New Asymptotics for Performance of Energy Detector

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***Abstract*—Performance analysis of the energy detector (ED) in fading channels has received enormous attention recently. However, averaging the generalized Marcum-Q function over fading statistics often results in complicated special functions and/or infinite series based expressions. Motivated by the need for simple expressions without compromising the accuracy, we propose a new representation for the probability density function (PDF) of the fading channel gain. This representation is then used to derive simple, unified expression for asymptotic miss-detection probability in closed-form. The derived expression is evaluated for several fading channels and antenna diversity schemes. Numerical results reveal its high accuracy over a wide range of signal-to-noise-ratio (SNR) (as low as $0\,\mathrm{dB}$) unlike the existing asymptotic expression which is accurate only for high SNR regime (say, SNR $\geq 20\,\mathrm{dB}$).**

***Index Terms*—Energy detector, fading, diversity combining.**

I. Introduction

Signal detection is a fundamental requirement in traditional wireless applications such as target detection in radar [1], as well as in emerging applications such as ultra-wide band (UWB) radio [2] and cognitive radio (CR) networks [3]. One of the most popular techniques used for signal detection is the energy detector (ED) which simply compares the energy of the received signal against a detection threshold to decide on the presence or absence of the signal of interest [4]. Thus, the ED is a non-coherent device which does not require any *a priori* information (such as modulation format) of the transmitted signal and has a low hardware complexity. These features have made the ED a highly active research topic [5]–[7].

The performance of ED was first considered in [4] by assuming a flat, band-limited Gaussian noise channel and deriving the detection probability (P_d) and the false alarm probability (P_f). However, in wireless multipath fading channels, P_d, which depends upon random fading gain (β), must be averaged over the distribution of the received signal-to-noise ratio (SNR) $\gamma = \overline{\gamma}\beta$, where $\overline{\gamma}$ is the average (unfaded) SNR, to quantify the overall detection performance. This problem has been extensively analysed recently. In [8], the average probability of detection ($\overline{P}_d$) is derived in closed-form for Rayleigh fading channels but the results for Nakagami-m and Rician fading channels involve a numerical integration and an infinite summation, respectively. In [9] and [10], closed-form expressions for $\overline{P}_d$ over various fading channel models without and with antenna diversity reception are derived. However, the results for multiple antenna reception are restricted only to Rayleigh fading channels. This limitation is overcome in [11] by utilizing an alternative representation of the Marcum-Q function along with the moment generating function (MGF) of the received SNR to derive $\overline{P}_d$ for several diversity combining schemes such as maximal ratio combining (MRC), equal gain combining (EGC) and selection combining (SC).

Although the aforementioned works are exact in the sense that no limiting assumptions on the critical parameters of interest such as the sample size and/or SNR are made, complicated special functions (e.g., confluent hypergeometric function, hypergeometric function of two variables, etc.) and/or infinite series sums appear in the final expressions and the impact of fading and diversity combining on the detection performance is not explicitly visible. To derive explicit expressions, [12] utilizes the asymptotic analysis originally presented in [13] to quantify the impact of the fading channels and antenna diversity combining on the ED performance at high SNR ($\overline{\gamma} \to \infty$). The key idea of [13] is to approximate the exact probability density function (PDF) of the fading channel gain β, denoted by $f(\beta)$, by a monomial term as

$$f(\beta) = a\beta^t + O(\beta^{t+\epsilon}), \tag{1}$$

where $O(x)$ is the error term as $x \to 0^+$, $\epsilon > 0$, a is a positive constant and parameter t represents the order of smoothness of $f(\beta)$ as $\beta \to 0^+$ (β tends to zero from above) [13]. The parameters a and t are completely determined by the fading channel model and can be derived from the exact PDF or MGF of β. By using (1), [12] derives asymptotic expression for average miss-detection probability ($\overline{P}_{md}$) which is highly accurate as $\overline{\gamma} \to \infty$ and can also be used for determining the "sensing gain". It is defined as the negative slope of $\overline{P}_{md}$ vs. $\overline{\gamma}$ in a log-log plot as $\overline{\gamma} \to \infty$ (which is equal to the negative exponent of $\overline{\gamma}$ occuring in $\overline{P}_{md}$ as $\overline{\gamma} \to \infty$) and thus serves as a single figure of merit to quantify the ED performance in fading and/or diversity combining at high SNR.

However, such analysis is accurate only for the high SNR regime (say, $\overline{\gamma} \geq 20\,\mathrm{dB}$) and fails to accurately capture the ED performance in moderate to low SNR, the region of interest for current wireless systems (e.g., the IEEE 802.16 local and metropolitan area networks operate at 3-20 dB SNR [14]). Hence, a more accurate asymptotic expression valid over a wide SNR range and without complicated special functions and/or infinite series is of interest to wireless engineers and researchers.

We thus propose a new representation for the exact PDF $f(\beta)$ which yields highly accurate (over $0 \leq \overline{\gamma}\,\mathrm{dB} < \infty$), unified asymptotic expression in closed-form which: (i) is a simple, finite order (without infinite series) expression, and (ii) reveals the sensing gain explicitly.

The basic energy detection model is described in Section II. The new representation for $f(\beta)$ is presented in Section III. Unified asymptotic expression for the average miss-detection probability is derived in Section IV. Application examples for analysing ED performance in fading channels and antenna diversity are presented in Section V. Concluding remarks and future directions are highlighted in Section VI.

II. Energy Detection over Fading channels

The problem of detecting the presence or absence of the signal-under-test (SUT) can be classified as a binary hypothesis test of the form, $y(t) = \begin{cases} w(t) & : H_0, \\ hS(t) + w(t) & : H_1, \end{cases}$ where $w(t)$ is the additive white Gaussian noise (AWGN) with mean zero and variance N_0, $S(t)$ is the unknown transmit SUT with energy E_s, h is the fading channel coefficient, and hypotheses H_0 and H_1 respectively denote the absence and presence of the transmit signal. In energy detection, the received signal $y(t)$ is passed through a noise pre-filter of bandwidth W, squared, and then fed to a finite-time integrator of duration T ($u = TW$ is called the time-bandwidth product). The result is then compared against a pre-determined threshold λ to decide on the presence/absence of the SUT [4] such that two fundamental performance metrics, the detection probability P_d and the false alarm probability P_f are given by [9]: $P_d(\gamma) = Q_u(\sqrt{2\gamma}, \sqrt{\lambda})$, and $P_f = \Gamma(u, \lambda/2)/\Gamma(u)$ respectively, where $Q_u(\cdot,\cdot)$ is the generalized Marcum-Q function of u-th order; $\Gamma(a, z) = \int_z^\infty x^{a-1}e^{-x}dx$ and $\Gamma(a) = \Gamma(a, 0)$ respectively denote the upper-incomplete and complete Gamma functions [15], and $\gamma \triangleq |h|^2 E_s/N_0 = \overline{\gamma}\beta$ is the received (instantaneous) SNR with $\overline{\gamma}$ being the average SNR. Equivalently, the ED performance can be characterized via miss-detection probability P_{md}, which is given by $P_{md}(\gamma) = 1 - P_d(\gamma) = 1 - Q_u(\sqrt{2\gamma}, \sqrt{\lambda})$. Clearly, it depends upon the instantaneous SNR γ, which in turn depends upon the random fading channel gain β. To quantify the ED performance over fading channels, the average miss-detection probability $\overline{P}_{md}$ is obtained by averaging $P_{md}(\gamma)$ over the fading PDF $f(\beta)$ as

$$\overline{P}_{md} = 1 - \int_0^\infty Q_u(\sqrt{2\overline{\gamma}\beta}, \sqrt{\lambda}) f(\beta) d\beta. \tag{2}$$

Due to complicated nature of the Marcum-Q function, closed-form solutions for (2) are limited to very few fading channel models and diversity combiners. Thus, existing solutions are either: (i) in closed-forms but involve complicated special functions (e.g., confluent hypergeometric function, hypergeometric function of two variables, etc.), or (ii) are expressed in terms of infinite series expansions, and sometimes even involve both (i) and (ii).

As well, since $\overline{P}_{md}$ is useful for obtaining the sensing gain which is critical to characterize the ED performance at high SNR, for numerical analysis, we are interested in the logarithmic plots of $\overline{P}_{md}$ vs. $\overline{\gamma}$, whose slopes at high SNR yield the sensing gains [12]. Since a low probability of false alarm is desirable (for example, in CR networks, a $P_f \leq 0.1$ is preferred [16]), the detection threshold λ is determined by solving $P_f = \Gamma(u, \lambda/2)/\Gamma(u)$ for a specific P_f (and u) and used for evaluating $\overline{P}_{md}$. We thus consider $\overline{P}_{md}$ vs. $\overline{\gamma}$ curves at a fixed $P_f = 0.01$ throughout our numerical analysis.

III. New Representation for $f(\beta)$

A performance analysis with better accuracy and without complicated special functions and/or infinite series is thus needed. With these goals in mind, we thus propose the following PDF $f^{\text{app}}(\beta)$ to represent the exact PDF $f(\beta)$ when $\beta \to 0^+$ as

$$f^{\text{app}}(\beta) = a\beta^t(e^{-\theta_1\beta} + e^{-\theta_2\beta}), \tag{3}$$

where $a > 0$, t, $\{\theta_1, \theta_2\} > 0$ are constants dependent upon the fading channel model under consideration and can be determined by using the exact PDF $f(\beta)$. Note that the PDF (1) is a special case of (3) with $\theta_1 = \theta_2 = 0$. The two main assumptions are stated below for completeness.

AS1) The instantaneous SNR at the receiver is expressed as $\gamma = \overline{\gamma}\beta$ where $\overline{\gamma}$ is the average (unfaded) SNR and $\beta > 0$ is a randomly varying channel gain whose PDF depends upon the fading model and/or the diversity combining scheme under consideration (same as [12], [13]).

AS2) The exact PDF of β can be expanded in a series-form $f(\beta) = \sum_{i=0}^{2} a_i\beta^{\tau+i-1} + O(\beta^{\tau+2})$ as $\beta \to 0^+$.

Next, we show how to determine the parameters a, t, θ_1 and θ_2 of (3). The approach is formally stated in Proposition 1 below.

Proposition 1. *Given the series expansion of the PDF $f(\beta)$ as $\beta \to 0^+$ in the form*

$$f(\beta) = a_0\beta^{\tau-1} + a_1\beta^\tau + a_2\beta^{\tau+1} + O(\beta^{\tau+2}), \tag{4}$$

the parameters of $f^{\text{app}}(\beta)$ are given by

$$t = \tau - 1, \;\; a = \frac{a_0}{2}, \text{and} \tag{5}$$

$$(\theta_1, \theta_2) = \left(\frac{b_1 + \sqrt{2b_2 - b_1^2}}{2}, \frac{b_1 - \sqrt{2b_2 - b_1^2}}{2}\right), \tag{6}$$

where $b_1 = -a_1/a$ and $b_2 = 2a_2/a$.

Proof: To determine the parameters of (3), we match the series expansion of $f^{\text{app}}(\beta)$ as $\beta \to 0^+$ with that of the exact PDF $f(\beta)$ given by (4). The series expansion of $f^{\text{app}}(\beta)$ (3) as $\beta \to 0^+$ followed by rearrangement of the resulting terms in ascending powers of β yields

$$f^{\text{app}}(\beta) = 2a\beta^t - (\theta_1+\theta_2)a\beta^{t+1} + \frac{(\theta_1^2+\theta_2^2)}{2}a\beta^{t+2} + O(\beta^{t+3}). \tag{7}$$

Then, matching the coefficients of (4) and (7), we find that $t = \tau - 1$, $a = a_0/2$ and $\{\theta_1, \theta_2\}$ satisfy

$$\theta_1 + \theta_2 = -\frac{a_1}{a} \triangleq b_1 \;\text{ and }\; \theta_1^2 + \theta_2^2 = \frac{2a_2}{a} \triangleq b_2. \tag{8}$$

The two simultaneous equations (8) can then be immediately solved to yield (6). □

$$\overline{P}_{md}^{\text{asy}} = \begin{cases} 1 - \sum_{i=1}^{2} \frac{a\Gamma(t+1)e^{-\lambda/2}}{(\theta_i+\overline{\gamma})^{t+1}} \left[\frac{1}{t!} \sum_{k=0}^{t} \binom{t}{k} \psi_{\eta_i}^{(k)} \phi_{\eta_i}^{(t-k)} + \frac{1}{(u-t-2)!} \sum_{k=0}^{u-t-2} \binom{u-t-2}{k} \chi_0^{(k)} \phi_0^{(u-t-k-2)} \right] & \text{for } u > t+1 \\ 1 - \sum_{i=1}^{2} \frac{a\Gamma(t+1)e^{-\lambda/2}}{(\theta_i+\overline{\gamma})^{t+1} t!} \sum_{k=0}^{t} \binom{t}{k} \psi_{\eta_i}^{(k)} \phi_{\eta_i}^{(t-k)} & \text{for } u \leq t+1. \end{cases} \quad (9)$$

IV. Average Probability of Miss-Detection

In order to derive the asymptotic miss-detection probability, the average (2) must be performed by replacing $f(\beta)$ with $f^{\text{app}}(\beta)$. The result is a unified asymptotic expression for the average miss-detection probability, denoted by $\overline{P}_{md}^{\text{asy}}$, in closed-form as given by Proposition 2.

Proposition 2. *If the PDF of the channel gain is given by $f^{app}(\beta)$, then the average miss-detection probability $\overline{P}_{md}^{asy}$ can be expressed as (9) shown on top of the page with*

$$\psi_{\eta_i}^{(k)} = \frac{(-1)^k}{\eta_i^{u-t-1+k}} \prod_{j=1}^{k} (u-t+j-2),$$
$$\chi_0^{(k)} = \frac{(-1)^{-(t+1)}}{\eta_i^{t+k+1}} \prod_{j=1}^{k} (t+j), \quad (10)$$

$$\phi_{z_0}^{(n)} = e^{\frac{\lambda}{2} z_0} \sum_{\nu=0}^{n} \frac{(\lambda/2)^{n-\nu}}{(1-z_0)^{\nu+1}} \frac{n!}{(n-\nu)!}, \quad (11)$$

where $\eta_i = \overline{\gamma}/(\theta_i + \overline{\gamma})$, $\forall i \in \{1, 2\}$ and $z_0 \in \{0, \eta_i\}$.

Proof: The proof is given in the Appendix A. □

An important observation can be made from (9). As $\overline{\gamma} \to \infty$, it can be shown that (9) can be expressed in the form $\overline{P}_{md}^{\text{asy}} \approx 1 - g(u, t, \lambda)\overline{\gamma}^{-(t+1)}$ where $g(u, t, \lambda)$ is a term independent of $\overline{\gamma}$. Thus, the sensing gain is equal to $(t + 1)$, which is in agreement with that of [12]. Hence, the parameter t fully characterizes the sensing gain for any given PDF $f(\beta)$ of the fading channel gain β.

It is important to note that the derived asymptotic (9) provides a unified expression for a wide range of fading channels and diversity reception schemes. In the following sections, we show that the parameters t, a, θ_1 and θ_2 of $f^{\text{app}}(\beta)$ can be readily determined for various fading and diversity combining cases by using Proposition 1. Once these parameters are obtained, Proposition 2 can be readily used to evaluate $\overline{P}_{md}^{\text{asy}}$ for each case.

V. Application Examples

A. Nakagami-m fading

For Nakagami-m fading, the instantaneous SNR is a Gamma random variable and thus the exact PDF of β is given by $f(\beta) = m^m/\Gamma(m)\beta^{m-1}e^{-m\beta}$, $\beta > 0$, with the parameter $m \geq 1/2$ representing the fading severity index. The series expansion of $f(\beta)$ as $\beta \to 0^+$ can be written as

$$f(\beta) \approx \frac{m^m}{\Gamma(m)}\beta^{m-1} - \frac{m^{m+1}}{\Gamma(m)}\beta^m + \frac{m^{m+2}}{2\Gamma(m)}\beta^{m+1} + O(\beta^{m+2}).$$

Then, using Proposition 1, we obtain the parameters of $f^{\text{app}}(\beta)$ to be $t = m - 1$, $a= m^m/[2\Gamma(m)]$, $a_1= -m^{m+1}/\Gamma(m)$ and

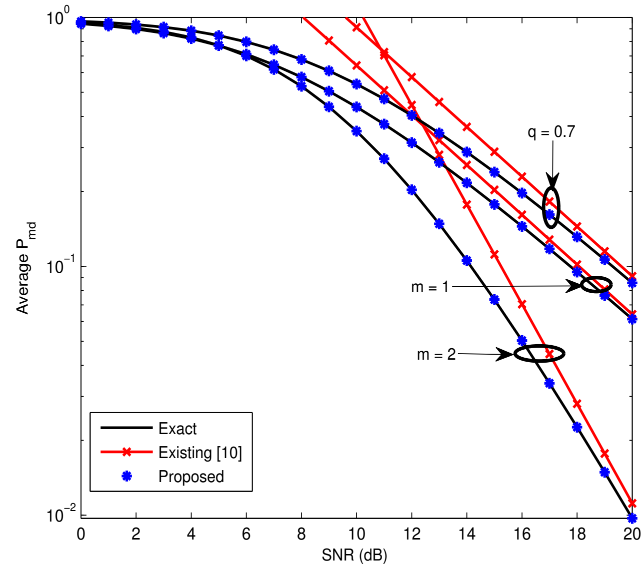

Fig. 1: $\overline{P}_{md}$ vs. $\overline{\gamma}$ in Nakagami-m fading for $u = 3$ and in Nakagami-q fading with $q = 0.7$, $u = 7$; exact, existing [12] and derived (9).

$a_2= m^{m+2}/[2\Gamma(m)]$. This gives $b_1 = 2m$ and $b_2 = 2m^2$, which substituted into (6) immediately yields $\theta_1 = \theta_2 = m$. Subsequently, Proposition 2 can be applied to evaluate $\overline{P}_{md}^{\text{asy}}$.

B. Nakagami-q (Hoyt) fading

The Nakagami-q distribution can typically model satellite links subject to strong ionospheric scintillation [17]. The PDF of β for this model is given by

$$f(\beta) = \frac{1+q^2}{2q} e^{-\frac{(1+q)^2}{4q^2}\beta} I_0\left(\frac{1-q^4}{4q^2}\beta\right) \quad (12)$$

where $I_0(\cdot)$ is the zero-th order modified Bessel function of first kind and q is the fading parameter which can range from 0 to 1. The series expansion of (12) as $\beta \to 0^+$ followed by re-arrangement of the terms in the ascending powers of β and comparison with (4) yields: $t = 0$, and

$$a_0 = \frac{1+q^2}{2q}, \; a_1 = -\frac{(1+q^2)}{2q}\left(\frac{q^4+2q^2+1}{4q^2}\right),$$
$$a_2 = \frac{(1+q^2)}{2q}\left(\frac{3q^8+8q^6+10q^4+8q^2+3}{64q^4}\right). \quad (13)$$

Then, the desired parameters of $f^{\text{app}}(\beta)$ can be readily obtained by using Proposition 1, and Proposition 2 yields $\overline{P}_{md}^{\text{asy}}$.

The derived asymptotic (9) is extremely accurate for both Nakagami-m and Nakagami-q fading (Fig. 1). For instance, (9) is virtually identical to the exact values over the entire SNR range (0-20 dB) while the asymptotic result of [12] is

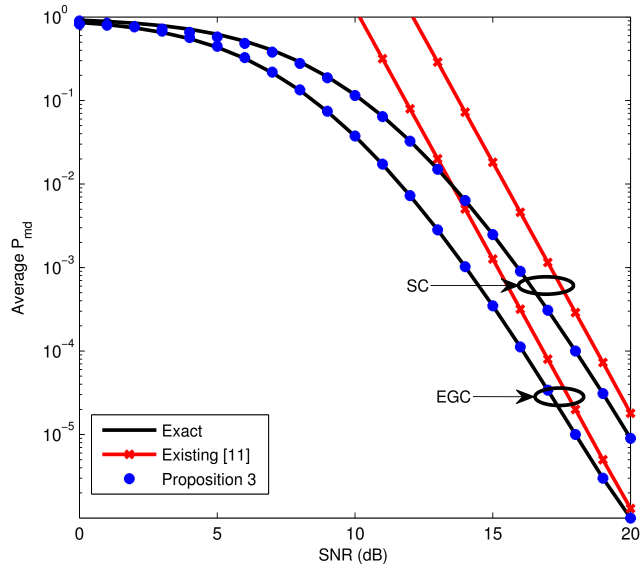

Fig. 2: $\overline{P}_{md}$ vs. $\overline{\gamma}$ for dual branch EGC and SC in Nakagami-3 fading with $u = 2$: exact, existing [12] and derived (9).

accurate only in the high SNR region (say, SNR $\geq$ 20 dB). Moreover, the gap between [12] and the exact value is 2 dB even at SNR = 17 dB for $m = 1$ while (9) is identical (7-significant digit accuracy) to the exact values over the whole SNR range (0-20 dB). Note that the exact $\overline{P}_{md}$ for Nakagami-m fading obtained from $\overline{P}_d$ derived in [11, (eq. 5)] involves an infinite series and its truncation to a finite number of terms to satisfy the desired precision requirement is needed. In contrast, (9) does not involve truncations or other approximations. Further, the negative slopes of the curves corresponding to cases $q = 0.7$, $m = 1$ and $m = 2$ at high SNR ($\geq$ 19 dB) are equal to 1, 1 and 2 respectively, which are equal to the corresponding sensing gains given by $(t + 1)$.

C. Dual branch EGC in Nakagami-m fading

In EGC scheme, the received signals at each branch are weighted by their corresponding phases and combined to yield a new signal which is fed to the ED input. For this case, the PDF of β can be easily obtained from that of γ given in [11, eq. (32)] using the transformation of random variables to be

$$f(\beta) = \frac{2^{2-2m}\sqrt{\pi}\Gamma(2m)m^{2m}}{\Gamma^2(m)\Gamma(2m+1/2)}\beta^{2m-1}e^{-2m\beta} \times {}_1F_1\big(2m; 2m+1/2; m\beta\big). \quad (14)$$

Performing the series expansion of (14) as $\beta \to 0^+$ and comparing the result with (4) yields $t = 2m - 1$, $a_0 = [2^{2-2m}\sqrt{\pi}\Gamma(2m)m^{2m}]/[\Gamma^2(m)\Gamma(2m + 1/2)]$, $a_1 = -[2m(2m + 1)]/[4m + 1]a_0$, $a_2 = [2m^2(4m^2 + 6m + 3)]/[(4m+1)(4m+3)]a_0$. Then, application of Proposition 1 followed by Proposition 2 gives $\overline{P}^{\text{asy}}_{md}$.

D. Dual branch SC in Nakagami-m fading

Unlike EGC, the SC only selects the antenna branch having the largest SNR and thus has a reduced complexity. For this case, using [11, eq. (50)] with $L = 2$, we can write the PDF of β after transformation of random variables, to be

$$f(\beta) = \frac{2m^m}{\Gamma^2(m)}\beta^{m-1}e^{-m\beta}\mathcal{G}(m, m\beta), \quad (15)$$

where $\mathcal{G}(a, z) = \int_0^z x^{a-1}e^{-x}dx$ is the lower-incomplete Gamma function [15]. Series expanding $f(\beta)$ as $\beta \to 0^+$ and comparing the resulting expression with (4), we get: $t = 2m - 1$, $a_0 = [2m^{2m-1}]/[\Gamma^2(m)]$, $a_1 = -[2m^{2m}(2m + 1)]/[(m+1)\Gamma^2(m)]$ and $a_2 = [2m^{2m+1}(2m^2+4m+1)]/[(m+1)(m+2)\Gamma^2(m)]$. Thus, Proposition 1 and Proposition 2 can be readily applied to yield $\overline{P}^{\text{asy}}_{md}$.

The remarkably high accuracy of our derived results for both EGC and SC is clear (Fig. 2). Note that the exact $\overline{P}_{md}$ for EGC and SC are obtained from $\overline{P}_d$ derived in [11, eq. (38)] and [11, eq. (55)] respectively, as infinite series expressions with hypergeometric function of two variables. Such expressions are highly complicated and do not reveal the sensing gain explicitly. In contrast, our expressions clearly show the sensing gain (given by $t + 1$) to be $2m$ (also given by negative slopes of the graphs in Fig. 2 at high SNR ($\geq$ 19 dB).

VI. Conclusion and Future Work

A new representation for the PDF of the fading channel gain has been proposed and utilized to derive simple, unified, closed-form asymptotic expression for the miss-detection probability of an ED. Several examples for fading channels and antenna diversity combining demonstrate the high accuracy (e.g., 7-significant digit accuracy over $0 \leq \overline{\gamma}\,\text{dB} < \infty$) of our analysis compared to the existing asymptotic [12] in addition to explicitly revealing the sensing gain. Our analysis may also be extended to cases when the PDF of β is unknown but its MGF is known, and to cases when cooperative and non-cooperative (interfering) transmissions take place. Further, our analysis may well be utilized to evaluate performance metrics such as the error probability, probability of outage and ergodic capacity which are often used to characterize the performance of coherent/non-coherent modulation schemes over fading channels. These results will be provided in a forthcoming paper.

Appendix

A. Derivation of $\overline{P}^{asy}_{md}$ (9)

Utilizing an alternative representation of the Marcum-Q function, the integral in (2) can in general be expressed in terms of a contour integral of the form [11]

$$\overline{P}_{md} = 1 - \frac{e^{-\frac{\lambda}{2}}}{2\pi\jmath}\oint_{\Delta}\mathcal{M}_\gamma\left(1 - \frac{1}{z}\right)\frac{e^{\frac{\lambda}{2}z}}{z^u(1-z)}dz, \quad (16)$$

where Δ is a circular contour of radius r such that $0 < r < 1$, and $\jmath$ denotes the imaginary unit. After expressing the MGF of γ, $\mathcal{M}_\gamma(s)$, in (16) in terms of the MGF of β using $\mathcal{M}_\gamma(s) =$

$\mathcal{M}_\beta(s\overline{\gamma})$, and substituting the MGF of $f^{\text{app}}(\beta)$: $\mathcal{M}_\beta^{\text{app}}(s) \triangleq \mathbb{E}\left[e^{-s\beta} f^{\text{app}}(\beta)\right]$, which can be derived to be

$$\mathcal{M}_\beta^{\text{app}}(s) = a\Gamma(t+1)\left[\frac{1}{(s+\theta_1)^{t+1}} + \frac{1}{(s+\theta_2)^{t+1}}\right], \quad (17)$$

into the resulting expression, followed by some steps of algebraic manipulations, we get

$$\overline{P}_{md}^{\text{asy}} = 1 - \frac{a\Gamma(t+1)e^{-\frac{\lambda}{2}}}{2\pi\jmath}\oint_\Delta [g_1(z)dz + g_2(z)]dz, \quad (18)$$

where for $i = \{1, 2\}$,

$$g_i(z) = \frac{1}{(\theta_i+\overline{\gamma})^{t+1}} \frac{e^{\lambda z/2}}{(z-\eta_i)^{t+1} z^{u-t-1}(1-z)},$$

and $\eta_i \triangleq \overline{\gamma}/(\theta_i+\overline{\gamma})$ represents the i-th pole of $g_i(z)$ having an order of $(t+1)$. To evaluate the contour integral in (18), residue of the pole at $z = 0$ of order $(u-t-1)$ and residues of poles at $z = \{\eta_1, \eta_2\}$ of order $(t+1)$ need to be evaluated. In general, two cases must be treated separately.

Case I: $u > t+1$: In this case, the function $g_i(z), \forall i = \{1, 2\}$ contains $(t+1)$-th ordered pole at $z = \eta_i$ and $(u-t-1)$ ordered pole at $z = 0$. Thus, $\overline{P}_{md}^{\text{asy}}$ for this case is given by

$$\overline{P}_{md}^{\text{asy}} = 1 - a\Gamma(t+1)e^{-\frac{\lambda}{2}} \times \sum_{i=1}^{2}\big[\text{Res}(g_i;\eta_i,t+1) + \text{Res}(g_i;0,u-t-1)\big], \quad (19)$$

where $\text{Res}(g_i; z_0, p)$ denotes the Residue of the function $g_i(z)$ at pole $z = z_0$ of order p which is defined as [18]

$$\text{Res}(g_i;z_0,p) \triangleq \frac{1}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}\left[g_i(z)(z-z_0)^p\right]\Big|_{z=z_0} = \frac{1}{(p-1)!}\xi_{z_0}^{(p)}, \quad (20)$$

where we use $\xi_{z_0}^{(p)}$ to denote the p-th order derivative of a function $\xi(z)$ with respect to z evaluated at $z = z_0$.

Case II: $u \le t+1$: For this case, there is no pole at $z = 0$ and only the poles at $z = \{\eta_1, \eta_2\}$ contribute to the integral which thus leads to

$$\overline{P}_{md}^{\text{asy}} = 1 - a\Gamma(t+1)e^{-\frac{\lambda}{2}}\sum_{i=1}^{2}\big[\text{Res}(g_i;\eta_i,t+1)\big]. \quad (21)$$

Thus, for both cases, evaluation of the respective residues is needed. The residues $\text{Res}(g_i; \eta_i, t+1)$, $\forall i = \{1, 2\}$ can be expressed in closed-form to be (Appendix B)

$$\text{Res}(g_i;\eta_i,t+1) = \frac{1}{(\theta_i+\overline{\gamma})^{t+1}t!}\sum_{k=0}^{t}\binom{t}{k}\psi_{\eta_i}^{(k)}\phi_{\eta_i}^{(t-k)}, \quad (22)$$

where $\psi(z) \triangleq 1/(z^{u-t-1})$ and $\phi(z) \triangleq e^{\lambda z/2}/(1-z)$. Following similar steps, $\text{Res}(g_i; 0, u-t-1)$ can be expressed as (similar details omitted for brevity)

$$\text{Res}(g_i;0,u-t-1) = \frac{1}{(\theta_i+\overline{\gamma})^{t+1}(u-t-2)!} \times \sum_{k=0}^{u-t-2}\binom{u-t-2}{k}\chi_0^{(k)}\phi_0^{(u-t-k-2)}, \quad (23)$$

where $\chi(z) \triangleq 1/(z-\eta_i)^{t+1}$. Then, (19) and (21) along with (22)-(23) yield (9).

B. Derivation of (22)

Using the definition of the Residue (20), $\text{Res}(g_i; \eta_i, t+1)$ can be expressed as

$$\text{Res}(g_i;\eta_i,t+1) = \frac{1}{(\theta_i+\overline{\gamma})^{t+1}t!}\frac{d^t}{dz^t}\left[\psi(z)\phi(z)\right]\Big|_{z=\eta_i}. \quad (24)$$

Then, applying the general Leibniz rule for t-th order derivative of a product [15, eq. (3.3.8)] on the right hand side of (24) followed by evaluating the resulting expression at $z = \eta_i$ yields (22). Note that $\psi_{\eta_i}^{(k)}$ (and $\chi_0^{(k)}$) can be derived by mathematical induction to be as in (10). For deriving $\phi_{\eta_i}^{(t-k)}$, the Leibniz rule can be re-applied after expressing $\phi(z)$ as a product of the terms $e^{\lambda z/2}$ and $1/(1-z)$ then following some steps of manipulations to yield (11).

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