Uniform approximations for wireless performance in fading, noise and interference

C. Tellambura, Y. Dhungana and M. Soysa
Department of Electrical and Computer Engineering
University of Alberta, Edmonton, Alberta T6G 2V4, Canada

***Abstract*—We derive simple uniform approximations (UAs) for the bit error rate (BER), the symbol error rate (SER), and the outage of wireless digital communication systems impaired by fading, noise, and interference. The striking feature of the UAs is their accuracy over the whole range of signal-to-noise ratio (SNR) values, whereas the existing high-SNR approximations break down as the SNR decreases. The UAs require slightly more information than that for high-SNR expressions. The additional information required in the case of error probabilities is the several moments (fractional) of channel gain, which can be extracted readily from the PDF, MGF or the Mellin transform of the PDF. The computation of the UA is simple and requires only the solution of a set of linear equations. Additionally, we also generalize the previous asymptotic results of Wang and Giannakis. The unified asymptotic results of the average of an arbitrary performance measure are thus derived. Various BER and SER expressions then become special cases of this unified approach.**

I. INTRODUCTION

Performance analysis of wireless systems over fading, noise and other forms of interference is extensive [1], [2]. This analysis typically requires averaging over the statistical distributions, and thus closed-form analysis of the bit error rate (BER), outage and ergodic capacity, for instance, has been pervasive. Although such works are extensive [1]–[3], simple yet accurate large signal-to-noise ratio (SNR) approximations have become popular recently. Large-SNR analysis provides direct insights into how channel and modulation parameters determine the diversity gain and SNR gain of various digital receiver techniques, feasible for problems that are otherwise analytically intractable. Important asymptotic SNR results may be found in [4]–[6].

Wang and Giannakis [5] use the local information from a first-order expansion of the probability density function (PDF) of the instantaneous SNR near the origin to determine the high-SNR error rate; i.e., just the first term of the Taylor expansion of the PDF determines the diversity order and coding gain. Importantly, the accuracy of this method is $O(\rho^{-G_d})$, where ρ is the unfaded link SNR, and G_d is the diversity order. This level of accuracy at high SNR is more than sufficient for wireless engineering applications. This approach has thus been widely used in recent research (cited over 380 times, see Google Scholar).

However, the accuracy of large SNR analysis [4]–[6] degrades as the SNR decreases. In some cases, the SNR must exceed, say, 20 dB for the approximations of [5] to be accurate. However, due to low power specifications or high energy-efficiency requirements, many communication systems may actually operate in the low SNR regime. Thus, accurate approximations over the range $-\infty < \rho < \infty$ dB are highly desirable. Is it possible to develop approximations that are valid over the entire SNR range?

In this paper, we give an affirmative answer to this question by deriving simple **uniform approximations** for the BER, the symbol error rate (SER), and the outage of digital communication systems impaired by fading and noise. The UAs require slightly more information than that for high-SNR expressions of [5]. The additional information required is the several moments (fractional) of channel gain. This information can be extracted from the PDF or moment-generating function (MGF) of the channel gain or from the Mellin transform of the PDF. The computation of the UA is extremely simple and requires only the solution of a set of linear equations. The UAs are highly accurate over the entire SNR range, $-\infty < \rho < \infty$ dB. Not surprisingly, the UA and the approximation of [5] coalesce into one in the high-SNR regime.

Additionally, we also generalize the results of [5] in two distinct ways. First, we show the case-by-case approach of [5] can be unified as the evaluation of the average of an arbitrary performance measure. Various BER and SER expressions then become special cases of this approach. In some cases, an improved accuracy level of $O(\rho^{-(G_d+1)})$ can even be achieved. Moreover, [5] does not immediately reveal how the coding gain relates to the modulation format. Our results clearly show that the coding gain is determined by the Mellin transform of the performance measure, which depends on the modulation format ([6] has derived high-SNR approximations for the SER of linear modulations over non-Gaussian noise and interference, via the Mellin transform of the PDF of the noise). Second, we also derive both low-SNR and high-SNR approximations for a generalized performance measure that is a weighted sum of either a finite or an infinite number of terms. For example, the union bound on coded systems and all digital modulations can be treated under this formulation.

II. PRELIMINARIES

A. Assumptions

We make the following assumptions where AS1 and AS2 are consistent with those in [5] and are stated for completeness.

AS1) The instantaneous SNR at the receiver is given by $\gamma = \rho\beta$, where ρ is the unfaded link SNR (aka the transmit SNR), or simply the SNR, and β is the channel-dependent, system-dependent nonnegative random variable. We will refer to parameter β as the channel gain.

AS2) Unless otherwise stated, the PDF of β may be expanded as $f(\beta) = a\beta^t + O(\beta^{t+1})$ as $\beta \to 0^+$. This assumption amounts to a first-order Taylor series expansion near the origin, as in [5]. The parameter t describes the rate of growth of the PDF near the origin. Note that this assumption is not limiting because this expansion is possible in all practical cases.

AS3) The system performance metric is denoted by $h(x)$; we assume $h(x)$ decays exponentially for large arguments; i.e., $h(x) \sim e^{-x}$ as $x \to \infty$. The instantaneous performance is $h(\gamma)$. The average value $\mathbb{E}[h(\gamma)]$ is the quantity of interest. The common performance metric function $h(x)$ includes $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, which is used to represent the BER or SER of various digital modulation schemes with a coherent demodulation process, while e^{-x} is used to represent the BER or SER of non-coherent demodulation schemes and others [1], [2].

B. Diversity order and SNR gain

It is well known that for transmission over flat fading channels impaired by Gaussian noise the SER P_E at high SNR can be approximated by

$$P_E \approx (G_c\rho)^{-G_d}$$

as $\rho \to \infty$, where ρ is the average SNR, and G_c and G_d are referred to as the coding gain (also known as the SNR gain or combining gain) and the diversity gain (diversity order), respectively. The Wang and Giannakis [5] main result is listed here for quick reference. The following high-SNR approximation, with the assumption AS2) above, is valid as $\rho \to \infty$:

$$\mathbb{E}\left[Q\left(\sqrt{\kappa\gamma}\right)\right] \sim \frac{2^t a\Gamma(t+\frac{3}{2})}{\sqrt{\pi}(t+1)} \frac{1}{(\kappa\rho)^{t+1}}, \quad \kappa > 0. \tag{1}$$

The error term associated with eq. (1) is found to be $O\left(\rho^{-(t+1)}\right)$. This approximation (eq. (1)) shows that the diversity order is equal to $t+1$; that is, the diversity order directly relates to the rate of growth of the PDF near the origin. This result is intuitively satisfying. Similarly, the coding gain is determined by both t and a. In sum, both the diversity order and coding gain require information from only the first-order expansion (AS2) of the PDF around the origin. In this paper, we will directly generalize eq. (1) to an arbitrary modulation format (see eq. (5)).

III. Average probability of error

In this section, we provide our main results in terms of three propositions. Their proofs are omitted for brevity and will be provided in a journal version of this paper.

The error rates, capacity, outage and other related measures are typically expressed as

$$\mathbb{E}[h(\gamma)] = \int_0^\infty h(\rho\beta) f(\beta)\, d\beta, \tag{2}$$

where $h(x)$ represents performance measures that need averaging over noise, fading and other effects. Clearly, (eq. (1)) is a special case of (eq. (2)) when $h(x) = Q(\sqrt{\kappa x})$.

A. High SNR approximation

Proposition 1. *The PDF of the channel gain is given by*

$$f(\beta) = \beta^t g(\beta) \quad as \quad \beta \to 0^+ \tag{3}$$

with $g(0) \neq 0$. The averaged performance metric eq. (2) can be approximated by

$$\mathbb{E}[h(\gamma)] \approx \frac{H(t+1)}{\rho^{t+1}}\, g\left(\frac{H(t+2)}{\rho H(t+1)}\right), \tag{4}$$

where $H(s)$ is the Mellin transform of $h(x)$, and the error in this approximation is $O(\rho^{-(t+2)})$.

Note that a special case of the above occurs when $g(\beta) = a$, that is, with the PDF model that appears in assumption AS2. (This case was treated in [5]). The average defined in eq. (2) can be approximated by the following as $\rho \to \infty$:

$$\mathbb{E}[h(\gamma)] \approx \frac{aH(t+1)}{\rho^{t+1}} = \left(\frac{1}{\sqrt[t+1]{aH(t+1)}}\,\rho\right)^{-(t+1)}. \tag{5}$$

This approximation (eq. (5)) shows that the coding gain depends on the Mellin transform of $h(x)$, whereas the diversity gain depends on the degree of the Taylor expansion of the PDF of β.

B. Low-SNR and High-SNR approximations - general case

In general, the performance metric can be the sum of either a finite or an infinite number of terms (e.g., the SER of digital modulations and the union bound of coded systems). The next proposition provides a means to derive both low-SNR and high-SNR approximations. The main idea is to transform eq. (2) via the Parseval formula into a contour integral in the complex plane (the vertical integration line lies in the so-called fundamental strip). The approximations to the integral are derived by considering the poles of $F(1-s)H(s)$, where $F(s)$ is the Mellin transform of the PDF of β. The poles on the right (left) of the fundamental strip yield the high-SNR (low-SNR) approximation. We refer the reader to [7] for more details on this process.

Proposition 2. *Consider a generalized performance metric given by the sum*

$$h(x) = \sum_k \lambda_k g(\mu_k x), \tag{6}$$

where $g(x)$ is a general base function, whose Mellin transform is $G(s)$. Let $\Lambda(s) = \sum_k \lambda_k \mu_k^{-s}$. Then the average defined in eq. (2) has the following asymptotics:

$$\mathbb{E}[h(\gamma)] \sim \pm \sum_{s\in H} Res\left\{\frac{1}{\rho^s} G(s)\Lambda(s)F(1-s)\right\}, \tag{7}$$

where, for an expansion as $\rho \to 0^+$, the sum is over the set H of poles to the left of the fundamental strip, and the sign *is '+'; for an expansion as $\rho \to \infty$, the sum is over the poles to the right of the fundamental strip, and the sign is '−'.*

Suppose the sequence $-t, -(t+p_1), -(t+p_2), \ldots$ is the left simple poles of $F(s)$ with $t \geq 0$ and $\{p_l\}$ $(l = 1, 2, \ldots)$ real, positive and strictly increasing ($p_0 = 0$); then

$$\mathbb{E}[h(\gamma)] \sim \sum_{l=0} \frac{\lim_{s \to 1+t+p_l} [(1+t+p_l-s)G(s)\Lambda(s)F(1-s)]}{\rho^{1+t+p_l}}. \tag{8}$$

By considering the first term of this asymptotic series, the diversity order is

$$G_d = 1+t,$$

and the coding gain is

$$G_c = \left\{ \lim_{s \to 1+t} [(1+t-s)G(s)\Lambda(s)F(1-s)] \right\}^{-\frac{1}{1+t}}.$$

The result in eq. (8) encompasses [5] as a special case. In order to understand this point, note that if $f(\beta) = a\beta^t$, then the Mellin transform is $F(s) = \int_0^\infty \beta^{s-1} f(\beta) d\beta = \frac{a}{s+t}$. Thus, $F(s)$ has a simple pole at $s = -t$, so that $F(s)$ is equivalent to $F(1-s)$ with a pole at $s = 1+t$. Thus, the main assumption AS2 used by [5] is equivalent to a simple pole in the Mellin transform.

C. Uniform approximation

The UA for the error rate is developed next. The main idea is to find a rational expression that matches with both the low-SNR and high-SNR approximations simultaneously [8].

Proposition 3. *As per AS2, the fading PDF is $f(\beta) = a\beta^t + O\left(\beta^{t+1}\right)$ as $\beta \to 0^+$ with $t \geq 0$. We assume that t is an integer. Define $x = \sqrt{\rho}$ and $\mu_z = \int_0^\infty \beta^z f(\beta)\, d\beta$. Recall that $\gamma = \rho\beta$ and substitute this in $\mathbb{E}\left[Q\left(\sqrt{\kappa\gamma}\right)\right]$. The average error rate is then given by*

$$\mathbb{E}\left[Q\left(\sqrt{\kappa\beta}x\right)\right] = \frac{1 + \sum_{l=1}^{L} a_l x^l}{2 + \sum_{k=1}^{K} b_k x^k} + E(x), \tag{9}$$

where $K = L + 2(t+1)$, and $L \geq 2$ is an integer. The coefficient vector $\mathbf{b} = (b_1, b_2, \ldots, b_K)^T$ is given by

$$\mathbf{b} = -2\mathbf{P}^{-1} \left(\tilde{c}(1) \quad \tilde{c}(2) \quad \ldots \quad \tilde{c}(K-1) \quad \tilde{c}(K)\right)^T, \tag{10}$$

where $\tilde{c}(l) = c(l+L-2)$, $g = \frac{2^t a \Gamma(t+3/2)}{\sqrt{\pi}(t+1)\kappa^{t+1}}$,

$$c(l) = \begin{cases} \frac{1}{2} & l = 0 \\ \dfrac{(-1)^{(l+1)/2}(\kappa/2)^{l/2}\mu_{l/2}}{\sqrt{\pi}\, l\, \Gamma\left[(l+1)/2\right]} & l = 1, 3, \ldots, \\ 0 & \textit{otherwise} \end{cases} \tag{11}$$

and $\mathbf{P} = \{p_{ij}\}$, $i = 1, \ldots, K$, $j = 1, \ldots, K$ with

$$p_{ij} = \begin{cases} \tilde{c}(i-j) - g & j = i + K - 2 \\ \tilde{c}(i-j) & \textit{otherwise.} \end{cases} \tag{12}$$

TABLE I

MELLIN TRANSFORMS OF FADING MODELS

Fading	$f(\beta)$	$F(s)$
Rayleigh	$e^{-\beta}$	$\Gamma(s)$
Nakagami-m	$\frac{m^m \beta^{m-1} e^{-m\beta}}{\Gamma(m)}$	$\dfrac{m^{1-s}\Gamma(s+m-1)}{\Gamma(m)}$
Weibull	$\dfrac{b\beta^{\frac{b}{2}-1}}{2a^{\frac{b}{2}}} e^{-\left(\frac{\beta}{a}\right)^{\frac{b}{2}}}$	$a^{s-1}\Gamma\left(\frac{2s+b-2}{b}\right)$
MRC	$\dfrac{\beta^{N_r-1}e^{-\beta}}{(N_r-1)!}$	$\dfrac{\Gamma(N_r+s-1)}{(N_r-1)!}$
SC	$N_r(1-e^{-\beta})^{N_r-1}e^{-\beta}$	$N_r \sum_{l=0}^{N_r-1} \dfrac{(-1)^l \binom{N_r-1}{l}\Gamma(s)}{(l+1)^s}$

The coefficient vector $\mathbf{a} = (a_1, a_2, \ldots, a_L)^T$ is given by

$$a_i = \begin{cases} 2c(i) + \sum_{k=1}^{i} b(k)c(i-k) & i = 1, \ldots L-2 \\ g b_{K-j} & j = 0, 1; i = L - j \end{cases} \tag{13}$$

for $i = 1, \ldots L$.

The approximation $E(x)$ of eq. (9) is $O\left(\rho^{K+L}\right)$ as $\rho \to 0$ (low-SNR region) and $O\left(\rho^{-(t+1)}\right)$ as $\rho \to \infty$ (high-SNR region).

D. BPSK performance in N_r branch MRC in iid Rayleigh

Maximal ratio combining (MRC) is an optimal diversity combining method [2, Chap. 9]. An exact analysis of the BER of MRC under independent and identical (iid) Rayleigh fading is given in [1, Sec. 14.4].

The PDF of β for the case of MRC with N_r diversity branches and its Mellin transform is given in Table I. We observe that $F(1-s)$ has simple poles at $s = N_r, N_r+1, \ldots$, which are on the right of the fundamental strip. As per eq. (7), these poles describe the asymptotic values of $\mathbb{E}[h(\gamma)]$ as $\rho \to \infty$.

For coherent BPSK, $h(x) = Q(\sqrt{2x})$. From Table 2, the Mellin transform of $h(x)$ has poles at $s = 0, -1/2, -3/2, \ldots$. Hence, by using Proposition 2, we obtain the following result:

$$\mathbb{E}[Q(\sqrt{2\gamma})] = \begin{cases} \frac{1}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(N_r+k+1/2)}{\sqrt{\pi}(2k+1)\Gamma(N_r)} \rho^{k+1/2} \\ \dfrac{\Gamma(N_r+1/2)}{2\sqrt{\pi}(N_r)\Gamma(N_r)\rho^{N_r}}, \end{cases} \tag{14}$$

where the upper expansion holds for $\rho \to 0^+$ and the other for $\rho \to \infty$. Low-SNR and high-SNR approximations in eq. (14) provide sufficient information to compute the UA for this case, which will be accurate over the whole range, $0 \leq \rho < \infty$. By converting the above to the uniform error rate format given in eq. (9), we can obtain an approximation valid for both the low and high regions of SNR.

Example 1: For the single branch case ($N_r = 1$), by using Proposition 3, the following simple UA can be obtained:

$$\mathbb{E}[Q(\sqrt{2\gamma})] = \frac{1 + x + 0.5\,x^2}{2 + 4\,x + 5\,x^2 + 4\,x^3 + 2\,x^4}, \tag{15}$$

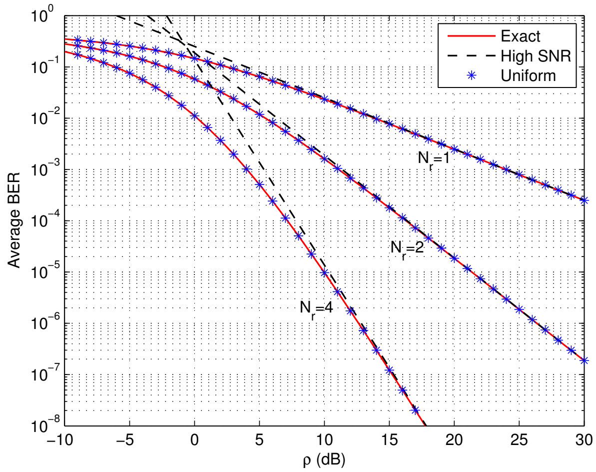

Fig. 1. The exact BER of MRC system, the high-SNR approximation eq. (1) and the UA eq. (9). In the UA, $L = 2$.

TABLE II
MELLIN TRANSFORMS OF PERFORMANCE MEASURES

Application	$h(x)$	$H(s)$
Coherent BPSK	$Q(\sqrt{2x})$	$\frac{\Gamma(s+1/2)}{2s\sqrt{\pi}}$
NCFSK	$\frac{1}{2}e^{-\frac{x}{2}}$	$2^{s-1}\Gamma(s)$
DPSK	$\frac{1}{2}e^{-x}$	$\frac{1}{2}\Gamma(s)$
Coherent FSK	$Q(\sqrt{x})$	$\frac{2^{s-1}\Gamma(s+1/2)}{s\sqrt{\pi}}$
Outage probability	$u(\gamma_T - x)$	$\frac{\gamma_T^s}{s}$

where $x = \sqrt{\rho}$. The exact average error rate for this case is well-known [1, Sec. 14.4]. Note that the expression eq. (15) matches the first three low-SNR terms and the high-SNR term in eq. (14). Similar UAs for any other N_r can be readily derived and are omitted for brevity. To test its accuracy, the UA is plotted along with the exact result [1, Sec. 14.4] and the high-SNR result of Wang and Giannakis (eq. (1)) in Figure 1. Notice that the UA coincides with the exact value for the entire range $-10 \le \rho < 30$ dB, while the high-SNR result of Wang and Giannakis, eq. (1), fails as the SNR decreases. Clearly, the UA provides an excellent approximation over the whole range of the SNR.

E. Approximation with Proposition 1

Consider the case of MRC with N_r diversity branches in iid Rayleigh fading. The PDF of β takes the form

$$f(\beta) = \beta^{N_r-1} g(\beta), \tag{16}$$

where $g(x) = \frac{e^{-x}}{(N_r-1)!}$. By using $H(s)$ from Table 2, the following approximation based on Proposition 1 can be obtained:

$$\mathbb{E}[Q(\sqrt{2\gamma})] \approx \frac{\Gamma(N_r + \frac{1}{2})}{2\sqrt{\pi}N_r!\rho^{N_r}} e^{-\frac{N_r(N_r+1/2)}{(N_r+1)\rho}}. \tag{17}$$

Figure 2 shows how this approximation compares against the Wang and Giannakis result, eq. (1). The relative error of both these approximations compared to the exact result is plotted. In terms of the relative error, the approximation of Proposition 1 is at least an order of magnitude better than eq. (1). However, this improved level of accuracy is not always achievable.

F. BPSK performance in N_r branch SC in iid Rayleigh fading

Selection combining (SC) is another classical diversity combining technique. It has less implementation complexity than MRC, but suffers a loss in performance. The PDF of β and its Mellin transform in this case are given in Table I and can be written as

$$\begin{aligned} F(1-s) &= \left[N_r \sum_{n=0}^{N_r-1} (-1)^n \binom{N_r-1}{n} \frac{1}{(n+1)^{1-s}} \right] \Gamma(1-s) \\ &= \nu(s)\Gamma(1-s). \end{aligned} \tag{18}$$

Given the term $\Gamma(1-s)$, it seems that $F(1-s)$ has simple poles at $s = 1, 2, \ldots$ and this suggests that the diversity order of the system is just one, which is incorrect. Surprisingly, it turns out that $\nu(s)$ has zeros at $s = 1, 2, \ldots, N_r - 1$, and these zeros cancel out the first $N_r - 1$ poles of $\Gamma(1-s)$. Therefore, the poles are at $s = N_r, N_r + 1, \ldots$, which are on the right of the fundamental strip. These are simple poles. As per Proposition 2, they describe the high-SNR approximation ($\rho \to \infty$).

As before, we consider the average error performance of BPSK, for which, $h(x) = Q(\sqrt{2x})$ and its Mellin transform is given in Table 2, with poles at $s = 0, -1/2, -3/2, \ldots$. Thereby, using Proposition 2, we find

$$P_e(\rho) = \begin{cases} \frac{1}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\nu(-1/2-k)\Gamma(k+3/2)}{\sqrt{\pi}(2k+1)} \rho^{k+1/2} \\ \frac{1}{(N_r)!} \frac{\nu(N_r)\Gamma(N_r+1/2)}{2\sqrt{\pi}(N_r)\rho^{N_r}}, \end{cases} \tag{19}$$

where the upper expansion holds for $\rho \to 0^+$ and the other for $\rho \to \infty$.

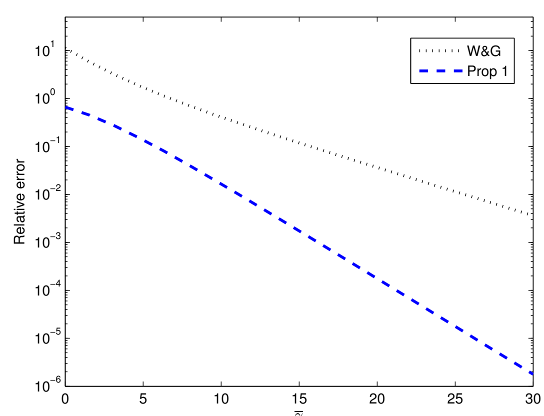

Fig. 2. The accuracy of eq. (1) and eq. (17).

G. Application of Proposition 3 with the MGF only

Proposition 3 requires quantities a, t and $\mu_{l/2}$, $l = 1, 3, 5, \ldots$. When the PDF is unavailable, we can use the MGF to extract this information. That a and t can be computed from the MGF is already known [5] and fortunately, so are the fractional moments [9]. Consider N_r branch MRC in iid Rayleigh fading as an example. The MGF of β is $M_\beta(s) = \mathbb{E}(e^{-s\beta}) = \frac{1}{(1+s)^{N_r}}$, which can be expanded for $s \to \infty$ as $M_\beta(s) = \frac{1}{(s)^{N_r}} + O\left(\frac{1}{(s)^{N_r+1}}\right)$, and hence, a and t are obtained to be $\frac{1}{\Gamma(N_r)}$ and $N_r - 1$ respectively [5].

The fractional moments of β, $\mu_{l/2}$, $l = 1, 3, 5, \ldots$ can be computed by using [9] as

$$\mu_{l/2} = \mathbb{E}[\beta^{l/2}] = \Gamma(\lambda)^{-1} \int_0^\infty t^{\lambda-1} \zeta(-t) dt, \qquad (20)$$

where λ is chosen to be $1/2$ such that $n = l/2 + \lambda$ is a positive integer while satisfying $0 < \lambda < 1$; $\zeta(s) = \frac{d^n M_\beta(s)}{ds^n}$.

By using eq. (20), $\mu_{l/2}$ can be obtained as

$$\begin{aligned} \mu_{l/2} &= \frac{\Gamma(N_r+n)}{\Gamma(1/2)\Gamma(N_r)} \int_0^\infty t^{-1/2}(1+t)^{-(N_r+n)} dt \\ &= \frac{\Gamma(N_r + l/2)}{\Gamma(N_r)}, \end{aligned} \qquad (21)$$

where eq. (21) is obtained by using [10, eq. (3.191.3)]. This example shows that the entire UA can be computed from the MGF only, facilitating the use of UA in many cases where the MGF is directly available.

H. Co-channel Interference

The analysis of such interference is based on the signal to interference and noise ratio (SINR). In most cases, the SINR(γ) is approximated by the ratio of the central chi-squared distributed random variables [11]. For the case of MRC with N_r diversity branches and N_I interferers, we write $\gamma = \rho\beta$, where $\beta = \frac{X}{Z}$ and X, Z are central chi-squared distributed random variables with $2N_r$ and $2N_i$ degrees of freedom, respectively.

The Mellin transform of the PDF of β can be obtained easily even without knowing the PDF itself, as $F(s) = F_X(s)F_Z(2-s)$. For this analysis, $F_X(s) = \Gamma(s + N_r - 1)/\Gamma(N_r)$ and $F_Z(s) = \Gamma(s + N_i - 1)/\Gamma(N_i)$. The poles of $F(s)$ can thus be readily determined, which, along with Proposition 2, enable the development of both low-SNR and high-SNR approximations. These can then be combined (i.e., as in Proposition 3) to develop a UA for the error rate.

I. Dual-Hop relay performance analysis

Figure 3 shows the BPSK error performance of a dual hop relay system over Rayleigh fading where the exact curve is plotted by using the results from [12]. By using [12, eq. (27)], the first-order expansion of the CDF of the end-to-end SNR, $F_{\gamma_{eq}}(\gamma)$ can be easily shown to be

$$F_{\gamma_{eq}}(\gamma) = \left(1 + \frac{1}{\zeta}\right)\frac{\gamma}{\rho_1} + O\left(\frac{\gamma}{\rho_1}\right), \qquad (22)$$

and hence, we obtain our required parameters $a = (1 + \frac{1}{\zeta})$ and $t = 0$, where $\zeta = \rho_2/\rho_1$, ρ_1 and ρ_2 are the average SNRs of the first and second hop respectively. If we say $\gamma = \rho_1\beta$, the required moments information for UA can be computed by using

$$\begin{aligned} \mu_z &= \int_0^\infty z\beta^{z-1}\left(1 - F_{\frac{\gamma_{eq}}{\rho_1}}(\beta)\right) d\beta \\ &= \frac{2z\sqrt{\pi}}{\sqrt{\zeta}} \frac{2B}{(A+B)^{z+2}} \frac{\Gamma(z+2)\Gamma(z)}{\Gamma(z+3/2)} \\ &\times {}_2F_1\left(z+2, \frac{3}{2}; z+\frac{3}{2}; \frac{A-B}{A+B}\right), \end{aligned} \qquad (23)$$

where $A = 1 + \frac{1}{\zeta}$, $B = 2\sqrt{\frac{1}{\zeta}}$, ${}_2F_1(.,.;.;.)$ is the Gaussian Hypergeometric function. eq. (23) is obtained directly by substituting [12, eq. (27)] into the top equation and solving the resultant integral by using [10, eq. (6.621.3)].

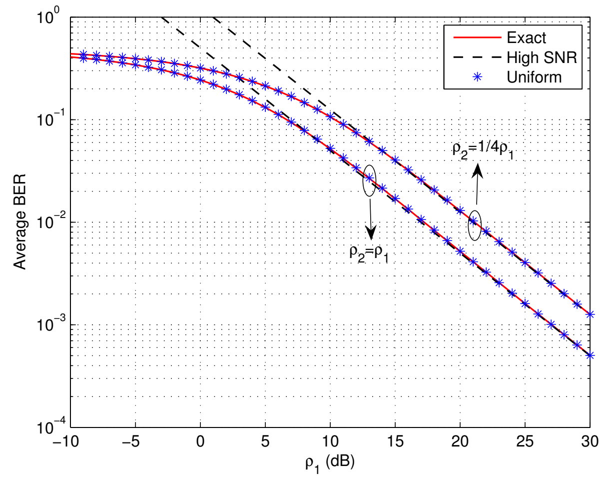

Fig. 3. The BER of a dual-hop relay system in Rayleigh fading.

J. Energy detection

The determination of the presence or absence of an unknown signal over a noisy channel through energy detection is of great interest [13]. Recently, energy detection has gained wide-spread attention for spectrum sensing in cognitive radio [14], [15]. For $2u$ samples, with λ threshold and γ instantaneous SNR, the average miss probability [15, Eq. 4] $P_m = \int_0^\infty (1 - Q_u(\sqrt{2x}, \sqrt{\lambda})) f_\gamma(x)\, dx$, where $Q_M(a, b)$ is the generalized Marcum-Q function. Let $h(x) = 1 - Q_u(\sqrt{2x}, \sqrt{\lambda})$. By using the contour integral representation of the generalized Marcum-Q function from [16], the Mellin transform $H(s)$ can be derived. For example, under Nakagami-m fading with fading parameter m, by using Proposition 3, we find

$$P_m = \begin{cases} 1 - \displaystyle\sum_{k=0}^\infty \sum_{n=0}^\infty \frac{(-1)^n \Gamma(u+k, \frac{\lambda}{2})\Gamma(m+k+n)}{k!n!\Gamma(u+k)m^{k+n}\Gamma(m)} \rho^{k+n} \\ \displaystyle\sum_{k=0}^\infty \frac{(-1)^k}{(k)!} \frac{m^{m+k}H(k+m)}{\Gamma(m)\rho^{k+m}}, \end{cases} \qquad (24)$$

where the upper expansion holds for $\rho \to 0^+$ and the other for $\rho \to \infty$. These expansions can be used to derive an UA for this case.

IV. Outage Probability

Outage probability, a common quality-of-service parameter for fading channel communication, is the probability that the

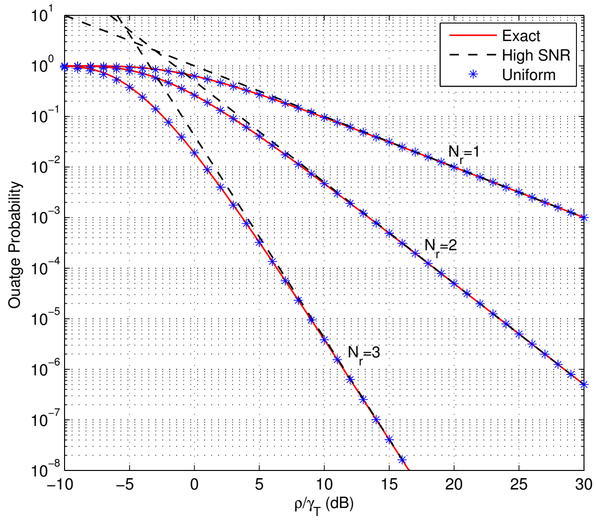

Fig. 4. The UA, exact outage and high-SNR approximation.

instantaneous SNR falls below threshold γ_T:

$$P_{out}(\gamma_T, \rho) = \Pr[\rho\beta \leq \gamma_T]. \tag{25}$$

The high-SNR approximation to the outage under AS2 can be readily obtained from (5) and Table 2, as

$$P_{out}(\gamma_T, \rho) \approx \frac{a}{t+1}\left(\frac{\gamma_T}{\rho}\right)^{t+1}, \tag{26}$$

which is nothing more than Proposition 5 given in [5].

We next give a UA for the outage.

Proposition 4. *Generalize AS2 as the following: the fading PDF is* $f(\beta) = \sum_{k=0}^{K} a_k \beta^{(k+t)} + O\left(\beta^{t+1}\right)$ *as* $\beta \to 0^+$ *with* $t \geq 0$. *We assume that* t *is an integer. The outage is then*

$$P_{out}(\gamma_T, \rho) = \frac{1 + \sum_{l=1}^{L} c_l \tilde{\rho}^l}{1 + \sum_{l=1}^{K} c_l \tilde{\rho}^l} + E(\tilde{\rho}), \tag{27}$$

where $K = L + (t+1) \geq 2$ *is an integer, and* $\tilde{\rho} = \rho/\gamma_T$. *The coefficient vector* $\boldsymbol{c} = (c_1, c_2, \ldots, c_K)^T$ *is given by* $\boldsymbol{c} = \boldsymbol{P}^{-1}\boldsymbol{e}_{K-L}$, *where* $\boldsymbol{e}_i$ *is a* $K \times 1$ *column vector whose* i^{th} *element is* 1, *all other elements being* 0 *and* $\boldsymbol{P} = \{p_{ij}\}, i = 1, \ldots, K, j = 1, \ldots, K$ *with*

$$p_{ij} = \begin{cases} -1 & i = K-n,\ j = L-n;\ n = 0, 1, \ldots, L-1 \\ b(j-i) & \text{otherwise,} \end{cases} \tag{28}$$

$b(n)$ *defined as,* $b(n) = \begin{cases} 0 & n < 0 \\ \frac{a_n}{n+t+1} & \text{otherwise.} \end{cases}$

Example 2: Consider a simple case of single-branch Rayleigh fading channel. Proposition 4 in this case yields the UA:

$$P_{out}(\gamma_T, \rho) \approx \frac{1 + 3\,\tilde{\rho} + 6\,\tilde{\rho}^2}{1 + 3\,\tilde{\rho} + 6\,\tilde{\rho}^2 + 6\,\tilde{\rho}^3}. \tag{29}$$

The UA (27), along with the high-SNR approximation (26) and the exact outage for the N_r branch MRC in iid Rayleigh fading, are plotted in Figure 4. The excellent accuracy of the UA over the entire range of SNR is clearly evident.

V. Conclusion

Simple uniform approximations for the BER, SER and the outage were derived. The UAs are accurate over the whole range of SNR values, unlike all existing high-SNR approximations. The additional information required to compute the UA of error probabilities is the several moments (fractional) of channel gain, which can be extracted readily from the PDF, MGF or the Mellin transform of the PDF. The computation of the UA is extremely simple. Additionally, in Proposition 1, we provided a simple generalization of the previous asymptotic results of [5], yielding unified asymptotic results of the average of an arbitrary performance measure. In Proposition 2, we provided a means to generate both low-SNR and high-SNR approximations simultaneously.

This paper seems to be the first one to develop the notion of UAs. They provide an alternative to closed-form analysis and also an excellent validation tool for simulations. While several applications were roughly outlined, we only scratched the surface, for the number of applications is vast.

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