Generalized Singular Value Decomposition for Coordinated Beamforming in MIMO systems

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Multiple Input Multiple Output (MIMO) Channel

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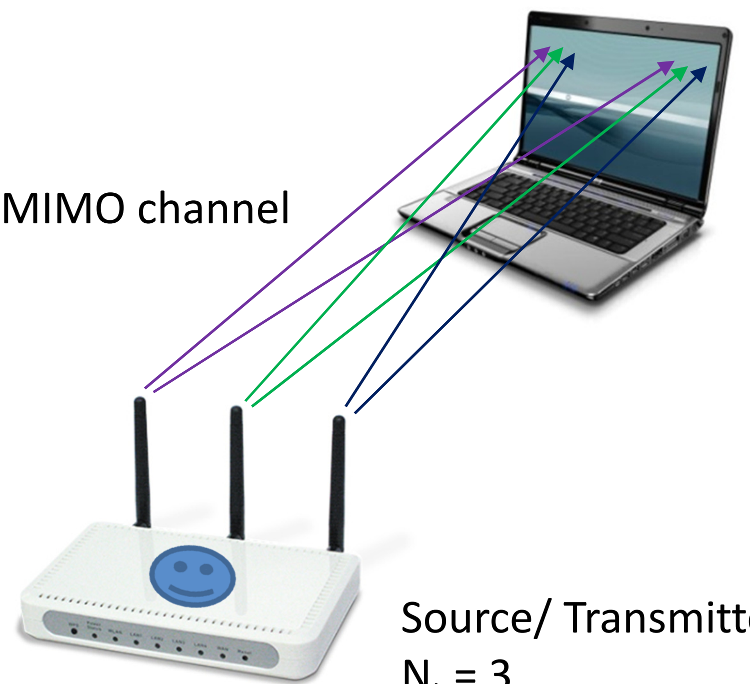
2 x 3 MIMO channel



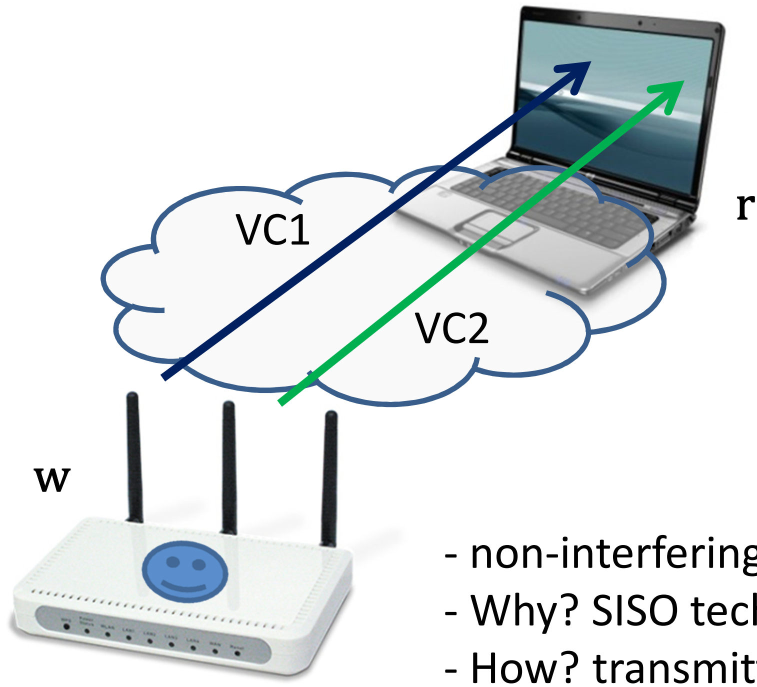
Destination/ Receiver
$N_r = 2$

Source/ Transmitter
$N_t = 3$

System Model: $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$

channel matrix $\boldsymbol{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix}_{2\times 3}$

Beamforming (+ Spatial Multiplexing)



- non-interfering, <u>point-to-point</u> virtual channels (VCs)
- Why? SISO techniques for modulation, coding...
- How? transmitter and/or receiver signal processing
    - transmit precoding matrix $\mathbf{w}$
    - receiver reconstruction matrix $\mathbf{r}$

$$\boldsymbol{y} = \boldsymbol{r}(\boldsymbol{H}\boldsymbol{w}\boldsymbol{x} + \boldsymbol{n})$$

$$\boldsymbol{rHw} = \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu} \end{bmatrix}_{2\times 2}$$

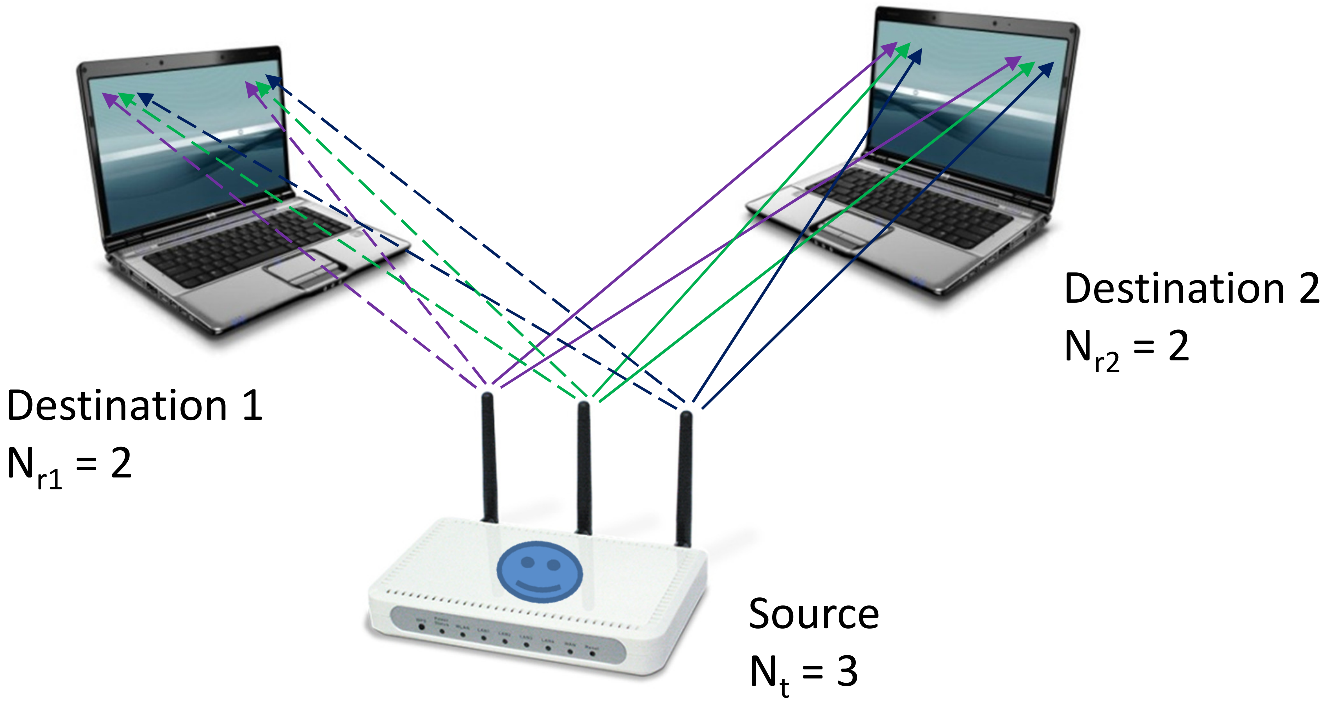
↑ VC1 ↑ VC2

<u>e.g.</u>

from linear algebra

- eigenmode transmission — SVD
- zero forcing — pseudo-inverse

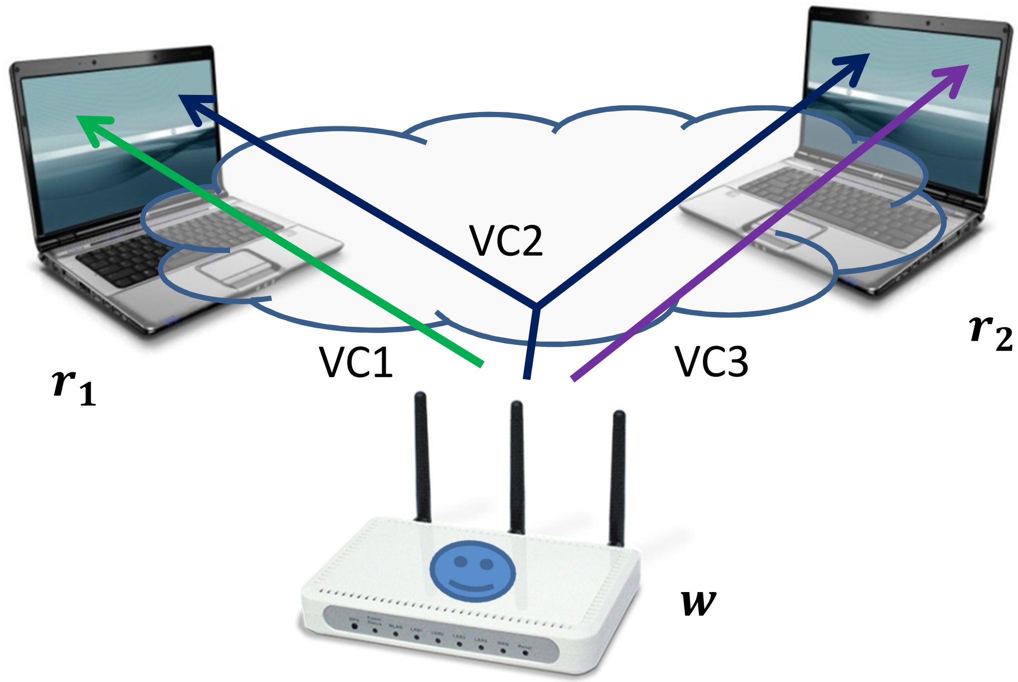
Two-User MIMO Channel



- Single Source, Two Destinations

$$\boldsymbol{y}_1 = \boldsymbol{H}_1 \boldsymbol{x} + \boldsymbol{n}_1$$
$$\boldsymbol{y}_2 = \boldsymbol{H}_2 \boldsymbol{x} + \boldsymbol{n}_2$$

Coordinated Beamforming



$$\boldsymbol{y}_1 = \boldsymbol{r}_1\boldsymbol{H}_1\boldsymbol{w}\boldsymbol{x} + \boldsymbol{n}_1$$
$$\boldsymbol{y}_2 = \boldsymbol{r}_2\boldsymbol{H}_2\boldsymbol{w}\boldsymbol{x} + \boldsymbol{n}_2$$

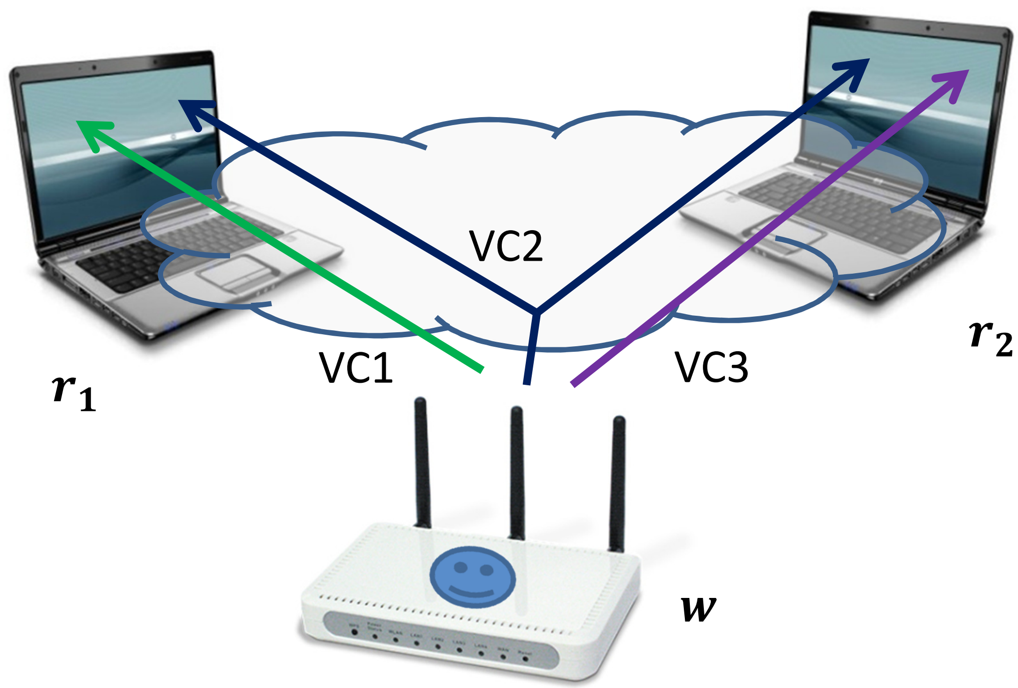
VC1, VC3 : point-to-point : private VCs
VC2 : point-to-2 point : common VC

$$\boldsymbol{r}_1\boldsymbol{H}_1\boldsymbol{w} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \mu_1 & 0 \end{bmatrix}_{2\times 3}$$

$$\boldsymbol{r}_2\boldsymbol{H}_2\boldsymbol{w} = \begin{bmatrix} 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}_{2\times 3}$$

VC1 **VC2** VC3

Coordinated Beamforming



GSVD-based Beamforming
- non-iterative
- coordinated b/f scheme
- for 2 users
- possibility hinted in **[Khisti2007]**

$$\boldsymbol{r}_1 \boldsymbol{H}_1 \boldsymbol{w} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \mu_1 & 0 \end{bmatrix}_{2\times 3}$$

$$\boldsymbol{r}_2 \boldsymbol{H}_2 \boldsymbol{w} = \begin{bmatrix} 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}_{2\times 3}$$

VC1 **VC2** VC3

Generalized Singular Value Decomposition (GSVD)

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- joint matrix decomposition technique

a restricted definition

- **Definition 1**: Van Loan form **[Loan1976]**

Consider two matrices, $\boldsymbol{H} \in \mathcal{C}^{m\times n}$ with $m \geq n$, and , $\boldsymbol{G} \in \mathcal{C}^{p\times n}$ having the same number $\boldsymbol{n}$ of columns. Let , $q = \min(p, n)$. $\boldsymbol{H}$ and $\boldsymbol{G}$ can be jointly decomposed as

$$\boldsymbol{H} = \boldsymbol{U\Sigma Q} \text{ and } \boldsymbol{G} = \boldsymbol{V\Lambda Q}$$

where

(i) $\boldsymbol{U} \in \mathcal{C}^{m\times m}, \boldsymbol{V} \in \mathcal{C}^{p\times p}$ are unitary,

(ii) $\boldsymbol{Q} \in \mathcal{C}^{n\times n}$ is non-singular,

(iii) $\boldsymbol{\Sigma} = \text{diag}(\sigma_1, \cdots, \sigma_n) \in \mathcal{C}^{m\times n}, \sigma_i \geq 0$, and

(iv) $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \cdots, \lambda_q) \in \mathcal{C}^{p\times n}, \lambda_i \geq 0$.

- use for b/f:

$$\mathbf{w} = \alpha \boldsymbol{Q}^{-1}, \boldsymbol{r}_1 = \boldsymbol{U}^H, \boldsymbol{r}_2 = \boldsymbol{V}^H \qquad (\boldsymbol{r}_1 \boldsymbol{H} \boldsymbol{w} = \alpha\boldsymbol{\Sigma}, \text{ and } \boldsymbol{r}_2 \boldsymbol{G} \boldsymbol{w} = \alpha\boldsymbol{\Lambda})$$

normalization coefficient

- **Definition 2**: Paige & Saunders form **[Paige1981]**

Consider two matrices, $\boldsymbol{H} \in \mathcal{C}^{m\times n}$ and $\boldsymbol{G} \in \mathcal{C}^{p\times n}$ having the same number *n* of columns. Let $\boldsymbol{C}^{\boldsymbol{H}} = [\boldsymbol{H}^{\boldsymbol{H}} \quad \boldsymbol{G}^{\boldsymbol{H}}] \in \mathcal{C}^{n\times(m+p)}$, t = rank($\boldsymbol{C}$), $\mathrm{r} = \mathrm{t} -$ rank($\boldsymbol{G}$), and s = rank($\boldsymbol{H}$) + rank($\boldsymbol{G}$) − t.

$\boldsymbol{H}$ and $\boldsymbol{G}$ can be jointly decomposed as

$$\boldsymbol{H} = \boldsymbol{U}[\boldsymbol{\Sigma} \quad \boldsymbol{0}_1]\boldsymbol{Q} = \boldsymbol{U\Sigma\widehat{Q}},$$
$$\boldsymbol{G} = \boldsymbol{V}[\boldsymbol{\Lambda} \quad \boldsymbol{0}_2]\boldsymbol{Q} = \boldsymbol{V\Lambda\widehat{Q}},$$

where $\widehat{\boldsymbol{Q}} = \{\boldsymbol{Q}\}_{R(t)}$ and

(i) $\boldsymbol{U} \in \mathcal{C}^{m\times m}, \boldsymbol{V} \in \mathcal{C}^{p\times p}$ are unitary,

(ii) $\boldsymbol{Q} \in \mathcal{C}^{n\times n}$ is non-singular,

(iii) $\boldsymbol{0_1} \in \mathcal{C}^{m\times(n-t)}, \boldsymbol{0_2} \in \mathcal{C}^{p\times(n-t)}$ are zero matrices

(iv) $\boldsymbol{\Sigma} \in \mathcal{C}^{m\times t}, \boldsymbol{\Lambda} \in \mathcal{C}^{p\times t}$ having structures

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{I_H} & & \\ & \widetilde{\boldsymbol{\Sigma}} & \\ & & \boldsymbol{0_H} \end{bmatrix}, \text{ and } \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{0_G} & & \\ & \widetilde{\boldsymbol{\Lambda}} & \\ & & \boldsymbol{I_G} \end{bmatrix}.$$

$\boldsymbol{I_H} \in \mathcal{C}^{r\times r}, \boldsymbol{I_G} \in \mathcal{C}^{(t-r-s)\times(t-r-s)}$ are unitary. $\boldsymbol{0_H} \in \mathcal{C}^{(m-r-s)\times(t-r-s)}, \boldsymbol{0_G} \in \mathcal{C}^{(p-t+r)\times r}$ are zero matrices possibly having no rows or no columns.

$\widetilde{\boldsymbol{\Sigma}} = \mathrm{diag}(\sigma_1, \cdots, \sigma_s), \widetilde{\boldsymbol{\Lambda}} = \mathrm{diag}(\lambda_1, \cdots, \lambda_s) \in \mathcal{C}^{s\times s}$

such that $1 \geq \sigma_1 \geq \cdots \geq \sigma_s \geq 0$ and ${\sigma_i}^2 + {\lambda_i}^2 = 1$.

- **Definition 2**: Paige & Saunders form **[Paige1981]**

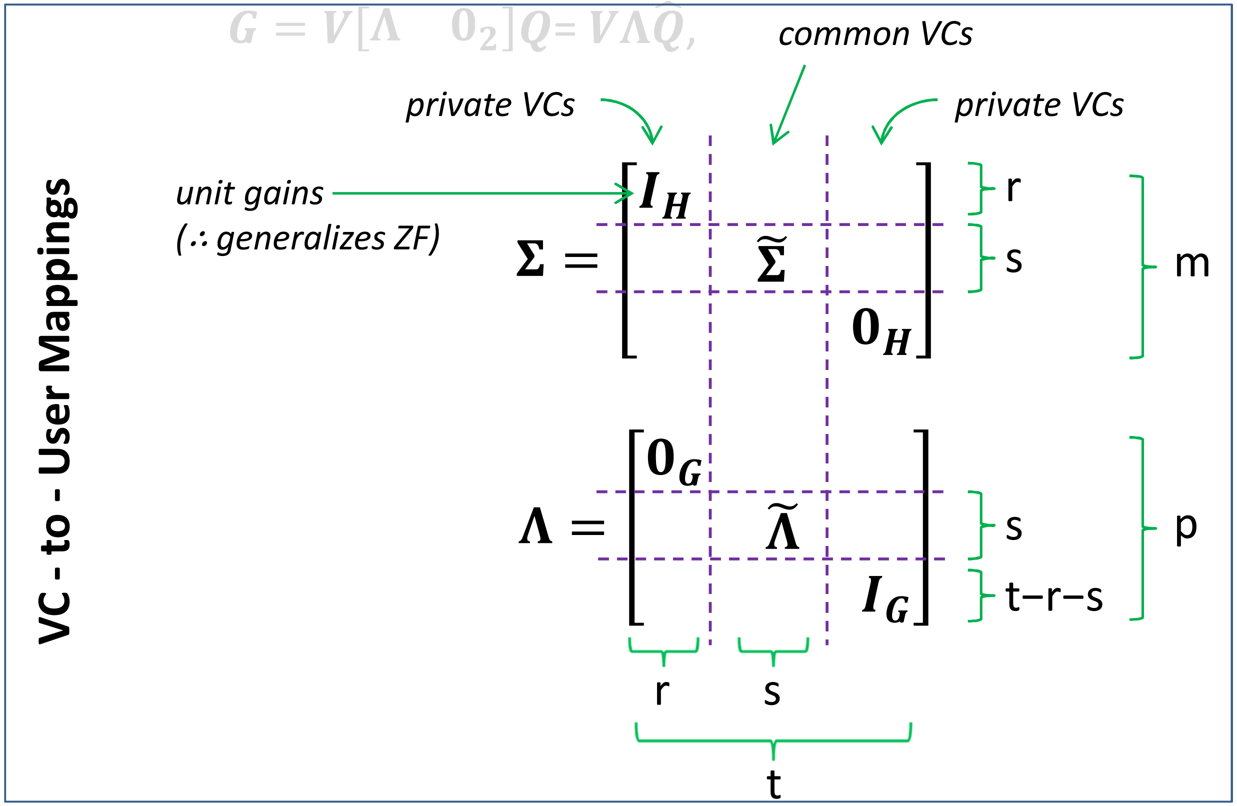
Consider two matrices, $H \in \mathcal{C}^{m\times n}$ and $G \in \mathcal{C}^{p\times n}$ having the same number n of columns. Let $C^H = [H^H \quad G^H] \in \mathcal{C}^{n\times(m+p)}$, t = rank($C$), r = t − rank($G$), and s = rank($H$) + rank($G$) − t.

holds for arbitrary ranks

H and G can be jointly decomposed as

$$H = U[\Sigma \quad 0_1]Q = U\Sigma\hat{Q},$$

$$G = V[\Lambda \quad 0_2]Q = V\Lambda\hat{Q},$$



$\tilde{\Sigma} = \mathrm{diag}(\sigma_1, \cdots, \sigma_s), \tilde{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_s) \in \mathcal{C}^{s\times s}$

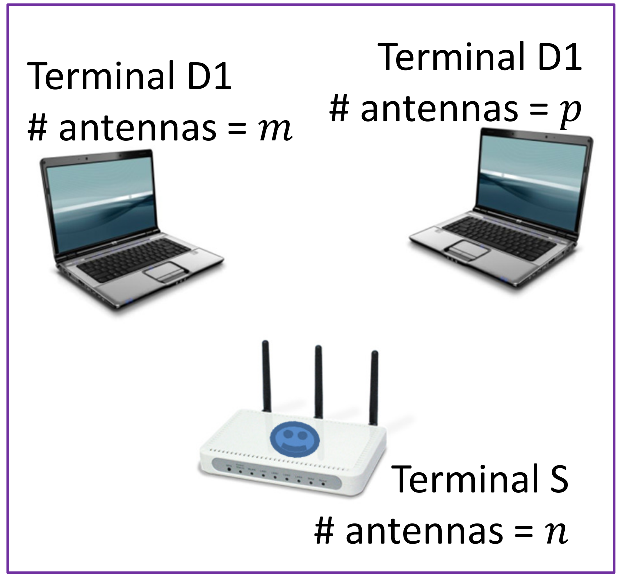
such that $1 \geq \sigma_1 \geq \cdots \geq \sigma_s \geq 0$ and $\sigma_i^2 + \lambda_i^2 = 1$.

common VC gains $\in (0,1)$

GSVD-based Beamforming

- b/f matrices using GSVD of channel matrices
- number of 'common' vs. 'private' VCs (assuming: no rank deficiency)

Configuration	# common channels S → {D1,D2}	# private channels S → D1	# private channels S → D2
$m > n, p \leq n$	p	$n - p$	0
$m \leq n, p > n$	m	0	$n - m$
$m \geq n, p \geq n$	n	0	0
$m < n, p < n$ $(m + p) > n$	$m + p - n$	$n - p$	$n - m$
$n \geq (m + p)$	0	m	p



- MATLAB implementation

```
% channel matrices
H = (randn(m,n)+i*randn(m,n))/sqrt(2);
G = (randn(p,n)+i*randn(p,n))/sqrt(2);
% D1, D2: diagonalized channels
[V,U,X,Lambda,Sigma] = gsvd(G,H);
w = X*inv(X'*X); C = [H' G']'; t = rank(C);
r = t - rank(G); s = rank(H)+rank(G)-t;
D1 = U(:,r+1:r+s)'*H*w(:,r+1:r+s);
D2 = V(:,1:s)'*G*w(:,r+1:r+s);
```

Applications

- MIMO subsystems
- single-source transmitting private/ common messages to 2 destinations
- each destination possibly requiring to process some of the messages

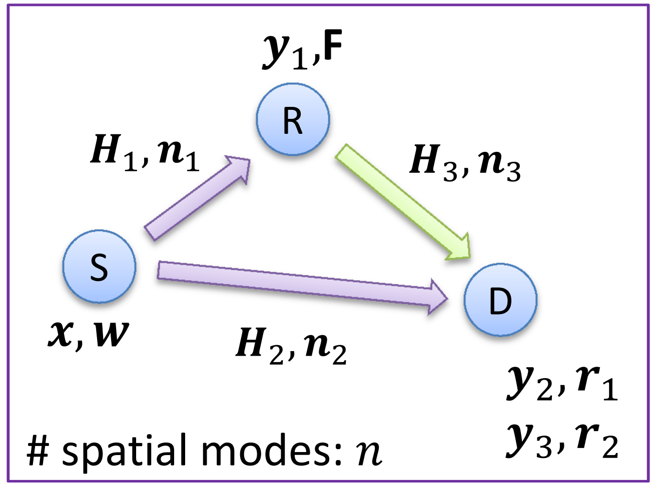
e.g.

- relay networks (AF, DF, CF)
- broadcasting
- physical layer multicasting
- two-way-relay networks

Example (AF Relaying)

Why?
SVD-based b/f is also possible if effective channel is point-to-point

TS1: $\boldsymbol{y}_1 = \boldsymbol{H}_1\boldsymbol{w}\boldsymbol{x} + \boldsymbol{n}_1, \boldsymbol{y}_2 = \boldsymbol{H}_2\boldsymbol{w}\boldsymbol{x} + \boldsymbol{n}_2$
TS2: $\boldsymbol{y}_3 = \boldsymbol{H}_3\boldsymbol{F}\boldsymbol{y}_1 + \boldsymbol{n}_3$



GSVD-based b/f (in TS1)

GSVD : $\boldsymbol{H}_1 = \boldsymbol{U}_1\boldsymbol{\Sigma}_1\boldsymbol{Q}, \boldsymbol{H}_2 = \boldsymbol{U}_2\boldsymbol{\Sigma}_2\boldsymbol{Q}$
SVD : $\boldsymbol{H}_3 = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{R}^H$

$\Rightarrow \boldsymbol{w} = \alpha\{\boldsymbol{Q}^{-1}\}_{C(n)}, \mathbf{F} = \boldsymbol{R}\boldsymbol{U}_1{}^H$
$\boldsymbol{r}_1 = \{\boldsymbol{U}_2{}^H\}_{R(n)}, \boldsymbol{r}_2 = \{\boldsymbol{V}^H\}_{R(n)}$
$\widehat{\boldsymbol{y}} = \boldsymbol{r}_1\boldsymbol{y}_2 + \boldsymbol{r}_2\boldsymbol{y}_3,$

Assumption: Van Loan form of GSVD holds
Note: an expression for α not known

SVD-based b/f (for comparison)

$$\widehat{\boldsymbol{y}} = \boldsymbol{r}\begin{bmatrix}\boldsymbol{y}_3\\ \boldsymbol{y}_2\end{bmatrix} = \boldsymbol{r}\left(\underbrace{\begin{bmatrix}\boldsymbol{H}_3\boldsymbol{F}\boldsymbol{H}_1\\ \boldsymbol{H}_2\end{bmatrix}}_{\widehat{\boldsymbol{H}}}\boldsymbol{w}\boldsymbol{x} + \begin{bmatrix}\boldsymbol{H}_3\boldsymbol{F}\boldsymbol{n}_1\\ \boldsymbol{0}\end{bmatrix} + \begin{bmatrix}\boldsymbol{n}_3\\ \boldsymbol{n}_2\end{bmatrix}\right)$$

SVD : $\widehat{\boldsymbol{H}} = \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^H,$
$\boldsymbol{H}_1 = \boldsymbol{U}_1\boldsymbol{\Sigma}_1\boldsymbol{V}_1{}^H, \boldsymbol{H}_3 = \boldsymbol{U}_3\boldsymbol{\Sigma}_3\boldsymbol{V}_3{}^H$

$\Rightarrow \mathbf{w} = \{\widehat{\boldsymbol{V}}\}_{C(n)}, \boldsymbol{F} = \boldsymbol{V}_3\boldsymbol{U}_1{}^H,$
$\boldsymbol{r} = \{\widehat{\boldsymbol{U}}^H\}_{R(n)} = \boldsymbol{r}_1 = \boldsymbol{r}_2$

Note: spatial modes not easily resolved at (R)
- not usable with DF
optimal $\boldsymbol{F}$ not always known

Example (AF Relaying)...

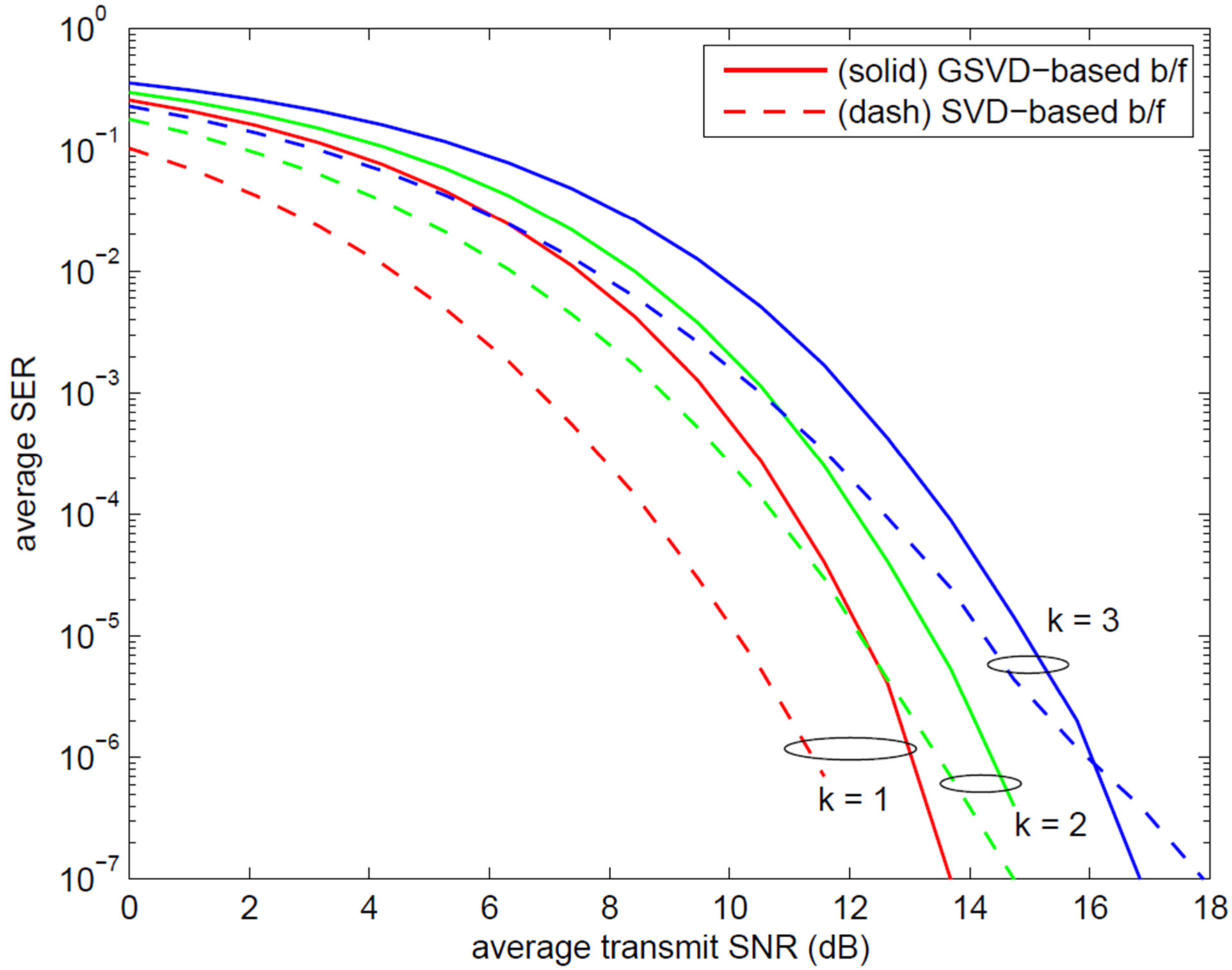


Fig. 6. The average SER of GSVD-based beamforming vs. SVD-based beamforming, for MIMO AF relaying with $N_s = 4, N_r = 3, N_d = 5$, for $n = 3$ common channels.

References

- **[Khisti2007]** A. Khisti, G. Wornell, A. Wiesel, and Y. Eldar, "On the Gaussian MIMO wiretap channel," in *Proc. Information Theory, IEEE International Symposium on*, Nice, France, Jun. 2007, pp. 2471–2475.
- **[Loan1976]** C. F. V. Loan, "Generalizing the singular value decomposition," *SIAM Journal on Numerical Analysis*, vol. 13, no. 1, pp. 76–83, Mar. 1976.
- **[Paige1981]** C. C. Paige and M. A. Saunders, "Towards a generalized singular value decomposition," *SIAM Journal on Numerical Analysis*, vol. 18, no. 3, pp. 398–405, Jun. 1981.

Conclusion

- the use of GSVD for coordinated b/f examined
    - a generalization of ZF (for single-source, two-destinations)
    - similarity with SVD-based b/f
- possible applications highlighted [see: paper]
- AF relaying example
    - a comparison with SVD-based b/f
    - SVD-based b/f possible only if the effective channel is point-to-point
        e.g. single-source, single-destination, AF relaying
- improvements
    - robustness
        - perfect CSI requirement
        - matrix inversion ($\boldsymbol{Q}^{-1}$)
    - extending random matrix theory for analysis

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GSVD-based b/f : another 'tool' like SVD-based b/f