Exact Solutions for Certain Weighted Sum-rate and Common-rate Maximization Problems

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***Abstract*—Weighted sum-rate and common-rate optimization problems have the general forms of $\max \ \sum_{i=1}^{N} a_i \log_2(1+\gamma_i)$ and $\max \ \min_i(\gamma_i)$, respectively, where γ_i represents the SNR (signal to noise ratio) of user i and a_i is a constant weight. In general, these problems are NP-hard problems [1]. In this paper, we propose an optimal solution framework for a selected class of such problems. Subject to some conditions on the region of feasible SNRs, we thus derive the optimal solutions utilizing the inequality of arithmetic and geometric means. We show that these solutions apply to several practical scenarios. For example, we derive optimal closed-form power allocations for a two-way relay network with a large antenna array relay. Numerical results and simulations verify the optimality of the analytical approach.**

***Index Terms*—Weighted sum-rate, common-rate, power allocation, optimization.**

I. Introduction

For wireless networks, weighted sum-rate (WSR), sum-rate (SR) and common-rate (CR) maximizations are extremely important. Applications include wireless resource management, cross-layer and beamforming design, link-scheduling, quality of service (QoS) and others. The general WSR problem is non-convex and NP-hard [1]. However, convex approximations may be utilized to take advantage of standard convex optimization methods. Even for fully convex cases, closed-form solutions are generally not available.

For instance, for single antenna transceivers, iterative algorithms for transmit power allocations to maximize WSR under QoS constraints have been presented in [2], [3]. In [4], [5] and [6], non-convex WSR and sum-rate maximization problems are converted into the equivalent convex problems for multi-antenna and single-antenna sources, respectively. For two single-antenna transceivers and multiple relays, [7] shows that maximizing the SR the two transceivers is equivalent to the CR maximization. The optimal solution is then developed under the total power constraint. While [7] offers analytical results, the solution is contingent on the specific system model.

CR maximization problems for amplify-and-forward multi-way relay channels and two-layer cellular radio systems have been studied in [7], [8]. Moreover, the optimal transmission power allocations under different QoS constraints for single antenna transmitters in a wireless multi-link system have been studied in [2], [6] and [5]. Algorithms for maximizing sum-rate and WSR of single antenna transceivers have been presented in [3] and [4], respectively.

In this letter, we develop a general solution framework for WSR and CR optimization problems. Sum rate maximization, in general, is an NP hard problem [1], however, we show that the developed framework is directly applicable to a wide range of cases (e.g. all triangle feasible regions,...) including non-convex feasible region of SNRs. The regions of feasible SNRs may exist, for example, due to power constraints. We develop the optimal solution when the feasible region of SNRs satisfies some conditions (but need not be convex).

Next, to illustrate the practicality and usefulness of our approach, we first show that it encompasses the results of [7] as a special case. Second, it allows us to derive the optimal WSR and CR power allocations for a two-way relay network which includes the two users and a massive multiple-input multiple output (massive MIMO) relay given the sum-power constraint. Finally, we present simulation results to justify our framework and validate the theoretical analysis. Note that our solution of the specific two-way relay network considered here, to the best of our knowledge, has not appeared in the literature before.

Notations: $\mathbf{h}^T$ and $\mathbf{h}^\dagger$ represent the transpose and the Hermitian of $\mathbf{h}$, respectively. Positive real numbers and integers are denoted by $\mathbb{R}_+$ and $\mathbb{Z}_+$.

II. Framework

Consider $n \geq 1$ communication links. For instance, this channel may arise as the uplink of a celluar base station with $n \geq 1$ users. The SNR of i-th ($i = 1, 2, \ldots, n$) link is γ_i. The WSR maximization problem can then be expressed as

$$\begin{cases} \max & \sum_{i=1}^{n} a_i \log_2(1+\gamma_i) \\ \text{s.t.} & \bar{\gamma} \in \mathbf{\Theta}, \end{cases} \tag{1}$$

where the vector $\bar{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}_+^n$ includes all links SNRs and $\mathbf{\Theta} \subset \mathbb{R}_+^n$ is a feasible set of the SNRs.

Since $a \log_2(x) = \log_2(x^a)$, this WSR maximization problem may be equivalently represented as

$$\begin{cases} \max & \prod_{i=1}^{n} X_i^{a_i} \\ \text{s.t.} & \mathbf{X} \in \mathbf{\Theta} \end{cases} \tag{2}$$

where $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbb{R}_+^n$. Similarly, the CR maximization may be expressed as

$$\begin{cases} \max & \min_i(X_i) \\ \text{s.t.} & \mathbf{X} \in \mathbf{\Theta}. \end{cases} \tag{3}$$

In order to develop optimal solutions of (2) and (3), we first define some useful notations.

Definition 1. For set $B = \{b_1, \ldots, b_n\} \subset \mathbb{R}_+^n$ and constant $K > 0$, we define $\mathbf{\Omega}_B(K)$ as the set of all points $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbb{R}_+^n$ such that

$\sum_{i=1}^{n} b_i X_i \le K$.

Definition 2. Let $A = \{a_1, a_2 ..., a_n\}$ and $B = \{b_1, b_2 ..., b_n\}$ be non-zero finite subsets of $\mathbb{Z}_+^n$ and $\mathbb{R}_+^n$, respectively, and K be a given positive constant. We define $\vartheta_{A,B}^K = (\vartheta_{a_1,b_1}^K, \vartheta_{a_2,b_2}^K, ..., \vartheta_{a_n,b_n}^K) \in \mathbb{R}_+^n$ where

$$\vartheta_{a_i,b_i}^K = \frac{a_i K}{b_i \sum_{i=1}^{n} a_i}, \quad \forall i \in \{1, 2, ..., n\}.$$

One can easily show that for any arbitrary $A \subset \mathbb{R}_+^n$, $\vartheta_{A,B}^K \in \mathbf{\Omega}_B(K)$. We next prove the following theorem.

Theorem 1. *Consider* $\mathbf{X} = (X_1, X_2, ..., X_n) \in \mathbb{R}_+^n$, $\mathbf{\Theta} \subset \mathbf{\Omega}_B(K)$ *and* $\vartheta_{A,B}^K \in \mathbf{\Theta}$. *Then,* $\mathbf{X} = \vartheta_{A,B}^K$ *is the optimal solution of maximization problem (2).*

Proof. Using **Definition 1** and the weighted geometric-mean arithmetic-mean inequality, we have

$$\begin{aligned} \frac{K}{\sum_{i=1}^{n} a_i} &\ge \frac{\sum_{i=1}^{n} a_i \frac{b_i X_i}{a_i}}{a} \\ &\ge \sqrt[a]{\prod_{i=1}^{n} (\frac{b_i}{a_i})^{a_i} X_i^{a_i}} \end{aligned} \tag{4}$$

where $a = \sum_{i=1}^{n} a_i$.
The geometric-mean achieves its upper bound if

$$\frac{b_i X_i}{a_i} = \frac{b_j X_j}{a_j} \quad \forall i, j \in \{1, 2, ..., n\}.$$

To maximize the geometric-mean, the arithmetic mean should achieve its upper bound (K) simultaneously. Therefore,

$$\sum_{i=1}^{n} b_i X_i = \sum_{i=1}^{n} a_i \frac{b_i X_i}{a_i} = \sum_{i=1}^{n} a_i t = K \Rightarrow t = \frac{K}{\sum_{i=1}^{n} a_i},$$

which completes the proof. □

Theorem 2 solves the CR maximization problems.

Theorem 2. *Let* $\mathbf{X} = (X_1, X_2, ..., X_n) \in \mathbb{R}_+^n$, $\mathbf{\Theta} \subset \mathbf{\Omega}_B(K)$ *and* $\vartheta_{A,B}^K \in \mathbf{\Theta}$. *Then, the optimal solution of problem (3) is* $\mathbf{X^{opt}} = (\frac{K}{\sum_{i=1}^{n} b_i}, ..., \frac{K}{\sum_{i=1}^{n} b_i})$ *if and only if* $\mathbf{X^{opt}} \in \mathbf{\Theta}$.

Proof. Clearly we find

$$\begin{aligned} \sum_{i=1}^{n} b_i \min\{X_i\} &\le \sum_{i=1}^{n} b_i X_i \le K \\ \min\{X_i\} &\le \frac{K}{\sum_{i=1}^{n} b_i} \end{aligned} \tag{5}$$

To achieve the optimal solution we claim $\min\{X_i\} = \frac{K}{\sum_{i=1}^{n} b_i}$.
Now if $\exists j \in \{1, 2, ..., n\} \quad X_j > \frac{K}{\sum_{i=1}^{n} b_i}$ then

$$\begin{aligned} \sum_{i=1}^{n} b_i X_i &\ge \sum_{i=1, i\neq j}^{n} b_i \min\{X_i\} + b_j X_j \\ &> \frac{K(\sum_{i=1,i\neq j}^{n} b_i)}{\sum_{i=1}^{n} b_i} + \frac{K b_j}{\sum_{i=1}^{n} b_i} = K \end{aligned}$$

which is a contradiction and completes the proof. □

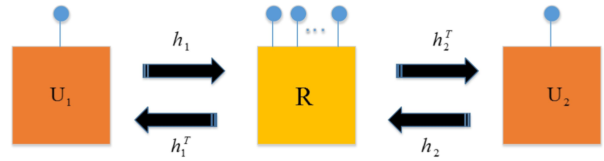

Fig. 1. Two-way relay network with a multiple-antenna relay.

To show the applicability of these two theorems, we obtain the main solutions of [7] using them. The two-way relay network in [7] consists of two end nodes and multiple relay nodes, where all nodes are single antenna terminals. By constraining the sum of transmit powers of all the nodes, optimal power allocation is derived in [7] to maximize the weighted sum rate and common rate.

Remark 1. *For the rate region* $\mathbf{\Theta}$ *obtained in [7],* $K = 2 + 2\gamma_{max}$, $b_i = 1 \ \forall i \in \{1, 2\}$, *using Theorem 1, the optimal solution for maximizing the sum-rate (*$a_i = 1$ *and* $X_i = 1 + SNR_i \ \forall i \in \{1, 2\}$ *) will be* $SNR_i = \gamma_{max} \ \forall i \in \{1, 2\}$ *which is equal to the optimal solution obtained in [7].*

Remark 2. *Using Theorem 2, the optimal solution for maximizing the common-rate (*$X_i = SNR_i \ \forall i \in \{1, 2\}$*) for the rate region* $\mathbf{\Theta}$ *proposed in [7], can be obtained as* $SNR_i = \gamma_{max} \ \forall i \in \{1, 2\}$ *which is equal to the optimal solution obtained in [7].*

Further applications are discussed next.

III. Applications

The first application is about bidirectional (two-way) relays. To the best of our knowledge, our optimization results for this example are new and have not appeared in the literature.

A. System model

We consider a two-way relay network (Fig. 1) where the realy has $N_r \ge 1$ antennas and users U_1 and U_2 are single antenna. The channel coefficients [1] for the two links $U_1 \leftrightarrow R$ and $U_2 \leftrightarrow R$ are $\mathbf{h}_1$ and $\mathbf{h}_2$, respectively. These channels are assumed to be reciprocal and the coefficients are independent. The transmit powers for users U_1, U_2 and relay R are P_1, P_2 and P_r, respectively. Also, the additive noise is Gaussian with mean zero and variance σ^2 for each hop. The communication protocol involves three time slots and is as follows. [2]

In the first time slot, both users transmit their signals to the relay. In the second time slot, the relay amplifies (with gain G) and employs maximal ratio transmission beamforming with weight $\mathbf{w}_1 = \frac{\mathbf{h}_1^* \mathbf{h}_2^\dagger}{\|\mathbf{h}_1\| \|\mathbf{h}_2\|}$ to forward the combined signals to U_1. In the third time slot, relay follows the same procedure as the

[1]We assume the availability of perfect channel state information.
[2]A two time slots protocol is considered in [9].

second time slot, with weight $\mathbf{w}_2 = \frac{\mathbf{h}_2^* \mathbf{h}_1^\dagger}{\|\mathbf{h}_1\|\|\mathbf{h}_2\|}$, to forward the received signal to U_2. Therefore, the received signal at R is:

$$y_r = \sqrt{P_1}\mathbf{h}_1 x_1 + \sqrt{P_2}\mathbf{h}_2 x_2 + \mathbf{n}_r, \tag{6}$$

where x_1 and x_2 are unit energy transmit signals and $\mathbf{n}_r$ is the relay AWGN term. The relay gain is

$$G = \sqrt{\frac{P_r}{P_1\|\mathbf{h}_1\|^2 + P_2\|\mathbf{h}_2\|^2 + \sigma^2}}$$

Using the transmission weight vectors and after self-interference cancellation by each user, the received signal at U_1 and U_2 can be expressed as:

$$\begin{aligned} \hat{y}_1 &= G\sqrt{P_2}\|\mathbf{h}_1\|\|\mathbf{h}_2\| x_2 + n_1 + \hat{n}_1 \\ \hat{y}_2 &= G\sqrt{P_1}\|\mathbf{h}_2\|\|\mathbf{h}_1\| x_1 + n_2 + \hat{n}_2, \end{aligned} \tag{7}$$

where $\hat{n}_1 = k\mathbf{w}_1\mathbf{h}_1^T\mathbf{n}_r$, $\hat{n}_2 = k\mathbf{w}_2\mathbf{h}_2^T\mathbf{n}_r$ and n_1, n_2 are the AWGN noises at users U_1 and U_2, respectively. Using (7), one can easily show that the SNRs at the receiver of each user are:

$$\begin{aligned} \gamma_2 &= \frac{P_1 P_r}{\sigma^2}\left[\frac{\|\mathbf{h}_1\|^2\|\mathbf{h}_2\|^2}{(P_2+P_r)\|\mathbf{h}_2\|^2 + P_1\|\mathbf{h}_1\|^2 + \sigma^2}\right] \\ \gamma_1 &= \frac{P_2 P_r}{\sigma^2}\left[\frac{\|\mathbf{h}_1\|^2\|\mathbf{h}_2\|^2}{(P_1+P_r)\|\mathbf{h}_1\|^2 + P_2\|\mathbf{h}_2\|^2 + \sigma^2}\right]. \end{aligned} \tag{8}$$

B. Common-rate

The CR maximization subject to the total power constraint can be formulated as

$$\begin{cases} \max\limits_{P_1,P_2,P_r} & \min(\gamma_1,\gamma_2) \\ \text{s.t.} & P_1 + P_2 + P_r \le P_t. \end{cases} \tag{9}$$

We next provide a proposition in order to reformulate (9).

Proposition 1. The constraint $P_1 + P_2 + P_r \le P_t$ is equivalent to

$$\gamma_1 + \gamma_2 \le \frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1} + \sqrt{\gamma_{2r}+1}\right)^2},$$

where $\gamma_{k,r} = \frac{P_t}{\sigma^2}\left|\left|\mathbf{h}_k\right|\right|^2, \quad k = 1, 2.$

Proof. We can define $P_1 = \alpha\beta P_t$, $P_2 = (1-\alpha)\beta P_t$ and $P_r = (1-\beta)P_t$ with $(0 \le \alpha, \beta \le 1)$. By substituting these definitions into the objective function $f(\alpha,\beta) = \gamma_1 + \gamma_2$ and forming the equations $\frac{\partial f}{\partial \beta} = 0$ and $\frac{\partial f}{\partial \alpha} = 0$, the global maximum of function $f(\alpha,\beta)$ can be obtained as $\frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1} + \sqrt{\gamma_{2r}+1}\right)^2}$. □

Hence, the problem (9) is equivalent to the following:

$$\begin{cases} \max\limits_{(\gamma_1,\gamma_2)\,\in\,\boldsymbol{\Theta}} & \min(\gamma_1,\gamma_2) \\ \text{s.t.} & \boldsymbol{\Theta} \subset \boldsymbol{\Omega}_B\left(\frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1}+\sqrt{\gamma_{2r}+1}\right)^2}\right), \end{cases} \tag{10}$$

which is a special case of the optimization problem (3).

We present the optimal solution for (10) using Theorem 2 and show that the solution is in $\boldsymbol{\Theta}$, therefore feasible.

Using Theorem (2) the optimal solution is

$$\gamma_1 = \gamma_2 = \frac{\gamma_{1r}\gamma_{2r}}{2\left(\sqrt{\gamma_{1r}+1} + \sqrt{\gamma_{2r}+1}\right)^2} \tag{11}$$

which results in

$$\begin{aligned} \beta^{opt} &= 0.5 \\ \alpha^{opt} &= \frac{-\gamma_{2r} - 1 + \sqrt{(\gamma_{2r}+1)(\gamma_{1r}+1)}}{\gamma_{1r} - \gamma_{2r}}, \end{aligned}$$

where both of them satisfy $0 \le \alpha, \beta \le 1$. Substituting β^{opt} and α^{opt} into P_1, P_2, P_r, we have:

$$\begin{aligned} P_1 &= \frac{P_t\left(-\gamma_{2r} - 1 + \sqrt{(\gamma_{2r}+1)(\gamma_{1r}+1)}\right)}{2(\gamma_{1r} - \gamma_{2r})} \\ P_2 &= \frac{P_t\left(\gamma_{1r} + 1 - \sqrt{(\gamma_{2r}+1)(\gamma_{1r}+1)}\right)}{2(\gamma_{1r} - \gamma_{2r})} \\ P_r &= \frac{P_t}{2} \end{aligned} \tag{12}$$

where $\gamma_{1r} = \frac{P_t}{\sigma^2}\left|\left|\mathbf{h}_1\right|\right|^2$, $\gamma_{2r} = \frac{P_t}{\sigma^2}\left|\left|\mathbf{h}_2\right|\right|^2$.

C. Weighted sum-rate

The WSR maximization (Fig. 1) subject to the total power constraint can be expressed as

$$\begin{aligned} R &= \frac{a_1}{2}\log_2(1+\gamma_1) + \frac{a_2}{2}\log_2(1+\gamma_2) \\ &= \frac{1}{2}\log_2[(1+\gamma_1)^{a_1}(1+\gamma_2)^{a_2}] \end{aligned} \tag{13}$$

Hence, the maximization problem can be reformulated as

$$\begin{cases} \max\limits_{P_1,P_2,P_r} & (1+\gamma_1)^{a_1}(1+\gamma_2)^{a_2} \\ \text{s.t.} & P_1 + P_2 + P_r \le P_t, \end{cases} \tag{14}$$

Using Proposition 1, optimization problem (14) turns to

$$\begin{cases} \max\limits_{(1+\gamma_1,1+\gamma_2)\in\boldsymbol{\Theta}} & (1+\gamma_1)^{a_1}(1+\gamma_2)^{a_2} \\ \text{s.t.} & \boldsymbol{\Theta} \subset \boldsymbol{\Omega}_B\left(2 + \frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1}+\sqrt{\gamma_{2r}+1}\right)^2}\right), \end{cases} \tag{15}$$

which is a special case of WSR problem (2).
From Theorem 1, the optimal solution of (15) occurs when

$$\begin{aligned} \gamma_1 &= \frac{a_1 - a_2}{a_1 + a_2} + \frac{a_1}{a_1 + a_2}\left(\frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1} + \sqrt{\gamma_{2r}+1}\right)^2}\right) \\ \gamma_2 &= \frac{a_2 - a_1}{a_1 + a_2} + \frac{a_2}{a_1 + a_2}\left(\frac{\gamma_{1r}\gamma_{2r}}{\left(\sqrt{\gamma_{1r}+1} + \sqrt{\gamma_{2r}+1}\right)^2}\right) \end{aligned}$$

For instance, assuming $a_1 = 2$, $a_2 = 1$, $P_t = 0$ dBm, $N_r = 100$, $\gamma_{1r} = 24$, and $\gamma_{2r} = 96$, the optimal solution will be $\gamma_1 = 7.3$ and $\gamma_2 = 3.15$ which translates to $P_1 = 0.1996, P_2 = 0.2362$ mW and $P_r = 0.5642$ mW.

IV. Numerical and Simulation results

We now present simulation results to verify that Theorems 1 and 2 yield optimal solutions.

In Figs. 1 and 2, we have plotted the achievable CR and WSR (Fig. 1) for eq. (12) and (15). To verify the optimality

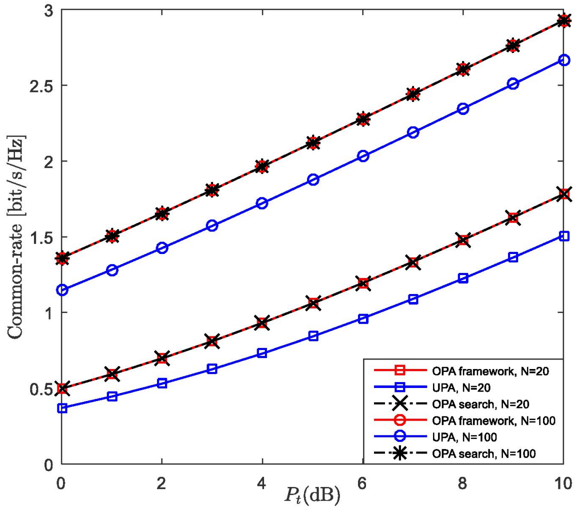

Fig. 2. Achievable CR for $a_1 = 2, a_2 = 1, \sigma_1^2 = 1/4$ and $\sigma_2^2 = \sigma^2 = 1$.

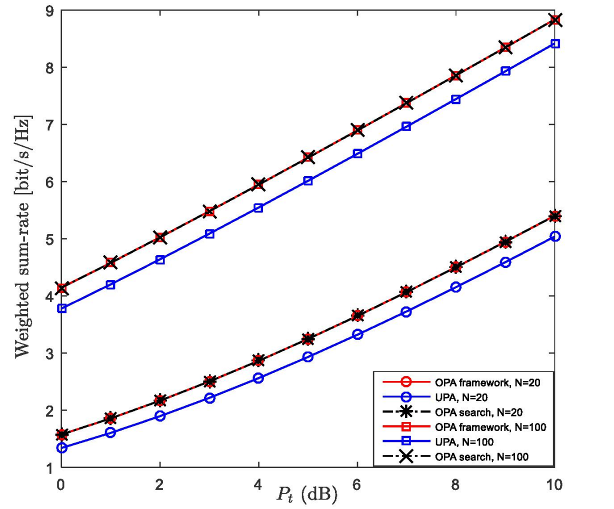

Fig. 3. Achievable WSR for $a_1 = 2, a_2 = 1, \sigma_1^2 = 0.25, \sigma_2^2 = \sigma^2 = 1$.

of these results, we also present an exhaustive discrete search (with step 0.001) over the feasible SNR region. This search has been performed for two cases of $N_r = 16$ and $N_r = 100$. Furthermore, as a benchmark, the achievable CR of uniform (i.e., equal) power allocation (UPA) has also been plotted. As can be seen, the theoretical results match well with the exhaustive search in both cases. Moreover, as expected, optimal power allocations significantly outperform UPA. In Fig. 4, we have plotted the CR and SR achieved by the users using our framework by setting $a_1 = a_2 = 1$ and $n = 2$ for the system in [7] when the number of relays is equal to one.

V. Conclusion

In this letter, we developed optimal solutions for WSR and CR maximization problems subject to certain conditions on the feasible SNR region. Our solutions do not restrict the region to be convex. The main advantage of the proposed method is the availability of analytical closed-form solutions for some convex and non-convex WSR and CR optimization problems. Moreover, closed-form solutions enable further investigations, e.g. deriving ergodic sum-rate, and may facilitate real time analysis of network performance. The proposed method generalizes the results in [7] to an arbitrary number of users, multiple-antenna devices and so on. To verify our analyses, we presented the optimal power allocations for a two-way relay network where the relay is equipped with a large-scale antenna array and the total transmit power is constant. We derived closed-form solutions for this problem and validated them simulation results.

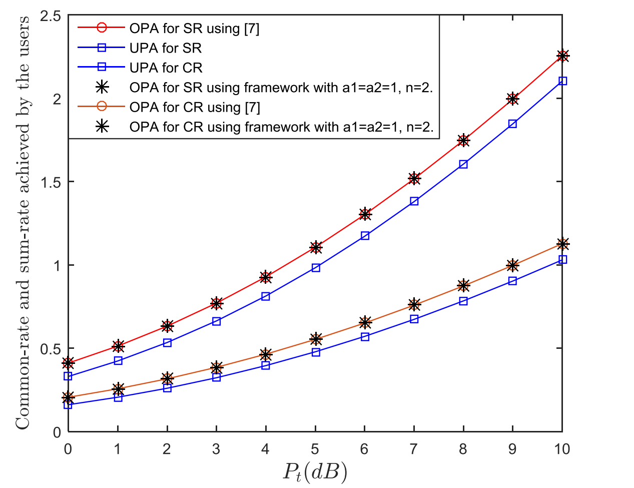

Fig. 4. Achievable CR and SR for the single-relay system [7], $\sigma_2^2 = \sigma^2 = 1$.

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