Generalized Asymptotic Measures for Wireless Fading Channels with a Logarithmic Singularity

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***Abstract*—In wireless channels, the received signal to noise ratio (SNR) can be represented as $\gamma = \beta\overline{\gamma}$, where $\overline{\gamma}$ is the average SNR and β is a random variable with probability density function (PDF) $f(\beta)$. In this paper, we analyze the high SNR performance of wireless channels with a logarithmic singularity. That is, $f(\beta) = a\beta^t + b\beta^\mu \log(\beta) + \cdots$ near $\beta = 0$. This logarithmic singularity (LS) is critically important in determining the high SNR performance and appears to have been completely overlooked. Important special cases include Gamma-Gamma and Generalized-K channels. For instance, the GG has been used to model scattering, reflection, and diffraction and optical, navigation and relay channels [1]. This versatility highlights the importance of LS wireless channels. Classical asymptotic or high SNR analysis is developed by expanding $f(\beta) = a\beta^t + \cdots$ near $\beta = 0$ and expressing the diversity and coding gain as direct functions of a and t. However, as this monomial expansion does not hold for LS channels, we develop generalized asymptotic performance measures for outage and error rates. The results show significantly improved accuracy in the SNR range of 10-25 dB. For this range, our new asymptotic expressions achieve much better accuracy than the conventional ones that ignore this singularity.**

***Index Terms*—Logarithmic singularity, fading channels, Gamma-Gamma distribution, outage probability, bit-error rate.**

I. Introduction

Analysis of critical performance metrics such as outage and error rates is contingent upon the proper statistical modeling of the wireless channel. For example, statistical models are used to design and optimize wireless transmitters and receivers and their antenna configurations too [2]. In this paper, we consider a class of channels called logarithmic singularity (LS) wireless channels. A LS wireless channel has that the characteristic property that its pdf has the expansion $f(\beta) = a\beta^t + b\beta^\mu \log(\beta) + \cdots$ near $\beta = 0$. Importance of analyzing LS channels is manifested by its versatility, as LS property is observed in popular generalized fading models, e.g., Gamma-Gamma (GG) channel and Generalized-K channel. Despite numerous notable works, the impact of logarithmic singularity (LS) at zero has been overlooked previously.

Moreover, the GG distribution is an alternative to popular models such as lognormal, Weibull, Rician, Nakagami, Gamma mixture [3] and K-distribution [4]. The GG distribution was introduced as a more flexible model than the K-distribution [5], it covers Gamma and K-distribution, and Nakagami-lognormal composite fading model as special cases. Thereafter, it is used to model multiple communication scenarios such as the single point-to-point channel with co-channel interferers [6]–[9], relay networks with amplify-and-forward (AF) and decode-and-forward (DF) strategies [10], wireless optical channels [11], and radar systems [12].

The GG model is also relevant for free-space optical (FSO) wireless communication, a complementary option to radio wireless access, which is being developed for high-speed short range links. The GG distribution is thus used to model the signal fluctuation in such links. Important performance measures of FSO links under turbulent conditions, e.g., achievable rate, bounds of average BER for different modulations, and FSO MIMO links are studied using the GG model [13]–[15]. Moreover, applications of GG model on the relay and point-to-point channels are extended to explore the spectral efficiency in heterogeneous networks (HetNets) [16], [17], which are highly relevant to the implementation of recent LTE release 12. It is also useful for analyzing composite fading channels with co-located interference, an important scenario of HetNets [18]. Thus, the importance of precise analysis of GG channel and correspondingly LS wireless channels is clearly evident.

A. High-SNR Analysis

To achieve simple, direct and insightful analytical results, it is common to develop asymptotic or high signal-to-noise ratio (SNR) analysis. In this paper, we thus consider the region characterized by $\overline{\gamma} \to \infty$ where $\overline{\gamma}$ denote the unfaded SNR. We also write the instantaneous fading SNR as $\gamma = \beta\overline{\gamma}$, where β is the random variable with the pdf $f(\beta)$. High-SNR analysis is then developed by extracting the first term of the Taylor series expansion of the pdf $f(\beta)$ near $\beta = 0$. Following this approach, outage probability and BER asymptotics were developed by Wang and Giannakis [19]. To do so, they expanded $f(\beta)$ as $f(\beta) = a\beta^t + R_{t+1}, \quad$ for $\beta \to 0^+$, where R_{t+1} is a remainder term that vanishes for $\beta \to 0^+$. The parameters $a > 0$ and $t \geq 0$ determine the SNR (coding) gain and diversity gain. The intuition of their work is that since the asymptotic performance may be given by $\int_0^{1/\overline{\gamma}} g(\beta)f(\beta)d\beta$ where $g(\cdot)$ is a rapidly decreasing function, what matters is a simple but accurate asymptote of $f(\beta)$ near $\beta = 0$. Moreover, they observed that classical coding and diversity gain model, given by (1) can be expressed as a function of a and t.

$$P_e(\overline{\gamma}) = (G_c\overline{\gamma})^{-G_d} + R(\overline{\gamma}) \tag{1}$$

where $R(\overline{\gamma})$ is the remainder term that vanishes as $\overline{\gamma} \to \infty$, and G_c and G_d are called coding gain and diversity gain respectively, which are important, widely-used parameters that are useful for wireless system design and optimization. For instance, from (1), we observe that on a log-log scale, $P(\overline{\gamma})$ varies linearly with $\overline{\gamma}$, which is a directly insightful representation of the system performance. Not only error probability, but also outage can be expressed in the same form as (1). Moreover, the parameters G_c and G_d for various wireless systems (e.g., multiple antenna transmission/reception) can be readily derived for popular fading channels models.

Since the seminal paper by Wang and Giannakis [19], many further improvements have been reported [20]–[24]. However, since $f(\beta) = a\beta^t + R_{t+1} \quad \text{for } \beta \to 0^+$ does not hold for the LS channels, the classical coding and diversity gains (1) fail to tightly approximate the LS wireless performance. Thus, in terms of asymptotic analysis, LS channels are fundamentally different from classical fading models. As the GG channel is the most popular special case of LS wireless channels, it is the focus of this paper.

B. Motivation and Contribution

Contributions of this paper are summarized below.

- We develop a new asymptotic performance measure for all wireless channels (3). It includes an exponential term, which leads to a generalization of the classical diversity gain and coding gain relationship (e.g., (1)).
- We derive new asymptotic expressions for error and outage probability considering GG fading channel. Closed-form asymptotics of BER are derived for different modulation schemes, Binary Phase Shift Keying (BPSK) and Differential Binary Phase Shift Keying (DBPSK).
- We evaluate the difference our asymptotic expressions and the existing ones (which are derived by ignoring the logarithmic singularity). We use the Kolmogorov-Smirnov test to measure the effects of using additional terms in (2).

C. Paper Organization

The rest of the paper is organized as follows. We provide the details of our asymptotic approach in Section II. Closed-form solutions for BER and outage probability for different scenarios are given in Section III. Numerical results for several diversity schemes and performance comparison with existing asymptotic approaches are given in Section IV. We give a brief overview and conclude our paper in Section V.

II. Main Results

In this section, we discuss the details of the new asymptotic performance measure. For this purpose, we consider the following expansion of LS channels:

$$f(\beta) = a\beta^t + b\beta^\mu \log(\beta) + R(\beta), \quad \text{for } \beta \to 0^+ \tag{2}$$

where a, b, t and μ are constants to be determined from the distribution of the fading channel, log is natural logarithm throughout the paper, and $R(\beta)$ is the vanishing remainder term. The singularity is determined by the term $b\beta^\mu \log(\beta)$.

The most important result given in the next theorem.

Theorem 1. *Now averaging a performance measure $g(\cdot)$ over the two terms in (2), we get a generalized version of asymptotic performance measure,*

$$P_e(\overline{\gamma}) = (G_c\overline{\gamma})^{-G_d} \exp\left(\frac{c' \log(\overline{\gamma})}{\overline{\gamma}^\alpha}\right) + R(\overline{\gamma}), \quad as \quad \overline{\gamma} \to \infty \tag{3}$$

where G_c, G_d, and $R(\overline{\gamma})$ are coding and diversity gain, and the remainder term, respectively.

Proof. Using (2) we can derive an expression for error probability similar to one obtained using Taylor series,

$$P_e(\overline{\gamma}) = \frac{c_1}{\overline{\gamma}^{t+1}} + \frac{c_2}{\overline{\gamma}^{\mu+1}} + \frac{c_3}{\overline{\gamma}^{\mu+1}} \log(\overline{\gamma}), \tag{4}$$

where c_1, c_2, and c_3 are constants depending on the modulation scheme and t, μ are the constants from (2). As we are interested in asymptotic solution,therefore, for $\overline{\gamma} \to \infty$ we can ignore the second term in (4) and applying elementary series approximations over (4) without the second term we reach the generalized form of error probability given by (3),

$$\begin{aligned} P_e(\overline{\gamma}) &= \frac{c_1}{\overline{\gamma}^{t+1}}\left(1 + \frac{c_3/c_1}{\overline{\gamma}^{\mu-t}} \log(\overline{\gamma})\right) \\ &= \frac{c_1}{\overline{\gamma}^{t+1}} \exp\left(\frac{c' \log(\overline{\gamma})}{\overline{\gamma}^{\mu-t}}\right), \end{aligned} \tag{5}$$

where $c' = c_3/c_1$. Understandably, (5) can be written in the form of (3) with $G_c = c_1^{-1/t+1}$, $G_d = t + 1$, and $\alpha = \mu - t$. □

Proposition 1. *However, the asymptote in (2) works only in the high SNR regime. To overcome this shortfall and extend the SNR range we add one more higher order term with (2) and develop a new asymptote,*

$$\begin{aligned} f(\beta) &= a\beta^t + a_1\beta^{t+1} + b\beta^\mu \log(\beta) + b_1\beta^{\mu+1} \log(\beta) + R(\beta) \\ &= a\beta^t \exp\left(-\frac{a_1}{a}\beta\right) + b\beta^\mu \log(\beta) \exp\left(-\frac{b_1}{b}\beta\right) + R(\beta) \end{aligned} \tag{6}$$

Similar to (2), $R(\cdot)$ is the remainder term.

The asymptote in (6) significantly improves the performance in the SNR range of 5-15 dB. Considering few more higher order terms and developing an asymptote similar to (6) can be a simple way to improve the accuracy of the asymptotic measure.

Comparing (1) and (3), we observe an additional exponential, correction term. This correction term can be found in (17) as the second and third term on the right hand side. In absence of logarithmic singularity, the $\log(\overline{\gamma})$ term vanishes and the expression reduces to the classical formula (1).

A. Application on Fading channels

We derive the expressions of the variables and constants mentioned in (2). Each of the variable and constant can be derived from the pdf of the GG distribution, and the pdf is given by,

$$f(\beta) = \frac{2\beta^{(m+k)/2-1}}{\Gamma(m)\Gamma(k)}\left(\frac{km}{\theta_0}\right)^{(m+k)/2} K_{k-m}\left(2\sqrt{\frac{km\beta}{\theta_0}}\right), \tag{7}$$

where $K_n(\cdot)$ is the modified Bessel function of second kind with order of $k-m$, m and θ_0 are the shaping parameter and the average local mean of β, respectively [1].

Since the asymptote of pdf $f(\beta)$ depends on the polynomial terms of β, we derive a series expansion of the GG pdf. A sum formula for the modified Bessel function with order of n is given by Abramowitz and Stegun [[25], 9.6.11],

$$K_n(x) = \frac{1}{2}\Big(\frac{x}{2}\Big)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(-\frac{x^2}{4}\Big)^k + (-1)^{n+1}\log\frac{x}{2} \times I_n(x) + (-1)^n\frac{1}{2}\Big(\frac{x}{2}\Big)^n\sum_{k=0}^{\infty}(\psi(k+1)+\psi(n+k+1)) \times \frac{(x^2/4)^k}{k!(n+k)!}, \tag{8}$$

where $\psi(\cdot)$ is the logarithmic derivative of the gamma function, commonly known as the digamma function and $I_n(\cdot)$ is the modified Bessel function of first kind. Now, we use the asymptotic approximation of the Bessel function of second kind, given in (8) to derive the expressions for a, b, t, and μ of (2). Replacing n with $k-m$ and x with $\left(2\sqrt{\frac{km}{\theta_0}}\right)$ in (8) and thereafter considering the first term in the series we obtain,

$$f(\beta) = \frac{(k-m-1)!}{\Gamma(m)\Gamma(k)}\left(\frac{km}{\theta_0}\right)^m\beta^{(3m-k)/2-1} + \frac{(-1)^{k-m+1}}{(k-m)!\Gamma(m)\Gamma(k)} \times\left(\frac{km}{\theta_0}\right)^k\beta^{k-1}\left(\gamma+\log\left(\frac{km}{\theta_0}\right)-\psi(k-m+1)\right) + \frac{(-1)^{k-m+1}}{(k-m)!\Gamma(m)\Gamma(k)}\left(\frac{km}{\theta_0}\right)^k\beta^{k-1}\log(\beta) + R(\beta) \tag{9}$$

We would like to mention that ($\psi(1) = -\gamma$) is a constant, where γ is the Euler-Mascheroni constant.

As mentioned earlier, only dominant terms are considered in asymptotic measures. Therefore, we can ignore the second term in (9), and determine the parameters of interest, a, b, t, and μ.

$$\begin{aligned} t &= \left(\frac{3m-k}{2}\right)-1 \\ \mu &= k-1 \\ a &= \frac{(k-m-1)!}{\Gamma(m)\Gamma(k)}\Big(\frac{km}{\theta_0}\Big)^m \\ b &= \frac{(-1)^{k-m+1}}{(k-m)!\Gamma(m)\Gamma(k)}\Big(\frac{km}{\theta_0}\Big)^k \end{aligned} \tag{10}$$

III. Closed-form solutions

In this section, we derive the closed-form expressions for two important performance measures: BER and outage probability.

A. BER

The average BER (or simply BER) is taken over the statistical distribution representing the wireless fading [26]. Integrating the conditional error probability (CEP), $P_b(\overline{\gamma})$ of a modulation scheme, over the pdf of the instantaneous SNR $f(\beta)$, we find

$$P_e(\overline{\gamma}) = \int_0^\infty P_b(\overline{\gamma})f(\beta)d\beta \tag{11}$$

Various expressions for the binary modulations are available in the literature, mainly using the Q-function, however, a general form of CEP for binary modulations that works for BPSK, DBPSK, and BFSK (binary frequency shift keying) is given in [27],

$$P_b(\overline{\gamma}) = \frac{\Gamma(\lambda, \nu\overline{\gamma}\beta)}{2\Gamma(\lambda)} \tag{12}$$

where $\Gamma(\lambda, \nu x)$ denote the upper incomplete gamma function, and λ, ν are the modulation dependent parameters. $\Gamma(\cdot,\cdot)$ is defined by,

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}dt \tag{13}$$

For BPSK, DBPSK, and BFSK modulation, (ν, λ) are (1,0.5), (1,1), and (0.5,0.5), respectively [27]. BER for binary modulations can be evaluated by substituting (12) in (11),

$$P_e(\overline{\gamma}) = \int_0^\infty \frac{\Gamma(\lambda, \nu\overline{\gamma}\beta)}{2\Gamma(\lambda)}\Big(a\beta^t + b\beta^\mu\log(\beta)\Big)d\beta \tag{14}$$

We use the following properties of incomplete gamma function [[28], 8.352.4, 8.359.3] to derive the closed form solutions for BER,

$$\begin{aligned} \Gamma(0.5, x^2) &= \sqrt{\pi}\Big(1-\Phi(x)\Big) \\ \Gamma(n, x) &= (n-1)!e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!} \qquad \text{for } n \in \mathbb{I} \end{aligned} \tag{15}$$

Applying the first property in (15), BER for BPSK modulation using (2) is given by,

$$P_e(\overline{\gamma}) = \frac{\sqrt{\pi}}{2\Gamma(\lambda)}\int_0^\infty\left(1-\Phi\left(\sqrt{\frac{\nu\overline{\gamma}\beta}{2}}\right)\right)(a\beta^t + b\beta^\mu\log(\beta))d\beta \tag{16}$$

Theorem 2. *Closed-form solution to evaluate BER for BPSK modulation is given by,*

$$P_e(\overline{\gamma}) = \frac{a\Gamma(t+3/2)}{(2t+2)(\nu\overline{\gamma}/2)^{2t}\Gamma(\lambda)} + \frac{b\Gamma(\mu+1/2)}{(\nu\overline{\gamma}/2)^{2\mu}\Gamma(\lambda)}\Big(\frac{1}{2\mu+2}+\frac{2\mu+1}{2} \times\psi\Big(\frac{2\mu+1}{2}\Big)\Big) - \frac{b\Gamma(\mu+3/2)}{(2\mu+2)(\nu\overline{\gamma}/2)^{2\mu}\Gamma(\lambda)}\log(\nu\overline{\gamma}/2) \tag{17}$$

Details of the derivation are given in Appendix A.

Theorem 3. *Applying the second property in (15), closed-form expression to evaluate BER for DBPSK is given by,*

$$P_e(\overline{\gamma}) = \frac{(\lambda-1)!}{2\Gamma(\lambda)}\sum_{m=0}^{\lambda-1}\frac{(\nu\overline{\gamma})^m}{m!}\Big(\frac{a\Gamma(t+m+1)}{(\nu\overline{\gamma})^{t+m+1}} + \frac{b\Gamma(\mu+m+1)}{(\nu\overline{\gamma})^{\mu+m+1}} \times\Big(\psi(\mu+m+1)-\log(\nu\overline{\gamma})\Big)\Big) \tag{18}$$

Details of the derivation are given in Appendix B.

B. Outage Probability

In addition to the BER, outage probability is another important performance measure for fading channels. In this paper we define outage probability as the probability of instantaneous SNR γ dropping below a certain threshold γ_{th}. Therefore, it can be evaluated using,

$$P_{out}(\gamma_{th}, \overline{\gamma}) = P[\gamma \leq \gamma_{th}] = P[\beta\overline{\gamma} \leq \gamma_{th}] \quad (19)$$

As SNR is always non-negative, therefore, the outage probability is simply the cdf of β. Now exact cdf of GG distribution is not computation friendly and therefore, it is important to use asymptotes of the pdf to obtain a computation friendly expression. Integrating the asymptote in (2) we obtain,

$$P_{out}(\gamma_{th}, \overline{\gamma}) = \frac{a(\gamma_{th}/\overline{\gamma})^{t+1}}{1+t} + \frac{b(\gamma_{th}/\overline{\gamma})^{\mu+1}((1+\mu)\log(\gamma_{th}/\overline{\gamma}) - 1)}{(1+\mu)^2} \quad (20)$$

Wang and Giannakis also observed that outage probability can be expressed as a function of outage diversity and coding gain, and similar to the error probability in (1), they expressed outage probability as [19],

$$P_{out} = (O_c\overline{\gamma})^{-O_d} + R(\overline{\gamma}), \quad (21)$$

where O_d and O_c are outage diversity and coding gain, respectively.

For LS wireless channels (21) does not hold, therefore, we express (20) in a more generalized form. Similar to (3), we derive a generalized form of outage probability using outage diversity and coding gain that works for channels with and without logarithmic singularity. Similar to Theorem 1 we can express outage probability as,

$$P_{out} = (O_c\overline{\gamma})^{-O_d} \exp\left(\frac{c'}{\overline{\gamma}^{\alpha}}\Big(\eta\log(\gamma_{th}/\overline{\gamma}) - 1\Big)\right), \quad (22)$$

where $O_c = \frac{1}{\gamma_{th}}\left(\frac{a}{t+1}\right)^{-1/(t+1)}$, $O_d = 1+t$, $c' = \frac{b\gamma_{th}^{\mu-t}(t+1)}{a(1+\mu)^2}$, $\alpha = \mu - t$, and $\eta = 1 + \mu$.

IV. Numerical Results

Numerical results are generated for BER and outage probability. Throughout this section, we will use the following legends, *Exact, Approx1, Approx2, Existing1,* and *Existing 2* to represent the exact solution, solution using the asymptote in (2), using the asymptote in (6), Taylor series based approach used by Wang and Giannakis, and the asymptotic solution given by Dhungana and Tellambura in [20], respectively. Performance is observed for SNR varying from $5-20$ dB.

Outage probability is calculated for two different cases, ($k=2$, $m=1$, $\theta=1$) and ($k=3$, $m=2$, $\theta=1$), and γ_{th} is assumed to be 0 dB. Performance of the proposed asymptotic measures are compared with two existing asymptotic approaches for SNR varying from $5-20$ dB. From Fig. 1 we observe that proposed approximations give a considerable improvement over the existing ones. As mentioned earlier, Approx2 gives better performance than Approx1 in the SNR range of $5-10$ dB. We also observe approximation by Dhungana and Tellambura gives better performance than Wang-Giannakis model but worse than Approx1 even after using two terms from the series expansion.

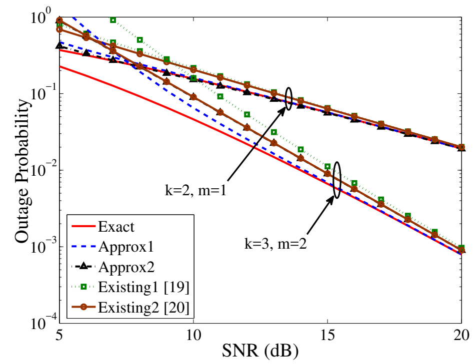

Fig. 1. Outage probability for varying SNR.

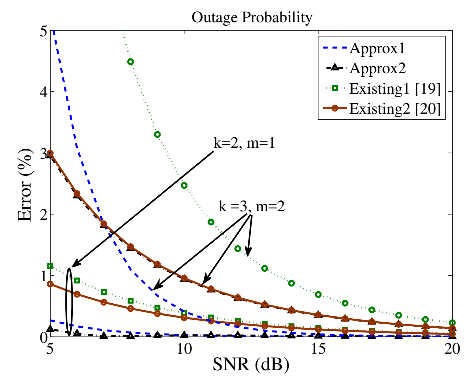

Fig. 2. Approximation error of outage probability.

We also plot the relative error for all the approximations, where relative error is calculated using (23),

$$\hat{e} = \frac{|\,approx - exact\,|}{exact} \quad (23)$$

Errors from the proposed and existing asymptotic measures are compared in Fig. 2. We observe significant improvement using our asymptotic approach in the range of $8-20$ dB.

Now we compare the asymptotic measures for BPSK and DBPSK modulation and observe the performance by varying SNR from $5-20$ dB. In Fig. 3 and 4 we observe Approx1 performs better for higher SNR, this is the primary shortfall of asymptotic approaches, which converges to the exact one in infinity. We also observe that Approx2 performs better than other asymptotes in the region of $10-14$ dB, and it is the effect of including the additional exponential function which reduces the divergence to a great extent in the lower SNR regime $(0-10)$ dB.

We should also keep in mind that only a single term from the series is used in Approx1, and therefore, the inclusion of more terms will certainly enhance the performance. Therefore,

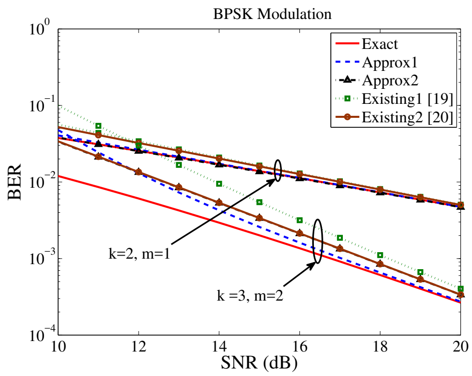

Fig. 3. BER for BPSK modulation with varying SNR.

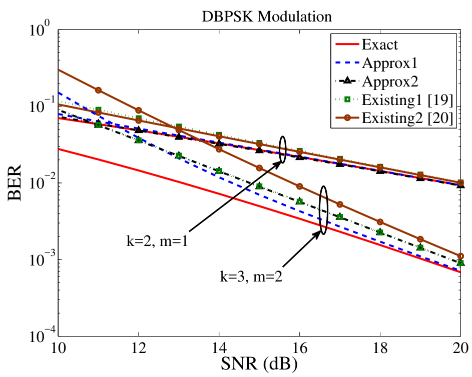

Fig. 4. BER for DBPSK modulation with varying SNR.

we observe the effects of including additional terms using a well-known statistical test.

A. KS Test

In this section we measure the closeness of our asymptotic measures, and effect of considering additional terms, using the Kolmogorov-Smirnov (KS) goodness-of-fit test [29]. KS test is a null hypothesis test where T-statistic is evaluated from the absolute difference between empirical cumulative distribution function (CDF) and approximated CDF. KS test statistic is defined as,

$$T \triangleq \max \mid F_\beta(z) - \hat{F}_\beta(z) \mid \tag{24}$$

Till now, all the results are generated by considering a single term from the series. Now, we explore the effect of including more terms in new asymptote, given by (2) and asymptotic approach by Wang and Giannakis. We compute the T-statistic for varying number of terms $(1-4)$ and varying z, and define a null hypothesis to determine the significance of both approaches.

- *Definition:* We define H_0 as the null hypothesis, that accepts a measurement with significance of 95% when corresponding T is less than a threshold (T_{max}), and similarly, rejects a measurement with 5% significance when T is greater than T_{max}.

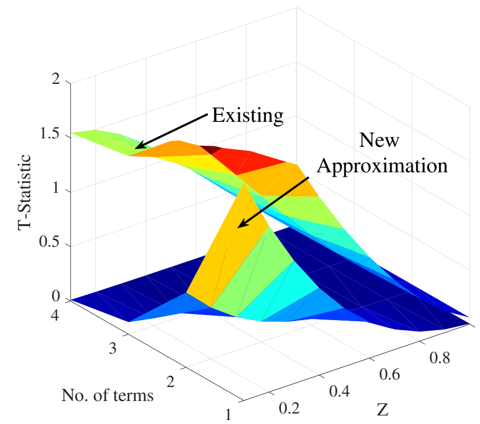

Fig. 5. KS test statistic.

We use the standard value of $T_{max} = 0.05$, to compare the performance. Test statistics for distribution parameter $k = 2$, $m = 1$, varying z from $0-1$ and number of terms from $1-4$, is given in Fig. 5. From the results, it is evident that the performance of the new asymptotic approach is significantly better than the existing approach, our proposed asymptote with one term outperforms the Taylor series based asymptote, even with four terms. We also observe the hypothesis H_0 does not accept any of the results using the existing asymptotic approach with 95% significance, whereas all the results using the proposed asymptote with four terms are accepted with 95% significance.

V. Conclusion

The impact of logarithmic singularity in certain wireless channels has been overlooked in all previous works. In LS channels, the Taylor series based asymptotic approaches are incorrect and the classical coding and diversity gain expressions fail. To circumvent these problems, we have proposed new asymptotic measures and developed a generalized coding and diversity gain model. It covers fading models with and without the logarithmic singularity. We also derived several asymptotic measures for error probability. We have provided numerical results for different modulation schemes. Results suggest significantly improved accuracy in the 10-25 dB SNR range. To increase the SNR range even further, we further developed the asymptotic approach which shows significant improvement in the 5-15 dB range.

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Appendix A
Proof of Theorem 1

Replacing Φ with error function and applying its odd function property in (16) gives,

$$P_e(\overline{\gamma}) = \frac{\sqrt{\pi}}{2\Gamma(\lambda)} \int_0^\infty \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{\nu\overline{\gamma}\beta}{2}}\right)(a\beta^t + b\beta^\mu \log(\beta))d\beta \tag{25}$$

Replacing $\nu\overline{\gamma}\beta$ with $2\tau^2$ and using [[30], 4.1.11,4.6.9] a closed-form solution of the integral in (25) is obtained,

$$\begin{aligned} P_e(\overline{\gamma}) = {} & \frac{a\sqrt{\pi}}{(\nu\overline{\gamma}/2)^{2t}\Gamma(\lambda)} \frac{\Gamma(t+3/2)}{\sqrt{\pi}(2t+2)} + \frac{2b\sqrt{\pi}}{(\nu\overline{\gamma}/2)^{2\mu}\Gamma(\lambda)} \frac{\Gamma(\mu+1/2)}{2\sqrt{\pi}} \\ & \times \left(\frac{1}{2\mu+2} + \frac{2\mu+1}{2}\psi\left(\frac{2\mu+1}{2}\right)\right) - \frac{b\sqrt{\pi}}{(\nu\overline{\gamma}/2)^{2\mu}\Gamma(\lambda)} \\ & \times \frac{\Gamma(\mu+3/2)}{\sqrt{\pi}(2\mu+2)} \log(\nu\overline{\gamma}/2) \end{aligned} \tag{26}$$

After some rearrangement of (26) leads to the closed-form solution using given in (17) of Theorem 2.

Appendix B
Proof of Theorem 2

Applying the second property in (15) over (14) we can write,

$$\begin{aligned} P_e(\overline{\gamma}) &= \frac{1}{2\Gamma(\lambda)} \int_0^\infty \left(a\beta^t + b\beta^\mu \log(\beta)\right)(\lambda-1)! e^{-\nu\overline{\gamma}\beta} \sum_{m=0}^{\lambda-1} \frac{(\nu\overline{\gamma}\beta)^m}{m!} d\beta \\ &= \frac{a(\lambda-1)!}{2\Gamma(\lambda)} \sum_{m=0}^{\lambda-1} \frac{(\nu\overline{\gamma})^m}{m!} \int_0^\infty \beta^{t+m} e^{-\nu\overline{\gamma}\beta} d\beta + \frac{b(\lambda-1)!}{2\Gamma(\lambda)} \\ &\quad \times \sum_{m=0}^{\lambda-1} \frac{(\nu\overline{\gamma})^m}{m!} \int_0^\infty \beta^{\mu+m} e^{-\nu\overline{\gamma}\beta} \log(\beta) d\beta \end{aligned} \tag{27}$$

Using [[28], 3.351.3, 4.352.1], (27) can be rewritten as,

$$\begin{aligned} P_e(\overline{\gamma}) = {} & \frac{a(\lambda-1)!}{2\Gamma(\lambda)} \sum_{m=0}^{\lambda-1} \frac{(\nu\overline{\gamma})^m}{m!} \Gamma(t+m+1)(\nu\overline{\gamma})^{-t-m-1} + \frac{b(\lambda-1)!}{2\Gamma(\lambda)} \\ & \times \sum_{m=0}^{\lambda-1} \frac{(\nu\overline{\gamma})^m}{m!} \frac{\Gamma(\mu+m+1)}{(\nu\overline{\gamma})^{\mu+m+1}} \left(\psi(\mu+m+1) - \log(\nu\overline{\gamma})\right) \end{aligned} \tag{28}$$

Rearranging (28) gives the closed-form solution given in Theorem 3.