1. Introduction
The purpose of this report is to summarize the work I have done in terms of an AR algorithm to track vortex shedding. Section 2 describes the advantages and disadvantages of the AR approach as compared to the DPLL approach. Section 3 describes the principal design issues involved in implementing the AR algorithm to track vortex shedding. Finally, Section 4 gives a brief summary and recommends how to proceed.

Since February, I have been developing a prototype of Tarek’s DPLL algorithm with the changes Professor Clarke made (e.g. removing heterodyning). A working Simulink prototype has been demonstrated on the xPC platform. I also have a C++ prototype that works in simulation (reading and writing data from and to the hard disk) on the VxWorks PC-104 platform. Once a data acquisition problem is solved, this prototype will be ready for experimentation. The DPLL algorithm does require more testing and refinement, particularly in terms of flow over a wide range of Reynolds number and in terms of the selection of parameter values. These may be addressed now that the rig has been configured to permit higher flow rates and to include the 1” vortex flow meter.

Because experiments on the rig and PC-104 prototyping involves coordination with other priorities of the group, I have had time to look at other things—principally, the use of autoregressive methods to track vortex shedding. Unlike the DPLL algorithm, which has been mostly a developmental effort for me due to previous work by Tarek and Professor Clarke, the AR algorithm has been one of research. Thus, there are some outstanding issues that need resolving before it would be ready for development as the signal processing core of a vortex flow meter. Nonetheless, I believe it is worth pursuing.

2. Autoregression versus dual phase-locked loop
An autoregressive (AR) model of order M represents the current output of a stochastic process by a linear combination of the M previous outputs plus a white noise process. Usually, the noise distribution is taken to be Gaussian with zero mean. Fitting an AR model consists of estimating the coefficients of the linear function and the variance of the noise from samples of the output. The AR model may then be used to estimate the power spectral density (PSD) of the stochastic process. In the case of vortex flow metering, the peaks in the PSD are used to find the vortex shedding frequency.

A phase-locked loop (PLL) is a feedback control loop that synchronizes the phase, and hence the frequency, of a locally-generated sinusoid with that of an external noisy sinusoid. The PLL consists of a phase detector, to estimate the phase difference between the sinusoids, a low-pass controller, to filter noise and achieve phase lock, and a programmable oscillator, to produce a clean sinusoid of known phase and frequency. The dual-PLL (DPLL) algorithm employs two PLLs, with different loop bandwidths, to track the vortex shedding frequency. One PLL is designed to track slowly-varying shedding frequency and the other is designed to track quickly-varying shedding frequency.
2.1. Windowing non-issues

In Chapter 3 of Tarek’s thesis on vortex signal processing, he notes “The problem we are dealing with is tracking a varying frequency sine wave signal.” He considers the advantages and disadvantages of various methods to do the same and says the following about autoregression: “AR modelling is a simple and reliable method of spectral estimation. It offers better frequency resolution than FFT spectral estimation. However, it suffers from the same problem [as FFT methods] of collecting N data points to estimate the spectral density, and poor performance in low SNR.”

The problems with AR modelling that Tarek mentions pertain to the Yule-Walker method, which windows the signal by assuming it is zero outside the given data record. Such windowing has drastic consequences in low SNR. The Burg, covariance and modified covariance methods do not window the signal. Of these, the Burg and modified covariance methods are better for short data records (relative to the frequencies of interest) and the latter is least affected by noise. Tarek also had the impression that AR modelling “requires large computations to obtain accurate estimates” but this is only true if very high order models are required, which is usually not the case.

One could argue that AR modelling of a signal, unlike tracking with a DPLL, uses a finite number of previous data points for spectral estimation. However, the bandwidths of the IIR filters in the DPLL imply an equivalence, given dithering caused by quantization error, to using a finite number of previous data points. Furthermore, Tarek wasn’t aware that recursive AR modelling is possible, which is IIR-type. Thus, windowing arguments do not favour the DPLL approach over the AR approach or vice versa.

2.2. Computational issues

The principal advantage of the AR method over the DPLL method is speed. The DPLL algorithm performs all of its calculations at the input frequency (e.g. 1 kHz for a 2” vortex flow meter). Once the vortex shedding frequency is estimated, it is decimated to the output frequency (e.g. 10 Hz for human users). The AR algorithm estimates the correlation matrix at the input frequency but performs the remaining calculations at the output frequency. Furthermore, the calculations at the input frequency are only multiply-accumulate operations, easily implemented in integer arithmetic.

With the AR method, computations at the input frequency take O(M²) operations and those at the output frequency take O(M³) operations. For some model order M, the AR algorithm would require more computation than the DPLL algorithm. Analysis of the vortex signal suggests a model order of six is sufficient to represent the dynamics. Though a detailed calculation has not been made, it appears that a model order greater than ten would be needed to exceed the computations required by the DPLL algorithm (not considering the use of integer arithmetic in either case).

Section 3.3 notes that it may be possible to perform input-related AR computations in O(M) time and output-related AR computations in O(M²) time. This would permit a much higher model order using the same resources as before or the same model order using much fewer resources than before.
2.3. *A priori* assumptions

The principal disadvantage of the AR method is that it makes *a priori* assumptions about the stochastic process to which it is applied. The first assumption it makes is that the process, on a short time scale, may be represented by an AR model of some order. The second assumption it makes concerns the order of the AR model. Some bias may be expected in the results of the method when either assumption is false.

On the first assumption, Haykin\(^1\) says “Within the family of linear stochastic models, the autoregressive (AR) model is often preferred over the moving average (MA) model and the autoregressive-moving-average (ARMA) model for an important reason: unlike an MA or ARMA model, computation of the AR coefficients is governed by a system of linear equations, namely, the Yule-Walker equations. Moreover, except for a predictable component, we may approximate a stationary discrete-time stochastic process by an AR model of sufficiently high order, subject to certain restrictions.”

According to Haykin, “Spectra computed using the parametric methods tend to have sharper peaks and higher resolution than those obtained from the nonparametric (classical) methods. The application of these parametric methods is therefore well suited for estimating the deterministic component [or the first moment] and, in particular, for locating the frequencies of periodic components in additive white noise when the signal-to-noise ratio is high.” The last phrase should be taken in the context that Haykin seems to define AR modelling by the Yule-Walker method.

The DPLL method is not free of assumptions. For instance, the algorithm has 27 parameters. The values of these parameters are defined by a combination of theory, experiment and intuition. One reason there are 27 parameters and not more or less is because there are two PLLs rather than some other quantity. It may prove that the assumptions on which the DPLL method is based are better than the assumptions on which the AR method is based but there is no *prima facie* case. An AR method of order two estimated the vortex frequency as well as the DPLL method on our rig.

2.4. *Model order mismatch*

Assuming the vortex signal is well represented by an AR model of order \(M_0\), the question arises what happens when the AR algorithm assumes an order \(M\) not equal to \(M_0\). If \(M_0\) is less than \(M\) the answer is one of numerical accuracy. It is easy to show that an AR model of order \(M\) is equivalent to an AR model of order \(M_0\) less than \(M\) by setting the trailing \(M\) minus \(M_0\) coefficients equal to zero. However, an algorithm will likely compute small nonzero values for these coefficients. These nonzero values may bias the PSD estimation, which is used to determine the vortex frequency.

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The problem of $M_0$ less than $M$ may be resolved by finding AR models for multiple orders from 1 to $M$ and selecting the ‘best’ one. The ‘best’ model, as Haykin says, is the one that minimizes a criterion such as Akaike’s Information Criterion or Rissanen’s Minimum Description Length, which have terms related to the modelling error and the number of parameters required. These criteria are easily computed. Section 3.1 explains that multiple order models may also be computed efficiently.

Suppose the vortex signal corresponds to an AR model of order $M_0$ but the AR algorithm computes multiple orders up to $M$ less than $M_0$. We can expect that Akaike’s and Rissanen’s information theoretic criteria would recommend the maximum order. The way the estimated model of order $M$ deviates from the actual model of order $M_0$ may be understood by recognizing the equivalence, according to Haykin, of the PSD estimated by the AR method and the maximum entropy method (MEM) of equal order.

“The basic idea of MEM is to choose the particular spectrum that corresponds to the most random or the most unpredictable time series whose autocorrelation function agrees with a set of known values. This condition is equivalent to an extrapolation of the autocorrelation function of the available time series by maximizing the entropy of the process” says Haykin. Using the AR method or MEM to estimate the PSD of a stochastic process to order $M$ involves an estimation of autocorrelations from lag 0 to lag $M$ and an assumption of maximum entropy to define the other autocorrelations.

In summary, the principle disadvantage of AR modelling is there will be a bias in the estimated PSD when either the process is unsuitable for AR modelling of any order or the required order is greater than the maximum order permitted by computational resources. Some insight may be found on the second point by noting an all-pole filter of order $M$ can have at most $M/2$, rounding down, resonant frequencies (which are defined coarsely by pairs of complex conjugate poles). Thus, the estimated PSD will be biased to some degree when the actual PSD has more than $M/2$ well-defined peaks.

### 2.5. Response to disturbance

Consider the AR and DPLL methods responding to a combination of a vortex sinusoid and a disturbance sinusoid. If the maximum order of the AR algorithm is two, there will be a constant bias in the vortex frequency estimated by the method. The bias is negligible if the disturbance amplitude is small or the disturbance frequency is near the vortex frequency. The DPLL method strongly filters the disturbance if the beat frequency is outside the bandwidth of the loop filter. However, when the beat frequency is small, the DPLL outputs an oscillating result. The mean value of this oscillation is unbiased so long as linearization conditions hold, which depends on the disturbance amplitude.

An AR algorithm with maximum order four could track two sinusoids without bias. If there is a second disturbance, bias would appear in the estimated PSD but it should be less than before because the incremental requirement of approximating a sixth order process by a fourth order one is less than that of approximating a fourth order process by a second order one (there is theory to support this). How will the DPLL method respond to two disturbances? Because the phase detector creates the product of each pair of the three frequencies, the beat frequency of each pair as compared to the bandwidth of the loop filter will determine the result, together with the disturbance amplitudes.
Thus, the accuracy of the AR method relies partly on the maximum order permitted, given limited computing power, with respect to the maximum number of disturbance frequencies expected. In the event of preamplifier saturation in the vortex meter, odd harmonics of the vortex frequency will appear in the output signal and some of these (as the amplitudes will decline) must be considered when accounting for disturbances. Low-pass pre-filtering, also used with the DPLL method, certainly helps with the AR method, especially for low Reynolds number. However, high-pass pre-filtering may be eliminated as the AR method can be formulated to reject the DC component.

3. Implementing the autoregressive method

Compared to the DPLL method, the principal advantage of the AR method is speed and the principal disadvantage is bias caused by model mismatch. Both of these factors are affected by the maximum model order. As the order increases, the impact of model mismatch decreases but the computational load increases. For the AR method to find a place in vortex signal processing, the computational load must be sufficiently minimized to permit reliable tracking under a wide range of conditions using a cheap processor.

3.1. Multiple order models

In his thesis, Rob Bowyer describes Niu et al’s algorithm to compute multiple orders from 1 to M for the covariance method without using more operations than it takes to solve the problem for only order M. Recently, I realized how to compute multiple orders $M, M-2, M-4, \ldots, 2$ or 1 (depending on whether M is even or odd) for the modified covariance method without using more operations than it takes to solve the problem for only order M. The advantage of the modified covariance method over the covariance method is that it has better performance in low signal-to-noise ratios. My algorithm also finds the former to be four times faster than the latter.

I exploited the persymmetry (symmetry across the anti-diagonal) of the correlation matrix estimated by the modified covariance method. Conventional algorithms, including the Matlab and Simulink ones, for both the covariance and modified covariance methods exploit only the symmetry. It seems impossible to compute a multiple order solution for the modified covariance method without exploiting the persymmetry. This may explain why a solution to this problem has not been published before (as far as I know).

Due to symmetries, a modified covariance AR problem of order M may be decomposed into two covariance AR sub-problems of order $(M-1)/2$, rounding up and rounding down respectively. Niu et al's method (with some refinements) may then be applied to find multiple order solutions of each sub-problem. The solutions of the sub-problems may be recombined to find a solution of the original problem. Because Niu et al's method takes $O(M^3)$ time (batch version), the number of operations needed for a modified covariance AR problem of order M is four times less than that needed for a covariance AR problem of order M. The overhead for decomposing and recombining proves negligible.

3.2. Pipelining of correlations

Apart from the maximum model order M, the computational requirements of the AR method depend on the number of samples L used to fit the model and the number of samples $L_0$ less than L that overlap from one data record to the next. The block length L
is chosen so that the data record includes one period at the minimum frequency of interest. For the 2” vortex meter, the minimum frequency is 1 Hz (corresponding to a Reynolds number of 2500), which means L is 1000 for a 1 kHz input frequency. Vortex shedding disappears at low Reynolds number (Tarek also did not track it below 1 Hz). The overlap length $L_0$ is chosen so that updates are available at the output rate. For the 2” vortex meter and a 10 Hz output frequency, $L_0$ equals 900.

Fitting an AR model of order M involves two steps: (1) computing a correlation matrix of size $M+1$ by $M+1$ and (2) computing the coefficients and noise variance for models up to order M. Done at the output frequency, the former takes $O(LM^2)$ operations and the latter takes $O(M^3)$ operations. For zero overlap, the correlation matrix may be accumulated at the input frequency using $O(M^2)$ operations. For nonzero overlap, let L and $L_0$ be chosen so L divides $L-L_0$ by an integer K (e.g. K equals 10 for L and $L_0$ equal to 1000 and 900). K gives the number of distinct correlation matrices to which each sample contributes, increasing the number of computations at the input frequency to $O(KM^2)$.

There are redundant calculations in the above approach for nonzero overlap, which may be exploited by pipelining. With pipelining, computations at the input frequency remain at $O(M^2)$ regardless of overlap. An extra $O(KM^2)$ operations are added at the output frequency. If integer calculations are used, which involve no rounding error, the extra calculations at the output frequency may be reduced to $O(M^2)$.

### 3.3. Exploiting Toeplitz structure

The possibility that the AR algorithm may be executed in $O(M)$ time at the input frequency and $O(M^2)$ time at the output frequency needs exploration because it would permit a much higher maximum order given the same computational resources. Fast algorithms for the single order problem have existed for decades (Haykin refers to papers by Marple from 1980/81) though they have not been implemented in Matlab or Simulink for the covariance and modified covariance methods. Recent papers have shown these algorithms to be numerically stable. What is not clear from the literature is whether the approach may be used to solve the multiple order problem. I suspect it is possible.

These fast algorithms rely on the fact that the correlation matrix in the covariance and modified covariance methods derive from the product of a Toeplitz matrix with itself. A Toeplitz matrix has equal elements on all diagonals running top-left to bottom-right. In the Yule-Walker method, the correlation matrix itself is Toeplitz, which may be exploited by the Levinson-Durbin recursion. The Burg method computes no correlation matrix but employs the Levinson-Durbin recursion. For sure, the Yule-Walker and Burg methods may be implemented in $O(M)$ time at the input frequency and $O(M^2)$ time at the output frequency. However, the covariance methods are better for spectral estimation.

### 3.4. Peak selection criteria

Once an AR model of order M is found to represent L samples of a vortex signal, the next step is to determine the vortex shedding frequency. As Section 3.5 shows, this is a simple problem to solve if we want the result with a fixed point precision of say 10 bits (which permits an accuracy of about 0.1% of full scale). It is a more difficult problem to solve if we want the result with a high fixed point or a floating point precision. In either case, the
shedding frequency is determined by finding the frequencies of all peaks in the PSD and choosing one according to some criterion.

The simplest selection criterion is to take the frequency of the highest peak. A better criterion is to select the peak that maximizes the PSD divided by frequency to the fourth power. Since the amplitude of the vortex process is proportional to its frequency squared, its power is proportional to the frequency squared twice. An even better selection criterion first compensates for the effects of pre-filtering on the PSDs of all peaks.

It is also necessary to establish a criterion to reject all peak frequencies when there is no vortex shedding. One that works well in practice, though more research is needed, is to dismiss all peaks with PSDs less than the PSD at zero frequency. The latter threshold does not depend on the DC component of the vortex signal, since the AR method is formulated to reject it, but on the average power of the AC component.

### 3.5. Peak finding methods

The PSD of the vortex process may be estimated by multiplying the squared amplitude response of the estimated AR filter of order $M$ and the estimated noise variance. The amplitude response of this all-pole IIR filter is the reciprocal of the amplitude response of its inverse, an all-zero FIR filter. The latter response, for $N$ angular frequencies from 0 to $\pi$ radians, equals the absolute value of the first half of a 2N-point FFT of the $M+1$ AR coefficients (i.e. the finite impulse response). Ordinarily, this FFT would take $O(N \log N)$ time, at the output frequency, but the substantial zero padding (assuming $N$ equals 1024 for 10-bit frequency resolution) may be exploited to reduce the time to $O(N \log M)$. It is possible to improve the performance a little further, since we are not interested in the phase information of the FFT, by using a discrete cosine transform (DCT).

The above is a poor way to achieve high fixed point or floating point precision since $N$ would be very large. Instead, note that peaks of the PSD occur at maxima of the squared amplitude response of the AR filter. Using Chebyshev theory, the maximization problem may be transformed into one of finding the roots of a real polynomial of degree $M−1$. Some roots may not correspond to maxima. Only those that are real and have an absolute value no greater than one are candidates. At most half the candidates, rounding up, may be maxima—the rest will be minima. Candidates may also correspond to inflections. Discriminating between maxima, minima and inflection points involves testing whether a related polynomial of degree $M−2$ is negative, positive or zero at the candidate roots. Once the maxima are found, it is easy to calculate the peak frequencies and PSDs.

Since analytic solutions exist for the roots of polynomials up to degree four, they may be employed for AR models up to order five. Beyond that, iterative solutions are required. One way to find the roots of a polynomial of degree $M−1$ is to create a companion matrix of size $M−1$ by $M−1$, which has eigenvalues equal to the polynomial roots. The real eigenvalues may then be found, with no complex arithmetic, by computing the Schur factorization of the companion matrix. This method takes $O(M^3)$ time, at the output frequency. Another method could first bracket the roots and then find them by iteration. For example, a DCT may be used to bracket the roots to 8-bit precision. One iteration of Newton-Rhapson’s algorithm should then suffice to double the precision to 16 bits.
4. Conclusion

The AR method to track vortex shedding uses AR modelling to estimate the PSD of the process, from which the shedding frequency is derived. The DPLL method uses two PLLs to lock a locally-generated sinusoid to the noisy vortex signal, thereby tracking the shedding frequency. On our rig, operating at low flow rates, the AR method of order two gave equally accurate results as the DPLL method using far fewer calculations. For now, both methods are worth pursuing. The DPLL algorithm works well and is at a further stage of development. The AR method permits a trade-off, according to model order, between the computational power needed and the ability to reject disturbance.

A number of tasks remain. The PC-104 prototype of the DPLL algorithm requires completion. Furthermore, its C++ implementation needs documentation. Ways to improve the DPLL algorithm may arise now that the rig has been configured to permit experiments Tarek was unable to do. The AR algorithm needs further documentation, starting with the multiple model order work. A C++ prototype of the AR algorithm for arbitrary order should be developed. At present, a Matlab prototype exists for arbitrary order and a Simulink prototype for the xPC platform exists for second order only. Further research on the AR algorithm will be required, especially to exploit Toeplitz structure.