equations that

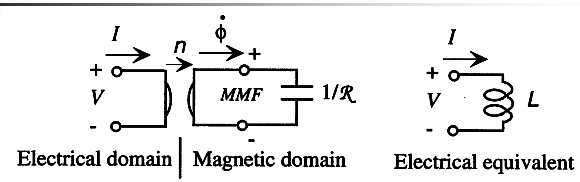
$$V = s\left(\frac{n^2}{R}\right) I$$

- Which is our familiar electrical inductor, with inductance *L* given by

$$L = \frac{n^2}{R}$$

- Thus, a completely equivalent representation is the simple electrical inductor shown on the right-hand side of Fig. 5.16.
- The stored energy in the inductor when carrying current *I* can be found by considering the stored potential energy in the capacitor. This energy is

The Electrical Inductor (ctnd.)



- The stored energy in the inductor when carrying current *I* can be found by considering the stored potential energy in the capacitor. This energy is

$$W_L = \frac{\phi^2}{2C} = \frac{\phi^2 R}{2}$$

- The corresponding co-energy is:

$$W_L^* = \frac{F_{MM}^2}{2R}$$

- If we use the gyrator relation between F_{MM} and *I*, we find that the co-energy is the familiar result:

$$W_L^* = \frac{1}{2} L I^2$$