LETTER

On Statistics of Log-Ratio of Arithmetic Mean to Geometric Mean for Nakagami-*m* Fading Power

Ning WANG[†], *Nonmember*, **Julian CHENG**[††a)], *Member*, *and* **Chintha TELLAMBURA**[†††], *Nonmember*

SUMMARY To assess the performance of maximum-likelihood (ML) based Nakagami m parameter estimators, current methods rely on Monte Carlo simulation. In order to enable the analytical performance evaluation of ML-based m parameter estimators, we study the statistical properties of a parameter Δ, which is defined as the log-ratio of the arithmetic mean to the geometric mean for Nakagami-m fading power. Closed-form expressions are derived for the probability density function (PDF) of Δ. It is found that for large sample size, the PDF of Δ can be well approximated by a two-parameter Gamma PDF.

key words: *Gamma distribution, maximum likelihood, moment generating function, Nakagami m parameter estimation*

1. Introduction

The Nakagami-m is an important fading model in wireless communications research. The probability density function (PDF) of the Nakagami-m fading envelope R is given by [1]

$$f_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1}\exp\left(-\frac{m}{\Omega}r^2\right), \quad r \geq 0 \tag{1}$$

where $\Omega = \mathbf{E}[R^2]$ is the scale parameter, and the shape parameter m, also known as the fading parameter, is defined as

$$m = \frac{\Omega^2}{\mathbf{E}[(R^2-\Omega)^2]}, \quad m \geq \frac{1}{2}. \tag{2}$$

In order to characterize wireless communication channels using the Nakagami-m distribution, it is often required to estimate parameter m from N random variates $R_1, R_2, \ldots, R_N$ drawn independently according to (1). The maximum likelihood (ML) based m parameter estimation can be solved from [2]

$$-\psi(m) + \ln(m) = \Delta \tag{3}$$

where $\psi(\cdot)$ is the digamma function defined as $\psi(x) = \Gamma'(x)/\Gamma(x)$ and where $\Gamma(\cdot)$ is the Gamma function. In (3), the parameter Δ is defined as [2]

$$\Delta \triangleq \ln\left[\frac{1}{N}\sum_{i=1}^{N} R_i^2\right] - \frac{1}{N}\sum_{i=1}^{N}\ln R_i^2 = \ln\left[\frac{\frac{1}{N}\sum_{i=1}^{N} R_i^2}{(\prod_{i=1}^{N} R_i^2)^{\frac{1}{N}}}\right] \tag{4}$$

and it can be interpreted as the log-ratio of the arithmetic mean to the geometric mean for the Nakagami-m fading power.

Since (3) is a transcendental equation which can not be solved analytically, all known ML-based m parameter estimators are approximate solutions to (3). Several ML-based estimators for m have been reported in literature, among which the most widely used is the Greenwood-Durand estimator (GDE) [3]. The GDE was developed in the framework of ML-based Gamma shape parameter estimation. Since the square of the Nakagami-m random variable (RV) gives the Gamma RV and the mapping is one-to-one, estimation of the shape parameter m in the Nakagami distribution or in the Gamma distribution is equivalent. More recently, Cheng and Beaulieu [2] proposed to use the first-order and the second-order approximations to $\psi(\cdot)$ and derived two approximate ML-based estimators for m. It was pointed out by Zhang [4] that similar estimators were reported earlier by Thom [5] in estimation problem for the Gamma distribution in another discipline.

It can be shown that all aforementioned ML-based m parameter estimators are functions of Δ only. Therefore it follows that if we know the PDF of Δ, we can assess the performance of ML-based estimators of the Nakagami m parameter without performing intensive computer simulations.

2. Statistical Properties of Δ

2.1 Nonnegative Property of Δ

According to the well-known Arithmetic-Geometric inequality, we have $\frac{1}{N}\sum_{i=1}^{N} R_i^2 \geq \left(\prod_{i=1}^{N} R_i^2\right)^{1/N}$, therefore; from (4) we must have $\Delta \geq 0$. When $m \to +\infty$ (static channel), it can be shown $\Delta = 0$ [2].

2.2 Complete Sufficient Statistic Property of Δ

We first show that the Nakagami-m distribution belongs to an exponential family by recalling the following theorem [6]:

Theorem: A family of density functions with parameters $\bar{\theta} = [\theta_1, \theta_2, \ldots, \theta_s]^T$ is said to form an s-parameter exponential family if it has the following form

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[†]The author is with the Department of Electrical and Computer Engineering at University of Victoria, Victoria, BC, Canada.
[††]The author is with the School of Engineering at the University of British Columbia, Kelowna, BC, Canada.
[†††]The author is with the Department of Electrical and Computer Engineering at the University of Alberta, Edmonton, AB, Canada.
a) E-mail: julian.cheng@ubc.ca

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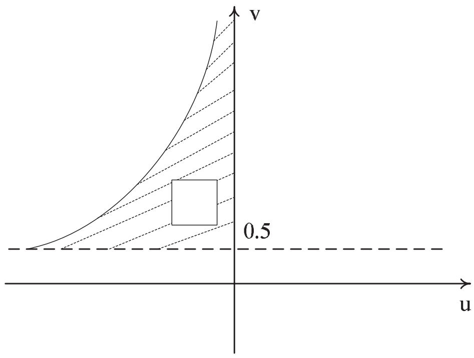

Fig. 1 The natural parameter space $\Xi = \{(-\frac{m}{\Omega}, m) : m > 1/2, \Omega > 0\}$ of the Nakagami-*m* distribution.

$$f_X(x;\bar{\theta}) = C(\bar{\theta}) \exp\left\{\sum_{i=1}^{s} \eta_i(\bar{\theta}) T_i(x)\right\} h(x) \quad (5)$$

where $C, \ \eta_1, \eta_2, \ldots, \eta_s, T_1, T_2, \ldots, T_s, h$ are real-valued functions. Furthermore, if N independent and identically distributed (i.i.d.) random samples $X_1, X_2, \ldots, X_N$ are drawn according to (5), then $\left(\sum_{i=1}^{N} T_j(X_i), j = 1, 2, \ldots, s\right)$ is a joint complete sufficient statistic for $\bar{\theta}$ provided that the natural parameter space $\Xi = \{(\eta_1(\bar{\theta}), \eta_2(\bar{\theta}), \ldots, \eta_s(\bar{\theta}))\} \subseteq R^s$, i.e., Ξ has full rank.

Now let $\bar{\theta} = [m, \Omega]^T$ and rewrite the PDF of the Nakagami-*m* distribution in (1) as

$$f_R(r; m, \Omega) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \exp\left\{-\frac{m}{\Omega} r^2 + m \ln r^2\right\} \frac{1}{r}. \quad (6)$$

If we take $C(m, \Omega) = \frac{2}{\Gamma(m)}(\frac{m}{\Omega})^m$, $\eta_1(m, \Omega) = -m/\Omega$, $\eta_2(m, \Omega) = m$, $T_1(r) = r^2$, $T_2(r) = \ln r^2$, by above theorem we conclude that the Nakagami-*m* distribution is a member of a two-parameter ($s = 2$) exponential family. As a result, the natural parameter space for the Nakagami-*m* distribution becomes $\Xi = \{(-m/\Omega, m) : m > 1/2, \Omega > 0\}$. Let $u = -m/\Omega$ and $v = m$, we plot the natural parameter space in Fig. 1 as the shaded area. Clearly, as shown, one can construct a two-dimensional rectangle inside this natural parameter space. Therefore, Ξ has full rank. It follows that $\left(\sum_{i=1}^{N} R_i^2, \sum_{i=1}^{N} \ln R_i^2\right)$ is a joint complete sufficient statistic of the Nakagami-*m* distribution. We observe from (4) that Δ is a function of this joint complete sufficient statistic.

2.3 Probability Density Function of Δ

To derive the probability density function of Δ, we first consider the moment generating function (MGF) of Δ by the definition

$$\begin{aligned}\Phi_\Delta(s) &= \mathbf{E}\left[e^{s\Delta}\right] \\ &= \frac{\left[\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s} \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{N} \prod_{i=1}^{N} r_i^{2m - \frac{2s}{N} - 1} \\ &\quad \times \left(\sum_{i=1}^{N} r_i^2\right)^s \exp\left(-\frac{m}{\Omega} \sum_{i=1}^{N} r_i^2\right) dr_1 \cdots dr_N. \quad (7)\end{aligned}$$

Let $d = m - s/N$ and substitute r_i^2 by x_i, we obtain

$$\begin{aligned}\Phi_\Delta(s) &= \frac{\left[\frac{1}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s} \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{N} \prod_{i=1}^{N} x_i^{d-1} \\ &\quad \times \left(\sum_{i=1}^{N} x_i\right)^s \cdot \exp\left(-\frac{m}{\Omega} \sum_{i=1}^{N} x_i\right) dx_1 \cdots dx_N. \quad (8)\end{aligned}$$

The multiple N integrals in (8) can be reduced into a single integral by invoking the following integral identity [7]

$$\begin{aligned}&\underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{N} x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} f\left(\sum_{i=1}^{n} x_i\right) dx_1 \cdots dx_n \\ &= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n)} \int_0^{+\infty} u^{\alpha_1 + \cdots + \alpha_{n-1}} f(u) du. \quad (9)\end{aligned}$$

Having $\alpha_1 = \alpha_2 = \ldots = \alpha_N = d$ and $f(x) = x^s \exp\left(-\frac{m}{\Omega}x\right)$ in (9), we obtain a compact form for the MGF of Δ as

$$\begin{aligned}\Phi_\Delta(s) &= \frac{\left[\frac{1}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s} \cdot \frac{[\Gamma(d)]^N}{\Gamma(Nd)} \cdot \int_0^{+\infty} u^{Nd-1} u e^{-\frac{m}{\Omega}u} du \\ &= \frac{\Gamma(mN)[\Gamma(m - s/N)]^N}{N^s [\Gamma(m)]^N \Gamma(mN - s)} \end{aligned} \quad (10)$$

where in obtaining the last step we used the definition of the Gamma function.

By applying an inverse Laplace transform to the MGF of Δ in (10), the PDF of Δ can be obtained as [8, Eq. (3.1.1.22)]

$$f_\Delta(\delta) = N\xi \cdot G_{N,N}^{0,N}\left(e^{N\delta} \middle| \begin{array}{ccc} 1-m & \cdots & 1-m \\ 1-m & \cdots & 1-m-\frac{N-1}{N} \end{array}\right) \quad (11)$$

where $G_{p,q}^{m,n}(\cdot)$ is the Meijer's G-function [9], and $\xi = [\Gamma(mN)] / \{(2\pi)^{\frac{1-N}{2}} N^{Nm - \frac{1}{2}} [\Gamma(m)]^N\}$. Alternatively, one can obtain (11) with an aid of Gauss multiplication theorem [10].

Computer simulations were carried out to generate empirical PDFs of Δ for different values of m and N, and they were used to compare the analytical PDFs obtained from (11). Figure 2 shows the analytical and empirical PDFs of Δ for $N = 10$ when m =0.5, 1, and 2. It is shown that the analytical PDFs of Δ have excellent agreement with the empirical PDF curves.

3. Gamma Approximation

When the sample size N becomes large, the latest version

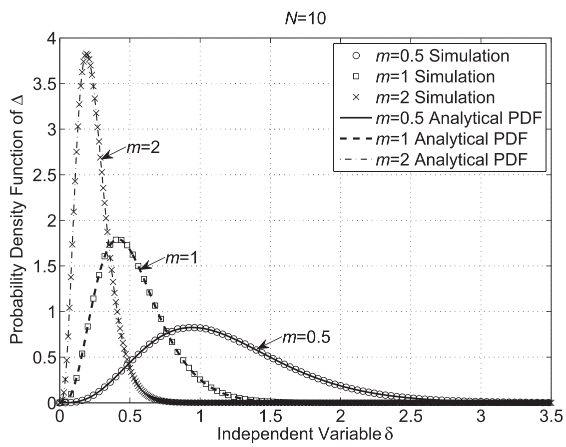

Fig. 2 Comparison of empirical PDFs and analytical PDFs of Δ for $m = 0.5, 1, 2$ with $N = 10$.

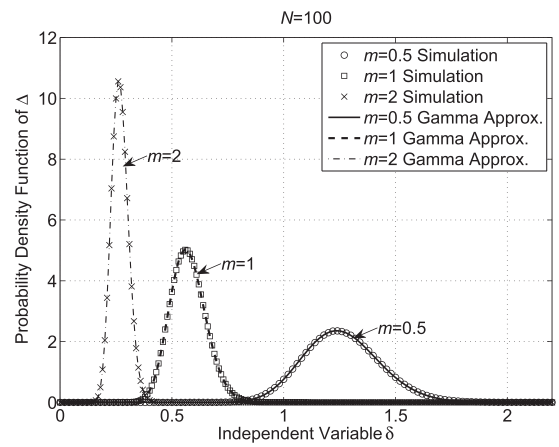

Fig. 3 Comparison of empirical PDFs and Gamma approximated PDFs of Δ for $m = 0.5, 1, 2$ with $N = 100$.

of commercial software tools such as MAPLE and MATHEMATICA are incapable of evaluating our analytical PDF expression in (11). To avoid the high computational complexity associated with the Meijer's G-function for large N, we are motivated to approximate (11) using another PDF which can be easily evaluated and is analytically tractable.

3.1 Gamma Approximation for PDF of Δ

By the nonnegative property discussed in Sect. 2.1, we propose a two-parameter Gamma PDF, whose support is also on $[0, +\infty)$, to approximate the PDF of Δ. To determine the scale parameter θ and the shape parameter k in the two-parameter Gamma PDF given by

$$f_X(x) = \frac{x^{k-1}e^{-x/\theta}}{\theta^k\Gamma(k)}, \quad x \geq 0;\ \theta, k > 0 \tag{12}$$

we simply match the first two moments of the Gamma distribution to those of Δ. The first two moments of the two-parameter Gamma distribution are well-known, which are $k\theta$ and $(k^2\theta^2 - k\theta^2)$; the first two moments of Δ can also be easily obtained by taking the first and the second derivatives of the MGF of Δ in (10) with respect to s and evaluating the resulting derivatives at $s = 0$. Solving the two moment equations for θ and k, we obtain

$$\theta = \frac{\frac{1}{N}\psi'(m) - \psi'(mN)}{-\psi(m) - \ln(N) + \psi(mN)} \tag{13a}$$

$$k = \frac{\frac{1}{N}\psi'(m) - \psi'(mN)}{[-\psi(m) - \ln(N) + \psi(mN)]^2}. \tag{13b}$$

The two-parameter Gamma approximation is desirable since this PDF has a simple exponential form, which can be easily evaluated and manipulated in practice.

3.2 Validating the Gamma Approximation

Computer simulations were also carried out to compare the approximated Gamma PDFs with the empirical PDFs of Δ.

Table 1 Kolmogorov-Smirnov test for Gamma approximation.

	$n = 100$			$n = 1,000$		
	D_{max}	D_{avg}	*Acpt. %*	D_{max}	D_{avg}	*Acpt. %*
			$N = 100$			
$m = 0.5$	0.212	0.085	98.95%	0.075	0.027	99.00%
$m = 1$	0.222	0.085	99.11%	0.066	0.027	98.91%
$m = 2$	0.211	0.085	99.00%	0.074	0.027	99.03%
$m = 5$	0.232	0.085	99.01%	0.068	0.027	98.88%
			$N = 10$			
$m = 0.5$	0.226	0.085	99.03%	0.067	0.027	99.05%
$m = 1$	0.218	0.085	98.87%	0.069	0.027	98.83%
$m = 2$	0.226	0.085	99.01%	0.072	0.027	99.00%
$m = 5$	0.232	0.085	99.11%	0.078	0.027	99.01%

Figure 3 shows the empirical PDFs of Δ and the approximated Gamma PDFs for $m = 0.5$, 1, and 2 with $N = 100$. The corresponding (θ, k) values are (0.023, 54.59), (0.011, 51.16), and (0.005, 49.93), respectively. We can observe that the Gamma approximations very well fit the empirical PDF curves for all m values considered here. It appears that this simple two-parameter Gamma PDF can approximate the PDF of Δ well.

The Kolmogorov-Smirnov (K-S) test for goodness-of-fit was conducted to validate the feasibility of approximating Δ as a Gamma RV. Case studies were conducted using test sample volume $n = 100$ and 1,000 for $m = 0.5$, 1, 2, and 5 with Δ generated from $N = 100$ i.i.d. Nakagami-m RVs. The significance level α was chosen to be 0.01, giving a 0.99 confidence level for the K-S test. The K-S test critical values D_n^α for prescribed α and n can be found in [11].

In our simulation, we performed 10,000 independent tests for each case studied. The acceptance rate values in Table 1 suggest that for every case, about 99% of the experiments accepted the hypothesis that Δ can be modeled as a Gamma RV at a confidence level of 0.99. For each case studied, we also define D_{avg} as the average value of 10,000 independent test statistics. We can observe that D_{avg} values in Table 1 are significantly below the corresponding critical values. In summary, the K-S test concludes that the two-parameter Gamma PDF can be used to accurately approxi-

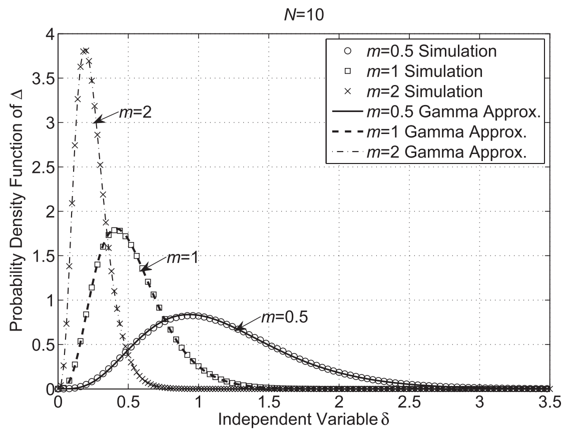

Fig. 4 Comparison of empirical PDFs and Gamma approximated PDFs of Δ for $m = 0.5, 1, 2$ with $N = 10$.

mate the PDF of Δ. Finally, we comment that $\Omega = 1$ is used to generate the simulation results in Figs. 2 and 3. However, the accuracy of our approximation as well as the K-S test does not depend on Ω, which is just a scaling factor and does not appear in the MGF of Δ.

To investigate the accuracy of the proposed Gamma approximation with a smaller sample size, computer simulation and the K-S test have also been performed when the sample size is $N = 10$. In the simulation, we have again chosen $\Omega = 1$ and $m = 0.5$, 1, and 2, and the corresponding (θ, k) values are (0.233, 5.004), (0.113, 4.670), and (0.054, 4.545), respectively. It can be easily observed from Fig. 4 that excellent fit to the empirical PDFs is achieved by the Gamma approximation with $N = 10$. The same conclusion can also be drawn from the K-S test results for the $N = 10$ case shown in the lower part of Table 1.

4. Conclusions

By investigating the statistical properties of parameter Δ, which is defined as the log-ratio of the arithmetic mean to the geometric mean for the Nakagami-*m* fading power, the MGF and the exact PDF of Δ have been derived. Gamma approximation of the PDF of Δ, which avoids the computational complexity of the exact PDF for large sample size N, has also been proposed. The validity of using the two-parameter Gamma PDF to approximate the PDF of Δ has been established using the K-S test.

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